Answer Set Programming

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Answer Set Programming

Winter Semester 2011/12

- Martin Gebser
- Torsten Schaub
- Marius Schneider

Information

- Lecture: 2h (weekly)
- Exercises: 2h (weekly)
- Credits: 6 **if**
  1. Written exam (at least "ausreichend")
  2. Two successful projects (= Implementation+Consultation)
- Mark: mark of written exam
- C(ourse)MS: `http://moodle.cs.uni-potsdam.de/`
- General Info: `http://www.cs.uni-potsdam.de/wv/lehre`
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Roadmap

- Introduction
- Modeling
- Language Extensions
- Operators, Algorithms, and Systems
- Applications

Resources

- Course material
  - http://www.cs.uni-potsdam.de/wv/lehre
  - http://moodle.cs.uni-potsdam.de
  - http://www.cs.uni-potsdam.de/~torsten/asp
- Systems
  - clasp http://potassco.sourceforge.net
  - dlv http://www.dbai.tuwien.ac.at/proj/dlv
  - smodels http://www.tcs.hut.fi/Software/smodels
  - gringo http://potassco.sourceforge.net
  - lparse http://www.tcs.hut.fi/Software/smodels
  - clingo http://potassco.sourceforge.net
  - iclingo http://potassco.sourceforge.net
  - oclingo http://potassco.sourceforge.net
  - asparagus http://asparagus.cs.uni-potsdam.de

Literature

Books [5], [65]

Surveys [59], [3], [47]

Articles [49], [50], [7], [71], [66], [58], [48], etc.

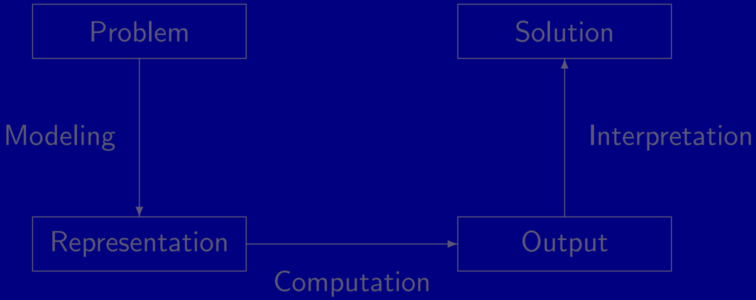
Motivation: Overview

1. Objective
2. Answer Set Programming
3. Historic Roots
4. Problem Solving
5. Applications
6. A First Example

Overview

Goal: Declarative problem solving

- *"What is the problem?"*
  instead of
- *"How to solve the problem?"*



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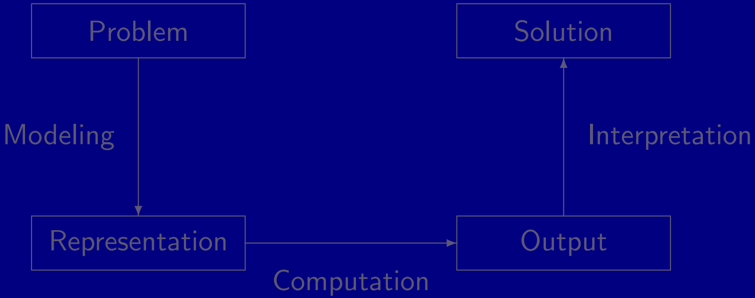

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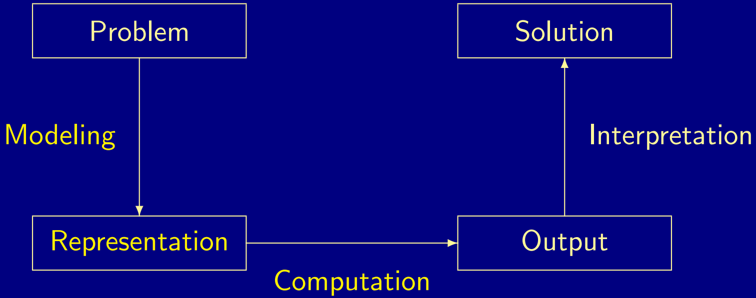

Overview

Answer Set Programming (ASP) *in a Nutshell*

- ASP is an approach to declarative problem solving, combining
  - a rich yet simple modeling language
  - with high-performance solving capacities

  tailored to Knowledge Representation and Reasoning
- ASP allows for solving all search problems in NP (and NP^{NP}) in a uniform way (being more compact than SAT)
- The versatility of ASP is reflected by the ASP solver `clasp`, winning first places at ASP'07/09/11, PB'09/11, and SAT'09/11
  - `http://potassco.sourceforge.net`
- ASP embraces many emerging application areas, eg.
  - second place at RoboCup@Home 2011 by USTC, Peking
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Logic Programming

- Algorithm = Logic + Control [55]
- Logic as a programming language
  - ➥ Prolog (Colmerauer, Kowalski)
- Features of Prolog
  - Declarative (relational) programming language
  - Based on SLD(NF) Resolution
  - Top-down query evaluation
  - Terms as data structures
  - Parameter passing by unification
  - Solutions are extracted from instantiations of variables occurring in the query

Prolog: Programming in logic

Prolog is great, it's almost declarative!

To see this, consider

```
above(X,Y) :- on(X,Y).
above(X,Y) :- on(X,Z),above(Z,Y).
```

and compare it to

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above(X,Y) :- above(Z,Y),on(X,Z).
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An interpretation in classical logic amounts to

$$\forall xy(on(x,y) \vee \exists z(on(x,z) \wedge above(z,y)) \rightarrow above(x,y))$$

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Model-based Problem Solving

Traditional approach (e.g. Prolog)

1. Provide a specification of the problem.
2. A solution is given by a derivation of an appropriate query.

Model-based approach (e.g. ASP and SAT)

1. Provide a specification of the problem.
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Automated planning, Kautz and Selman [53]

Represent planning problems as propositional theories so that models not proofs describe solutions (e.g. `Satplan`)

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Model-based Problem Solving

Specification	Associated Structures
constraint satisfaction problem	assignment
propositional horn theories	smallest model
propositional theories	models
propositional theories	minimal models
propositional theories	stable models
propositional programs	minimal models
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first-order theories	models
default theories	extensions
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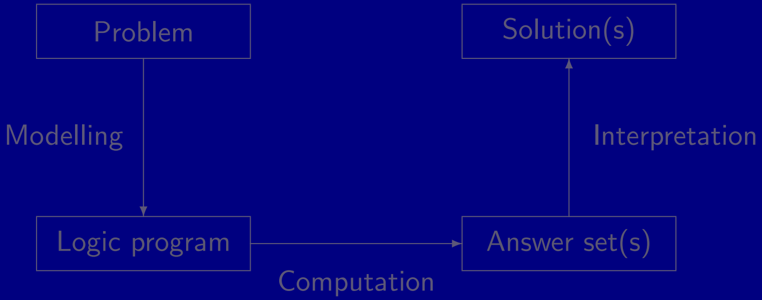
ASP as High-level Language

- Basic Idea:
  - Encode problem (class+instance) as a set of rules
  - Read off solutions from answer sets of the rules



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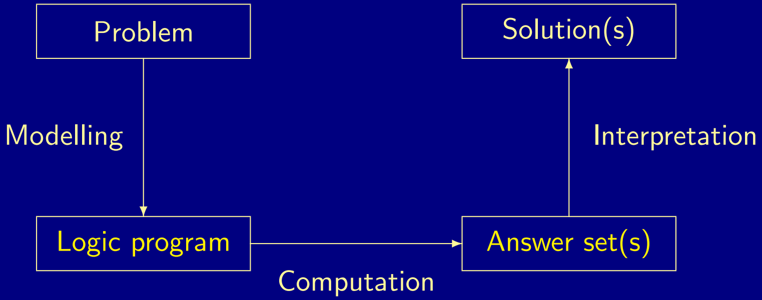

ASP as Low-level Language

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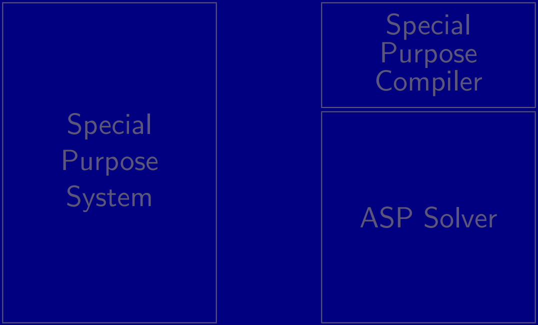

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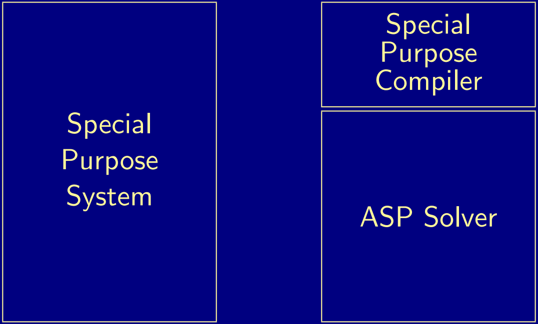

Overview

What is ASP good for?

- Combinatorial search problems
  (some with substantial amount of data):
  - For instance, auctions, bio-informatics, computer-aided verification, configuration, constraint satisfaction, diagnosis, information integration, planning and scheduling, security analysis, semantic web, wire-routing, zoology and linguistics, and many more
- My favorite: Using ASP as a basis for a decision support system for NASA's space shuttle (Gelfond et al., Texas Tech)
- Our own applications:
  - Automatic synthesis of multiprocessor systems
  - Inconsistency detection, diagnosis, repair, and prediction
    in large biological networks
  - Home monitoring for risk prevention in ambient assisted living
  - General game playing

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What does ASP offer?

- Integration of KR, DB, and search techniques
- Compact, easily maintainable problem representations
- Rapid application development tool
- Easy handling of dynamic, knowledge intensive applications
  (including: data, frame axioms, exceptions, defaults, closures, etc.)

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$$\text{ASP} = \text{KR} + \text{DB} + \text{Search}$$

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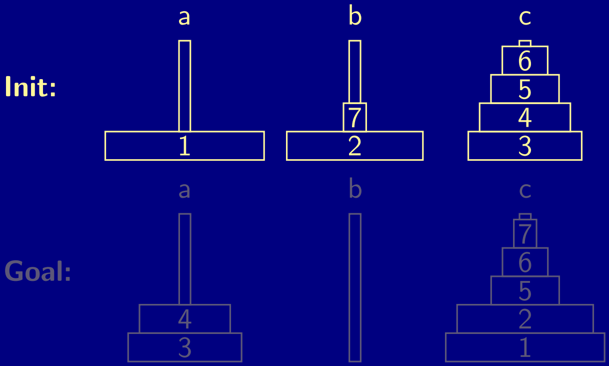
An instance of Towers of Hanoi




```
peg(a;b;c).
init_on(1,a).
init_on(2;7,b).
init_on(3;4;5;6,c).
```

```
disk(1..7).
goal_on(3;4,a).
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moves(70).
```
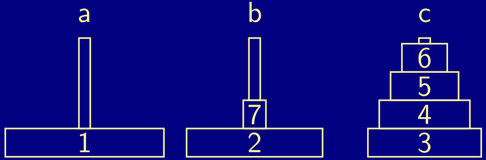
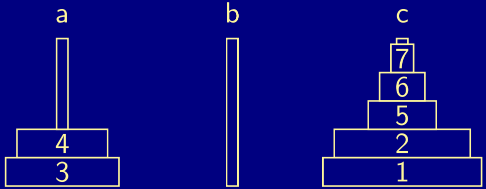

An instance of Towers of Hanoi

Init:



Goal:

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1 { move(D,P,T) : disk(D) : peg(P) } 1 :- moves(M), T = 1..M.
move(D,T)    :- move(D,_,T).

on(D,P,T)    :- move(D,P,T).
on(D,P,T+1) :- on(D,P,T), not move(D,T+1), not moves(T).

blocked(D-1,P,T+1) :- on(D,P,T), not moves(T).
blocked(D-1,P,T)   :- blocked(D,P,T), disk(D).

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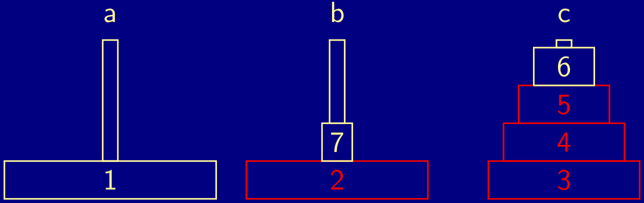


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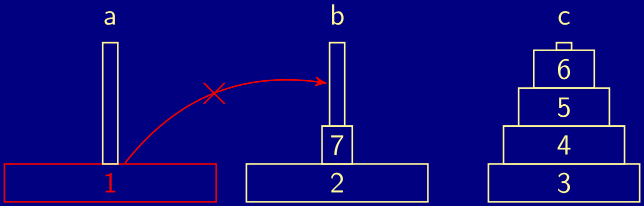


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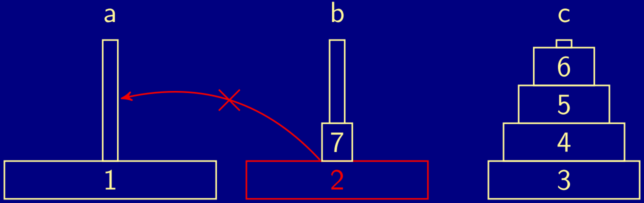


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move(D,T)    :- move(D,_,T).

on(D,P,T)    :- move(D,P,T).
on(D,P,T+1) :- on(D,P,T), not move(D,T+1), not moves(T).

blocked(D-1,P,T+1) :- on(D,P,T), not moves(T).
blocked(D-1,P,T)   :- blocked(D,P,T), disk(D).

:- move(D,P,T), blocked(D-1,P,T).
:- move(D,T), on(D,P,T-1), blocked(D,P,T).
:- not 1 { on(D,P,T) } 1, disk(D), moves(M), T = 1..M.
:- goal_on(D,P), not on(D,P,M), moves(M).
```

Let it run!

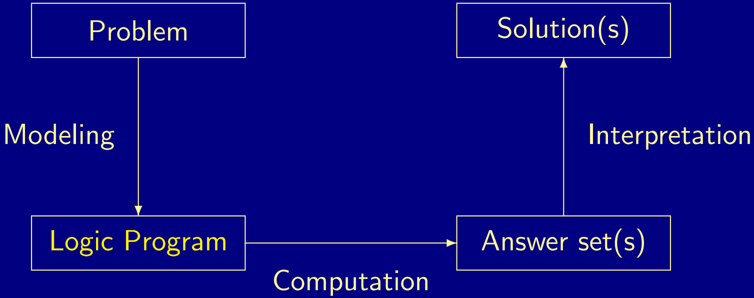
```
torsten@raz > gringo toh_instance.lp toh_encoding.lp | clasp --stats
clasp version 1.3.5
Reading from stdin
Solving...
Answer: 1
peg(a) peg(c) peg(b) init_on(1,a) init_on(2,b) ...
move(6,a,1)  move(7,a,2)  move(5,b,3)  move(7,c,4)
move(6,b,5)  move(7,b,6)  move(4,a,7)  move(7,a,8) ...
move(2,c,63) move(7,c,64) move(6,b,65) move(7,b,66)
move(5,c,67) move(7,a,68) move(6,c,69) move(7,c,70)
move(7,70) move(6,69) move(7,68) move(5,67) move(7,66) ...
SATISFIABLE

Models       : 1+
Time         : 3.280s (Solving: 3.23s 1st Model: 3.23s Unsat: 0.00s)
Choices      : 130907
Conflicts    : 35738
Restarts     : 12
```

Introduction: Overview

Overview

Problem solving in ASP: Syntax



Normal logic programs

- A (normal) rule, r, is an ordered pair of the form

$$A_0 \leftarrow A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n,$$

  where $n \geq m \geq 0$, and each A_i $(0 \leq i \leq n)$ is an atom.
- A (normal) logic program is a finite set of rules.
- Notation

$$\begin{aligned} head(r) &= A_0 \\ body(r) &= \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\} \\ body^+(r) &= \{A_1, \dots, A_m\} \\ body^-(r) &= \{A_{m+1}, \dots, A_n\} \end{aligned}$$

- A program is called positive if $body^-(r) = \emptyset$ for all its rules.

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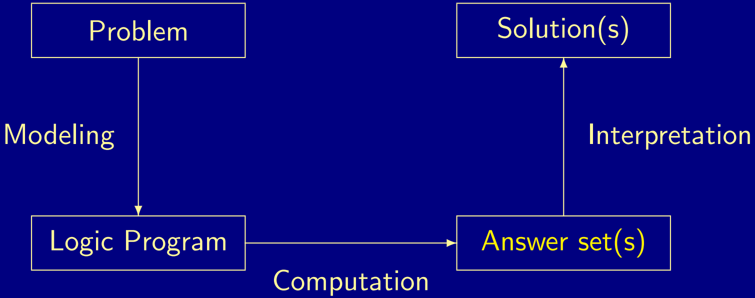

Answer set: Formal Definition

Positive programs

- A set of atoms X is closed under a positive program Π iff for any $r \in \Pi$, $head(r) \in X$ whenever $body^+(r) \subseteq X$.
  - ➥ X corresponds to a model of Π (seen as a formula).
- The smallest set of atoms which is closed under a positive program Π is denoted by $Cn(\Pi)$.
  - ➥ $Cn(\Pi)$ corresponds to the $\subseteq$-smallest model of Π (ditto).
- The set $Cn(\Pi)$ of atoms is the answer set of a *positive* program Π.

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Some “logical” remarks

- Positive rules are also referred to as definite clauses.
  - Definite clauses are disjunctions with exactly one positive atom:

$$A_0 \vee \neg A_1 \vee \cdots \vee \neg A_m$$

  - A set of definite clauses has a (unique) smallest model.
- Horn clauses are clauses with at most one positive atom.
  - Every definite clause is a Horn clause but not vice versa.
  - A set of Horn clauses has a smallest model or none.
- This smallest model is the intended semantics of a set of Horn clauses.
  - ☞ Given a positive program Π, $Cn(\Pi)$ corresponds to the smallest model of the set of definite clauses corresponding to Π.

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Answer set: Basic idea

Consider the logical formula Φ and its three (classical) models:

Φ $\boxed{q \ \wedge \ (q \wedge \neg r \rightarrow p)}$

$\{p, q\}, \{q, r\}$, and $\{p, q, r\}$

Formula Φ has one stable model, called answer set:

Π_Φ $\boxed{\begin{array}{lll} q & \leftarrow & \\ p & \leftarrow & q, \ not \ r \end{array}}$

$\{p, q\}$

Informally, a set X of atoms is an answer set of a logic program Π

if X is a (classical) model of Π and

if all atoms in X are justified by some rule in Π

(rooted in intuitionistic logics HT (Heyting, 1930) and G3 (Gödel, 1932))

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$$\begin{array}{|ccc|}\hline p & \mapsto & 1 \\ q & \mapsto & 1 \\ r & \mapsto & 0 \\ \hline\end{array}$$

Π_Φ

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  Recall: $Cn(\Pi^X)$ is the $\subseteq$–smallest (classical) model of Π^X.

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A closer look at Π^X

In other words, given a set X of atoms from Π,

Π^X is obtained from Π by deleting

1. each rule having a *not* A in its body with $A \in X$ and then
2. all negative atoms of the form *not* A in the bodies of the remaining rules.

Overview

A first example

$\Pi = \{p \leftarrow p,\ q \leftarrow not\ p\}$

X	Π^X	$Cn(\Pi^X)$
$\emptyset$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$
$\{p\}$	$p \leftarrow p$	$\emptyset$
$\{q\}$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$
$\{p,q\}$	$p \leftarrow p$	$\emptyset$

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$$\Pi = \{p \leftarrow p,\ q \leftarrow not\ p\}$$

X	Π^X	$Cn(\Pi^X)$	
$\emptyset$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$	✗
$\{p\}$	$p \leftarrow p$	$\emptyset$	✗
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A second example

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$\{p, q\}$		$\emptyset$

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Answer set: Some properties

- A logic program may have zero, one, or multiple answer sets!
- If X is an answer set of a logic program Π,
  then X is a model of Π (seen as a formula).
- If X and Y are answer sets of a *normal* program Π,
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Answer set: Alternative Definition

Let Π be a normal program and X a set of atoms.

- The set of generating rules of X relative to Π is defined by

$$\Pi_X = \{r \in \Pi \mid body^+(r) \subseteq X \text{ and } body^-(r) \cap X = \emptyset\}.$$

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The second example revisited

$$\Pi = \{p \leftarrow not\ q,\ \ q \leftarrow not\ p\}$$

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$\emptyset$	$p \leftarrow not\ q$ $q \leftarrow not\ p$	$p \vee q$	$\{p\}, \{q\}$	✗
$\{p\}$	$p \leftarrow not\ q$	$p \vee q$	$\{p\}, \{q\}$	✔
$\{q\}$	$q \leftarrow not\ p$	$p \vee q$	$\{p\}, \{q\}$	✔
$\{p, q\}$		$\top$	$\emptyset$	✗

The second example revisited

$\Pi = \{p \leftarrow not\ q,\ q \leftarrow not\ p\}$

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$\{q\}$	$q \leftarrow not\ p$	$p \vee q$	$\{p\}, \{q\}$	✔
$\{p, q\}$		$\top$	$\emptyset$	✘

A closer look at Cn
Inductive characterization

Let Π be a positive program and X a set of atoms.

- The immediate consequence operator T_Π is defined as follows:

$$T_\Pi X = \{head(r) \mid r \in \Pi \text{ and } body(r) \subseteq X\}$$

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Theorem

For any positive program Π, we have

- $Cn(\Pi) = \bigcup_{i \geq 0} T^i_\Pi \emptyset$,
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A closer look at Cn
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Let's iterate T_Π

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T_\Pi^0\emptyset & = & \emptyset & & & & \\
T_\Pi^1\emptyset & = & \{p,q\} & = & T_\Pi T_\Pi^0\emptyset & = & T_\Pi\emptyset \\
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To see that $Cn(\Pi) = \{p,q,r,t,s\}$ is the smallest fixpoint of T_Π, note that $T_\Pi\{p,q,r,t,s\} = \{p,q,r,t,s\}$ and $T_\Pi X \neq X$ for every $X \subseteq \{p,q,r,t,s\}$.

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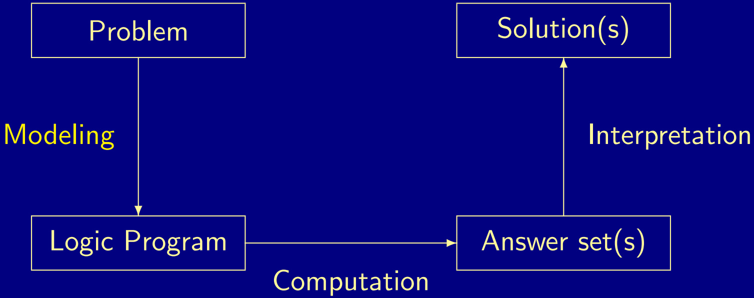
Overview

Problem solving in ASP: Modeling



(Rough) notational convention

We sometimes use the following notation interchangeably in order to stress the respective view:

	if	and	or	negation as failure	classical negation
source code	`:-`	,	|	`not`	`-`
logic program	$\leftarrow$	,	;	*not*/$\sim$	$\neg$
formula	$\rightarrow$	$\wedge$	$\vee$	$\sim/(\neg)$	$\neg$

Language Constructs

Variables (over the Herbrand Universe)

`p(X) :- q(X)` over constants $\{a, b, c\}$ stands for

```
p(a) :- q(a), p(b) :- q(b), p(c) :- q(c)
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Disjunction

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p(X) | q(X) :- r(X)
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Integrity Constraints

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:- q(X), p(X)
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Choice

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2 { p(X,Y) : q(X) } 7 :- r(Y)
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Aggregates

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s(Y) :- r(Y), 2 #count { p(X,Y) : q(X) } 7
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Overview

Programs with Variables

Let Π be a logic program.

- Herbranduniverse U^{Π}: Set of constants in Π
- Herbrandbase B^{Π}: Set of (variable-free) atoms constructible from U^{Π}
  ☞ We usually denote this as $\mathcal{A}$, and call it alphabet.
- Ground Instances of $r \in \Pi$: Set of variable-free rules obtained by replacing all variables in r by elements from U^{Π}:

  $$ground(r) = \{r\theta \mid \theta : var(r) \rightarrow U^{\Pi}\}$$

  where $var(r)$ stands for the set of all variables occurring in r; θ is a (ground) substitution.
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An example

$$\Pi = \{\ r(a,b) \leftarrow,\ \ r(b,c) \leftarrow,\ \ t(X,Y) \leftarrow r(X,Y)\ \}$$

$$U^\Pi = \{a, b, c\}$$

$$B^\Pi = \left\{ \begin{array}{l} r(a,a), r(a,b), r(a,c), r(b,a), r(b,b), r(b,c), r(c,a), r(c,b), r(c,c), \\ t(a,a), t(a,b), t(a,c), t(b,a), t(b,b), t(b,c), t(c,a), t(c,b), t(c,c) \end{array} \right\}$$

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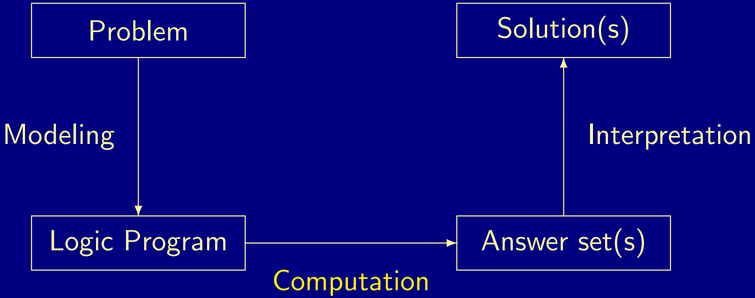
Answer sets of programs with Variables

Let Π be a normal logic program with variables.

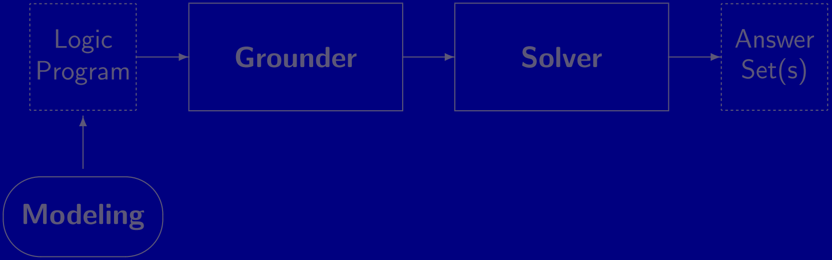
We define a set X of (ground) atoms as an answer set of Π
if $Cn(ground(\Pi)^X) = X$.

Overview

Problem solving in ASP: Computation



ASP Solving Process



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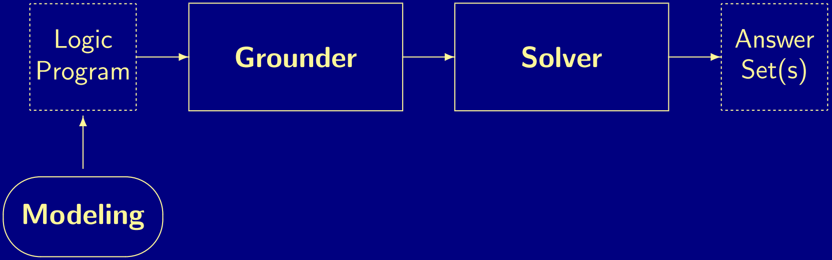

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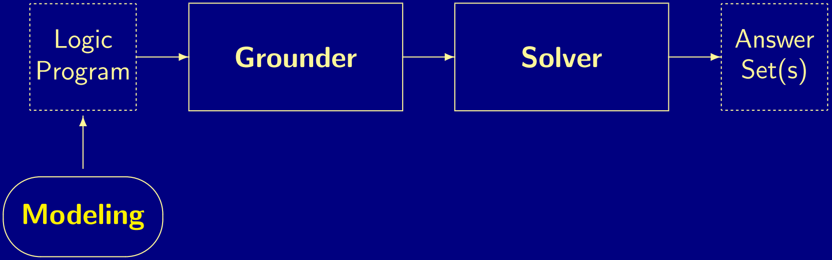

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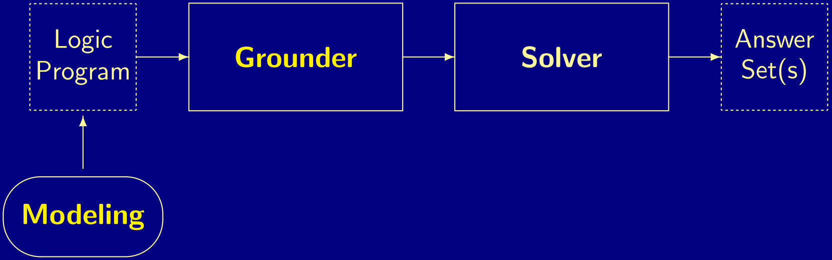

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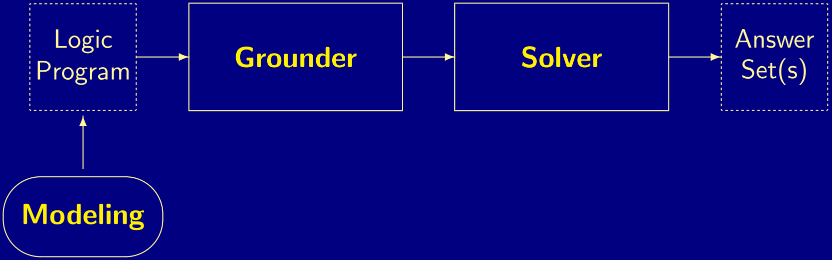

Traditional Solving Procedure

Global parameters: Logic program Π and its set $\mathcal{A}$ of atoms.

$solve_\Pi(X, Y)$

1. $(X, Y) \leftarrow propagate_\Pi(X, Y)$
2. **if** $(X \cap Y) \neq \emptyset$ **then fail**
3. **if** $(X \cup Y) = \mathcal{A}$ **then return**(X)
4. **select** $A \in \mathcal{A} \setminus (X \cup Y)$
5. $solve_\Pi(X \cup \{A\}, Y)$
6. $solve_\Pi(X, Y \cup \{A\})$

Comments:

- (X, Y) is supposed to be a 3-valued model such that $X \subseteq Z$ and $Y \cap Z = \emptyset$ for any answer set Z of Π.
- Key operations: $propagate_\Pi(X, Y)$ and '**select** $A \in \mathcal{A} \setminus (X \cup Y)$'
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Overview

Reasoning Modes

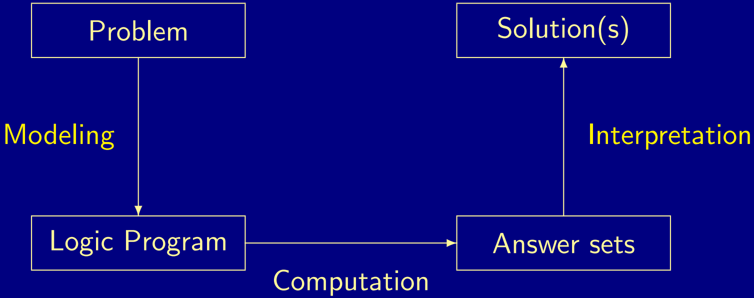
- Satisfiability
- Enumeration†
- Projection†
- Intersection‡
- Union‡
- Optimization
- Sampling

† without solution recording

‡ without solution enumeration

Basic Modeling: Overview

Modeling and Interpreting



Modeling

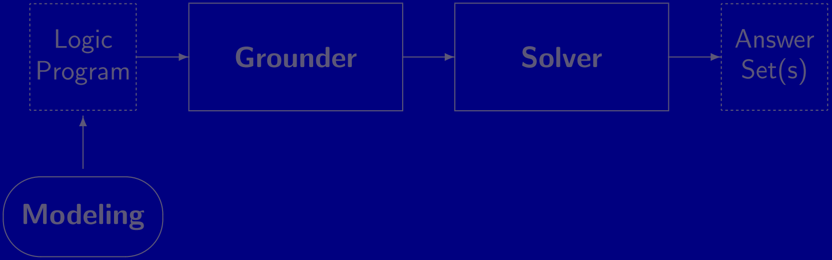
For solving a problem class P for a problem instance I,
encode

1. the problem instance I as a set $C(I)$ of facts and
2. the problem class P as a set $C(P)$ of rules

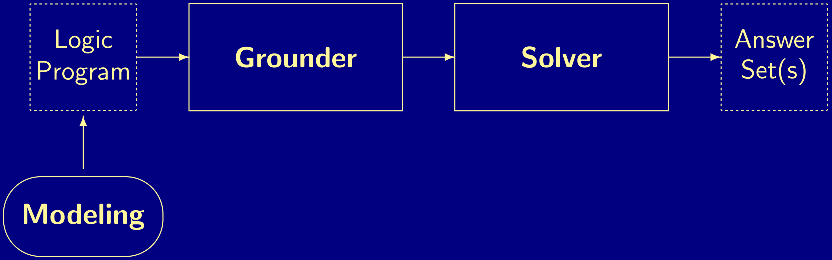
such that the solutions to P for I can be (polynomially) extracted from the answer sets of $C(I) \cup C(P)$.

Overview

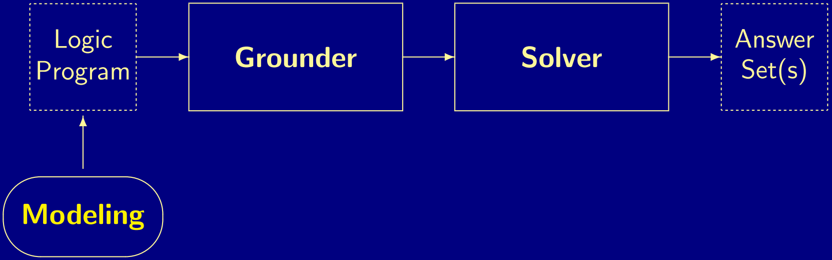
ASP Solving Process



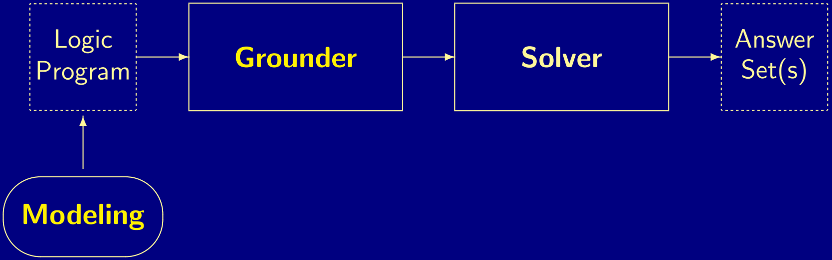
ASP Solving Process



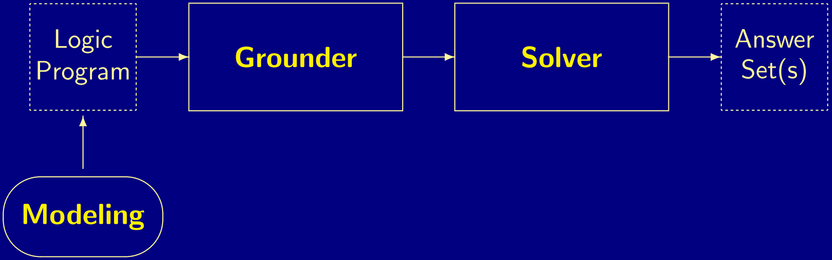
ASP Solving Process



ASP Solving Process



ASP Solving Process



Overview

Graph Coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).

1 {color(X,C) : col(C)} 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

Graph Coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).

1 {color(X,C) : col(C)} 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

Graph Coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).

1 {color(X,C) : col(C)} 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

Graph Coloring

```
node(1..6).

edge(1,2).  edge(1,3).  edge(1,4).
edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).
edge(4,1).  edge(4,2).
edge(5,3).  edge(5,4).  edge(5,6).
edge(6,2).  edge(6,3).  edge(6,5).

col(r).   col(b).   col(g).

1 {color(X,C) : col(C)} 1 :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

Graph Coloring: Grounding

$ gringo -t color.lp

```
node(1).  node(2).  node(3).  node(4).  node(5).  node(6).

edge(1,2).  edge(1,3).  edge(1,4).  edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).  edge(4,1).  edge(4,2).  edge(5,3).
edge(5,4).  edge(5,6).  edge(6,2).  edge(6,3).  edge(6,5).

col(r).  col(b).  col(g).

1 {color(1,r), color(1,b), color(1,g)} 1.
1 {color(2,r), color(2,b), color(2,g)} 1.
1 {color(3,r), color(3,b), color(3,g)} 1.
1 {color(4,r), color(4,b), color(4,g)} 1.
1 {color(5,r), color(5,b), color(5,g)} 1.
1 {color(6,r), color(6,b), color(6,g)} 1.

 :- color(1,r), color(2,r).  :- color(2,g), color(5,g). ...  :- color(6,r), color(2,r).
 :- color(1,b), color(2,b).  :- color(2,r), color(6,r).       :- color(6,b), color(2,b).
 :- color(1,g), color(2,g).  :- color(2,b), color(6,b).       :- color(6,g), color(2,g).
 :- color(1,r), color(3,r).  :- color(2,g), color(6,g).       :- color(6,r), color(3,r).
 :- color(1,b), color(3,b).  :- color(3,r), color(1,r).       :- color(6,b), color(3,b).
 :- color(1,g), color(3,g).  :- color(3,b), color(1,b).       :- color(6,g), color(3,g).
 :- color(1,r), color(4,r).  :- color(3,g), color(1,g).       :- color(6,r), color(5,r).
 :- color(1,b), color(4,b).  :- color(3,r), color(4,r).       :- color(6,b), color(5,b).
 :- color(1,g), color(4,g).  :- color(3,b), color(4,b).       :- color(6,g), color(5,g).
 :- color(2,r), color(4,r).  :- color(3,g), color(4,g).
 :- color(2,b), color(4,b).  :- color(3,r), color(5,r).
 :- color(2,g), color(4,g).  :- color(3,b), color(5,b).
```

Graph Coloring: Grounding

$ gringo -t color.lp

```
node(1).  node(2).  node(3).  node(4).  node(5).  node(6).

edge(1,2).  edge(1,3).  edge(1,4).  edge(2,4).  edge(2,5).  edge(2,6).
edge(3,1).  edge(3,4).  edge(3,5).  edge(4,1).  edge(4,2).  edge(5,3).
edge(5,4).  edge(5,6).  edge(6,2).  edge(6,3).  edge(6,5).

col(r).  col(b).  col(g).

1 {color(1,r), color(1,b), color(1,g)} 1.
1 {color(2,r), color(2,b), color(2,g)} 1.
1 {color(3,r), color(3,b), color(3,g)} 1.
1 {color(4,r), color(4,b), color(4,g)} 1.
1 {color(5,r), color(5,b), color(5,g)} 1.
1 {color(6,r), color(6,b), color(6,g)} 1.

 :- color(1,r), color(2,r).  :- color(2,g), color(5,g). ...  :- color(6,r), color(2,r).
 :- color(1,b), color(2,b).  :- color(2,r), color(6,r).      :- color(6,b), color(2,b).
 :- color(1,g), color(2,g).  :- color(2,b), color(6,b).      :- color(6,g), color(2,g).
 :- color(1,r), color(3,r).  :- color(2,g), color(6,g).      :- color(6,r), color(3,r).
 :- color(1,b), color(3,b).  :- color(3,r), color(1,r).      :- color(6,b), color(3,b).
 :- color(1,g), color(3,g).  :- color(3,b), color(1,b).      :- color(6,g), color(3,g).
 :- color(1,r), color(4,r).  :- color(3,g), color(1,g).      :- color(6,r), color(5,r).
 :- color(1,b), color(4,b).  :- color(3,r), color(4,r).      :- color(6,b), color(5,b).
 :- color(1,g), color(4,g).  :- color(3,b), color(4,b).      :- color(6,g), color(5,g).
 :- color(2,r), color(4,r).  :- color(3,g), color(4,g).
 :- color(2,b), color(4,b).  :- color(3,r), color(5,r).
 :- color(2,g), color(4,g).  :- color(3,b), color(5,b).
```

Graph Coloring: Solving

```
$ gringo color.lp | clasp 0
```

```
clasp version 1.2.1
Reading from stdin
Reading      : Done(0.000s)
Preprocessing: Done(0.000s)
Solving...
Answer: 1
color(1,b) color(2,r) color(3,r) color(4,g) color(5,b) color(6,g) node(1) ... edge(1,2) ... col(r) ...
Answer: 2
color(1,g) color(2,r) color(3,r) color(4,b) color(5,g) color(6,b) node(1) ... edge(1,2) ... col(r) ...
Answer: 3
color(1,b) color(2,g) color(3,g) color(4,r) color(5,b) color(6,r) node(1) ... edge(1,2) ... col(r) ...
Answer: 4
color(1,g) color(2,b) color(3,b) color(4,r) color(5,g) color(6,r) node(1) ... edge(1,2) ... col(r) ...
Answer: 5
color(1,r) color(2,b) color(3,b) color(4,g) color(5,r) color(6,g) node(1) ... edge(1,2) ... col(r) ...
Answer: 6
color(1,r) color(2,g) color(3,g) color(4,b) color(5,r) color(6,b) node(1) ... edge(1,2) ... col(r) ...

Models      : 6
Time        : 0.000  (Solving: 0.000)
```

Graph Coloring: Solving

```
$ gringo color.lp | clasp 0
```

```
clasp version 1.2.1
Reading from stdin
Reading      : Done(0.000s)
Preprocessing: Done(0.000s)
Solving...
Answer: 1
color(1,b) color(2,r) color(3,r) color(4,g) color(5,b) color(6,g) node(1) ... edge(1,2) ... col(r) ...
Answer: 2
color(1,g) color(2,r) color(3,r) color(4,b) color(5,g) color(6,b) node(1) ... edge(1,2) ... col(r) ...
Answer: 3
color(1,b) color(2,g) color(3,g) color(4,r) color(5,b) color(6,r) node(1) ... edge(1,2) ... col(r) ...
Answer: 4
color(1,g) color(2,b) color(3,b) color(4,r) color(5,g) color(6,r) node(1) ... edge(1,2) ... col(r) ...
Answer: 5
color(1,r) color(2,b) color(3,b) color(4,g) color(5,r) color(6,g) node(1) ... edge(1,2) ... col(r) ...
Answer: 6
color(1,r) color(2,g) color(3,g) color(4,b) color(5,r) color(6,b) node(1) ... edge(1,2) ... col(r) ...

Models      : 6
Time        : 0.000  (Solving: 0.000)
```

Overview

Basic Methodology

Generate and Test (or: Guess and Check) approach

Generator	Generate potential answer set candidates (typically through non-deterministic constructs)
Tester	Eliminate invalid candidates (typically through integrity constraints)

Nutshell

$$\text{Logic program} \quad = \quad \text{Data} + \text{Generator} + \text{Tester} \\ (+\ \text{Optimizer})$$

Basic Methodology

Generate and Test (or: Guess and Check) approach

Generator: Generate potential answer set candidates
(typically through non-deterministic constructs)

Tester: Eliminate invalid candidates
(typically through integrity constraints)

Nutshell

Logic program $=$ Data $+$ Generator $+$ Tester
($+$ Optimizer)

Satisfiability

- Problem Instance: A propositional formula ϕ.
- Problem Class: Is there an assignment of propositional variables to true and false such that a given formula ϕ is true.

- Example: Consider formula $(a \vee \neg b) \wedge (\neg a \vee b)$.
- Logic Program:

Generator	**Tester**	**Answer sets**
$\{a,b\} \leftarrow$	$\leftarrow$ not a, b	$X_1 = \{a,b\}$
	$\leftarrow$ a, not b	$X_2 = \{\}$

Satisfiability

- Problem Instance: A propositional formula ϕ.
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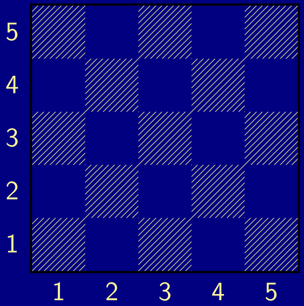
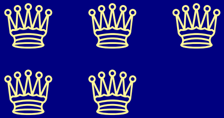
- Logic Program:

Generator	**Tester**	**Answer sets**
$\{a,b\} \leftarrow$	$\leftarrow$ not a, b	$X_1 = \{a,b\}$
	$\leftarrow$ a, not b	$X_2 = \{\}$

The n-Queens Problem



- Place n queens on an $n \times n$ chess board
- Queens must not attack one another



Defining the Field

queens.lp

```
row(1..n).
col(1..n).
```

- Create file `queens.lp`
- Define the field
  - n rows
  - n columns

Defining the Field

Running ...

```
$ clingo queens.lp -c n=5
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
SATISFIABLE

Models       : 1
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Placing some Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I) : col(J) }.
```

- Guess a solution candidate
- Place some queens on the board

Placing some Queens

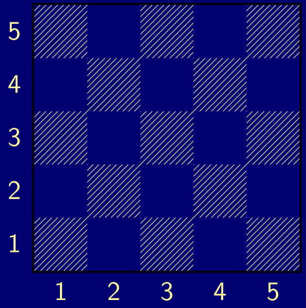
Running ...

```
$ clingo queens.lp -c n=5 3
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(1,1)
Answer: 3
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(2,1)
SATISFIABLE

Models      : 3+
```

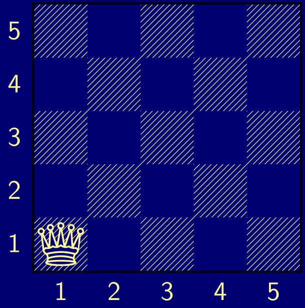
Placing some Queens: Answer 1

Answer 1



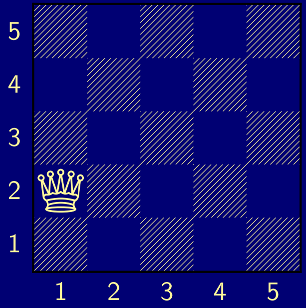
Placing some Queens: Answer 2

Answer 2



Placing some Queens: Answer 3

Answer 3



Placing n Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I) : col(J) }.
:- not { queen(I,J) } == n.
```

- Place exactly n queens on the board

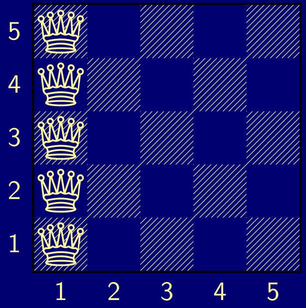
Placing n Queens

Running ...

```
$ clingo queens.lp -c n=5 2
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,1) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(1,2) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
...
```

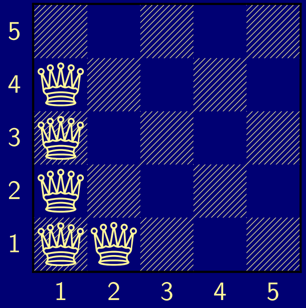
Placing n Queens: Answer 1

Answer 1



Placing n Queens: Answer 2

Answer 2



Horizontal and vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I) : col(J) }.
:- not { queen(I,J) } == n.
:- queen(I,J), queen(I,JJ), J != JJ.
:- queen(I,J), queen(II,J), I != II.
```

- Forbid horizontal attacks
- Forbid vertical attacks

Horizontal and vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I) : col(J) }.
:- not { queen(I,J) } == n.
:- queen(I,J), queen(I,JJ), J != JJ.
:- queen(I,J), queen(II,J), I != II.
```

- Forbid horizontal attacks
- Forbid vertical attacks

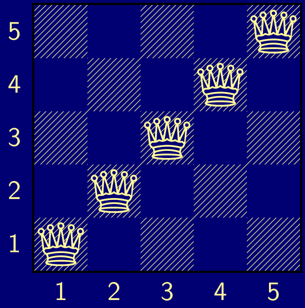
Horizontal and vertical Attack

Running ...

```
$ clingo queens.lp -c n=5
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,5) queen(4,4) queen(3,3) \
queen(2,2) queen(1,1)
...
```

Horizontal and vertical Attack: Answer 1

Answer 1



Diagonal Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I) : col(J) }.
:- not { queen(I,J) } == n.
:- queen(I,J), queen(I,JJ), J != JJ.
:- queen(I,J), queen(II,J), I != II.
:- queen(I,J), queen(II,JJ), (I,J) != (II,JJ),
   I-J == II-JJ.
:- queen(I,J), queen(II,JJ), (I,J) != (II,JJ),
   I+J == II+JJ.
```

- Forbid diagonal attacks

Diagonal Attack

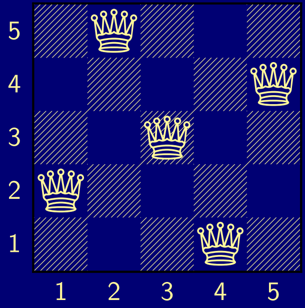
Running ...

```
$ clingo queens.lp -c n=5
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(4,5) queen(1,4) queen(3,3) \
queen(5,2) queen(2,1)
SATISFIABLE

Models       : 1+
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Diagonal Attack: Answer 1

Answer 1



Optimizing

queens-opt.lp

```
{ queen(I,1..n) } == 1 :- I = 1..n.
{ queen(1..n,J) } == 1 :- J = 1..n.
:- { queen(D-J,J) } >= 2, D =   2..2*n.
:- { queen(D+J,J) } >= 2, D = 1-n..n-1.
```

- Encoding can be optimized
- Much faster to solve

- See Section *Tweaking N-Queens*

Reviewer Assignment
by Ilkka Niemelä

```
reviewer(r1). paper(p1). classA(r1,p1). classB(r1,p2). coi(r1,p5).
reviewer(r2). paper(p2). classA(r1,p3). classB(r1,p4). coi(r1,p6).
...

3 { assigned(P,R) : reviewer(R) } 3 :-  paper(P).

 :- assigned(P,R), coi(R,P).
 :- assigned(P,R), not classA(R,P), not classB(R,P).
 :- 9 { assigned(P,R) : paper(P) }  , reviewer(R).
 :-   { assigned(P,R) : paper(P) } 6, reviewer(R).

assignedB(P,R) :-  classB(R,P), assigned(P,R).
 :- 3 { assignedB(P,R) : paper(P) }, reviewer(R).

#minimize { assignedB(P,R) : paper(P) : reviewer(R) }.
```

Reviewer Assignment
by Ilkka Niemelä

```
reviewer(r1). paper(p1). classA(r1,p1). classB(r1,p2). coi(r1,p5).
reviewer(r2). paper(p2). classA(r1,p3). classB(r1,p4). coi(r1,p6).
...

3 { assigned(P,R) : reviewer(R) } 3 :-  paper(P).

 :- assigned(P,R), coi(R,P).
 :- assigned(P,R), not classA(R,P), not classB(R,P).
 :- 9 { assigned(P,R) : paper(P) }  , reviewer(R).
 :-   { assigned(P,R) : paper(P) } 6, reviewer(R).

assignedB(P,R) :-  classB(R,P), assigned(P,R).
 :- 3 { assignedB(P,R) : paper(P) }, reviewer(R).

#minimize { assignedB(P,R) : paper(P) : reviewer(R) }.
```

Reviewer Assignment
by Ilkka Niemelä

```
reviewer(r1). paper(p1). classA(r1,p1). classB(r1,p2). coi(r1,p5).
reviewer(r2). paper(p2). classA(r1,p3). classB(r1,p4). coi(r1,p6).
...

3 { assigned(P,R) : reviewer(R) } 3 :-  paper(P).

 :- assigned(P,R), coi(R,P).
 :- assigned(P,R), not classA(R,P), not classB(R,P).
 :- 9 { assigned(P,R) : paper(P) }  , reviewer(R).
 :-   { assigned(P,R) : paper(P) } 6, reviewer(R).

assignedB(P,R) :-  classB(R,P), assigned(P,R).
 :- 3 { assignedB(P,R) : paper(P) }, reviewer(R).

#minimize { assignedB(P,R) : paper(P) : reviewer(R) }.
```

Reviewer Assignment
by Ilkka Niemelä

```
reviewer(r1). paper(p1). classA(r1,p1). classB(r1,p2). coi(r1,p5).
reviewer(r2). paper(p2). classA(r1,p3). classB(r1,p4). coi(r1,p6).
...

3 { assigned(P,R) : reviewer(R) } 3 :-  paper(P).

 :- assigned(P,R), coi(R,P).
 :- assigned(P,R), not classA(R,P), not classB(R,P).
 :- 9 { assigned(P,R) : paper(P) }  , reviewer(R).
 :-   { assigned(P,R) : paper(P) } 6, reviewer(R).

assignedB(P,R) :-  classB(R,P), assigned(P,R).
 :- 3 { assignedB(P,R) : paper(P) }, reviewer(R).

#minimize { assignedB(P,R) : paper(P) : reviewer(R) }.
```

Reviewer Assignment
by Ilkka Niemelä

```
reviewer(r1). paper(p1). classA(r1,p1). classB(r1,p2). coi(r1,p5).
reviewer(r2). paper(p2). classA(r1,p3). classB(r1,p4). coi(r1,p6).
...

3 { assigned(P,R) : reviewer(R) } 3 :-  paper(P).

 :- assigned(P,R), coi(R,P).
 :- assigned(P,R), not classA(R,P), not classB(R,P).
 :- 9 { assigned(P,R) : paper(P) }  , reviewer(R).
 :-   { assigned(P,R) : paper(P) } 6, reviewer(R).

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Simplistic STRIPS Planning

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fluent(p).     fluent(q).    fluent(r).
action(a).     pre(a,p).     add(a,q).     del(a,p).
action(b).     pre(b,q).     add(b,r).     del(b,q).
init(p).       query(r).

time(1..k).    lasttime(T) :- time(T), not time(T+1).

holds(P,0) :- init(P).

1 { occ(A,T) : action(A) } 1 :- time(T).
 :- occ(A,T), pre(A,F), not holds(F,T-1).

ocdel(F,T) :- occ(A,T), del(A,F).
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Simplistic STRIPS Planning with iASP

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#base.

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init(p).       query(r).

holds(P,0) :- init(P).

#cumulative t.

1 { occ(A,t) : action(A) } 1.
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#volatile t.

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```

Disjunctive logic programs: Overview

Overview

17 Syntax

18 Semantics

19 Examples

Disjunctive logic programs

- A disjunctive rule, r, is an ordered pair of the form

$$A_1 \,; \ldots ; A_m \leftarrow A_{m+1}, \ldots, A_n, \mathit{not}\ A_{n+1}, \ldots, \mathit{not}\ A_o,$$

  where $o \geq n \geq m \geq 0$, and each A_i $(0 \leq i \leq o)$ is an atom.
- A disjunctive logic program is a finite set of disjunctive rules.
- (Generalized) Notation

$$\begin{aligned}
\mathit{head}(r) &= \{A_1, \ldots, A_m\}\\
\mathit{body}(r) &= \{A_{m+1}, \ldots, A_n, \mathit{not}\ A_{n+1}, \ldots, \mathit{not}\ A_o\}\\
\mathit{body}^+(r) &= \{A_{m+1}, \ldots, A_n\}\\
\mathit{body}^-(r) &= \{A_{n+1}, \ldots, A_o\}
\end{aligned}$$

- A program is called positive if $\mathit{body}^-(r) = \emptyset$ for all its rules.

Overview

Answer sets

- Positive programs:
  - A set X of atoms is closed under a positive program Π iff for any $r \in \Pi$, $head(r) \cap X \neq \emptyset$ whenever $body^+(r) \subseteq X$.
    - ➥ X corresponds to a model of Π (seen as a formula).
  - The set of all $\subseteq$-minimal sets of atoms being closed under a positive program Π is denoted by $\min_\subseteq(\Pi)$.
    - ➥ $\min_\subseteq(\Pi)$ corresponds to the $\subseteq$-minimal models of Π (ditto).
- Disjunctive programs:
  - The reduct, Π^X, of a disjunctive program Π relative to a set X of atoms is defined by

$$\Pi^X = \{head(r) \leftarrow body^+(r) \mid r \in \Pi \text{ and } body^-(r) \cap X = \emptyset\}.$$

  - A set X of atoms is an answer set of a disjunctive program Π if $X \in \min_\subseteq(\Pi^X)$.
- FYI: The alternative definition on Page 104 is applicable as well.

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Overview

A "positive" example

$$\Pi = \left\{ \begin{array}{lcr} a & \leftarrow & \\ b\,;c & \leftarrow & a \end{array} \right\}$$

- The sets $\{a, b\}$, $\{a, c\}$, and $\{a, b, c\}$ are closed under Π.
- We have $\min_{\subseteq}(\Pi) = \{\ \{a, b\},\ \{a, c\}\ \}$.

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3-colorability revisited

```
node(1..6).

edge(1,2;3;4).  edge(2,4;5;6).  edge(3,1;4;5).
edge(4,1;2).    edge(5,3;4;6).  edge(6,2;3;5).

colored(X,r) | colored(X,b) | colored(X,g) :- node(X).
:- edge(X,Y), color(X,C), color(Y,C).
```

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```

More Examples

- $\Pi_1 = \{a\,;b\,;c \leftarrow\}$ has answer sets $\{a\}$, $\{b\}$, and $\{c\}$.
- $\Pi_2 = \{a\,;b\,;c \leftarrow\ ,\ \leftarrow a\}$ has answer sets $\{b\}$ and $\{c\}$.
- $\Pi_3 = \{a\,;b\,;c \leftarrow\ ,\ \leftarrow a\ ,\ b \leftarrow c\ ,\ c \leftarrow b\}$ has answer set $\{b, c\}$.
- $\Pi_4 = \{a\,;b \leftarrow c\ ,\ b \leftarrow \mathit{not}\ a, \mathit{not}\ c\ ,\ a\,;c \leftarrow \mathit{not}\ b\}$ has answer sets $\{a\}$ and $\{b\}$.

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Answer set: Some properties

- A disjunctive logic program may have zero, one, or multiple answer sets.
- If X is an answer set of a disjunctive logic program Π, then X is a model of Π (seen as a formula).
- If X and Y are answer sets of a disjunctive logic program Π, then $X \not\subset Y$.
- If $A \in X$ for some answer X set of a disjunctive logic program Π, then there is a rule $r \in \Pi_X$ such that $\{A\} = head(r) \cap X$.

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An example with variables

$$\Pi = \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(X)\,;c(Y) & \leftarrow & a(X,Y), not\ c(Y) \end{array} \right\}$$

$$ground(\Pi) = \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\,;c(1) & \leftarrow & a(1,1), not\ c(1) \\ b(1)\,;c(2) & \leftarrow & a(1,2), not\ c(2) \\ b(2)\,;c(1) & \leftarrow & a(2,1), not\ c(1) \\ b(2)\,;c(2) & \leftarrow & a(2,2), not\ c(2) \end{array} \right\}$$

For every answer set X of Π, we have

- $a(1,2) \in X$ and
- $\{a(1,1), a(2,1), a(2,2)\} \cap X = \emptyset$.

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Nested logic programs: Overview

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Nested logic programs

- Formulas are formed from
  - propositional atoms and
  - $\top$ and $\bot$

  using
  - negation-as-failure (*not*),
  - conjunction (,), and
  - disjunction (;).
- A nested rule, r, is an ordered pair of the form $F \leftarrow G$ where F and G are formulas.
- A nested program is a finite set of rules.
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Satisfaction relation

- The satisfaction relation $X \models F$ between a set of atoms and a formula F is defined recursively as follows:
  - $X \models F$ if $F \in X$ for an atom F,
  - $X \models \top$,
  - $X \not\models \bot$,
  - $X \models (F, G)$ if $X \models F$ and $X \models G$,
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- A set X of atoms satisfies a nested program Π, written $X \models \Pi$, iff for any $r \in \Pi$, $X \models \mathit{head}(r)$ whenever $X \models \mathit{body}(r)$.
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  - For $X = \emptyset$, we get
    - $\Pi_1^{\emptyset} = \{(p\,;\top) \leftarrow \top\}$
    - $\min_{\subseteq}(\Pi_1^{\emptyset}) = \{\emptyset\}$. ✔
  - For $X = \{p\}$, we get
    - $\Pi_1^{\{p\}} = \{(p\,;\bot) \leftarrow \top\}$
    - $\min_{\subseteq}(\Pi_1^{\{p\}}) = \{\{p\}\}$. ✔
- $\Pi_2 = \{p \leftarrow not\ not\ p\}$
  - For $X = \emptyset$, we get $\Pi_2^{\emptyset} = \{p \leftarrow \bot\}$ and $\min_{\subseteq}(\Pi_2^{\emptyset}) = \{\emptyset\}$. ✔
  - For $X = \{p\}$, we get $\Pi_2^{\{p\}} = \{p \leftarrow \top\}$ and $\min_{\subseteq}(\Pi_2^{\{p\}}) = \{\{p\}\}$. ✔
- In general,
  - $F \leftarrow G,\ not\ not\ H$ is equivalent to $F\,;not\ H \leftarrow G$
  - $F\,;\ not\ not\ G \leftarrow H$ is equivalent to $F \leftarrow H, not\ G$
  - $not\ not\ not\ F$ is equivalent to $not\ F$

➥ Intuitionistic Logics HT (Heyting, 1930) and G3 (Gödel, 1932)

Some more examples

$$
\begin{array}{rcl}
\Pi_3 & = & \{p \leftarrow (q, r) \,;\, (\mathit{not}\ q, \mathit{not}\ s)\} \\
\Pi_4 & = & \{(p \,;\, \mathit{not}\ p), (q \,;\, \mathit{not}\ q), (r \,;\, \mathit{not}\ r) \leftarrow \top\} \\
\Pi_5 & = & \{(p \,;\, \mathit{not}\ p), (q \,;\, \mathit{not}\ q), (r \,;\, \mathit{not}\ r) \leftarrow \top,\ \bot \leftarrow p, q\}
\end{array}
$$

Propositional Theories: Overview

Overview

Propositional theories

- Formulas are formed from
  - propositional atoms and
  - $\bot$

  using
  - conjunction ($\wedge$),
  - disjunction ($\vee$), and
  - implication ($\rightarrow$).
- Notation

$$\begin{array}{rcll} \top & = & (\bot \rightarrow \bot) & \\ \sim F & = & (F \rightarrow \bot) & \quad (\text{or: } \mathit{not}\ F) \end{array}$$

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Reduct

- The satisfaction relation $X \models F$ between a set X of atoms and a (set of) formula(s) F is defined as in propositional logic.
- The reduct, F^X, of a formula F relative to a set X of atoms is defined recursively as follows:
  - $F^X = \bot$ if $X \not\models F$
  - $F^X = F$ if $F \in X$
  - $F^X = (G^X \circ H^X)$ if $X \models F$ and $F = (G \circ H)$ for $\circ \in \{\wedge, \vee, \rightarrow\}$
  - ➥ If $F = \sim G = (G \rightarrow \bot)$, then $F^X = (\bot \rightarrow \bot) = \top$, if $X \not\models G$, and $F^X = \bot$, otherwise.
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Answer sets

- The set of all $\subseteq$-minimal sets of atoms satisfying a propositional theory $\mathcal{F}$ is denoted by $\min_\subseteq(\mathcal{F})$.
- A set X of atoms is an answer set of a propositional theory $\mathcal{F}$ if $X \in \min_\subseteq(\mathcal{F}^X)$.
- If X is an answer set of $\mathcal{F}$, then
  - $X \models \mathcal{F}$ and
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Overview

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- $\mathcal{F}_1 = \{p \vee (p \rightarrow (q \wedge r))\}$
  - For $X = \{p, q, r\}$, we get
    $\mathcal{F}_1^{\{p,q,r\}} = \{p \vee (p \rightarrow (q \wedge r))\}$ and $\min_{\subseteq}(\mathcal{F}_1^{\{p,q,r\}}) = \{\emptyset\}$.
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Overview

Relationship with logic programs

- The translation, $\tau[(F \leftarrow G)]$, of a (nested) rule $(F \leftarrow G)$ is defined recursively as follows:
  - $\tau[(F \leftarrow G)] = (\tau[G] \rightarrow \tau[F])$,
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  - $\tau[\mathit{not}\ F] = \sim\tau[F]$,
  - $\tau[(F, G)] = (\tau[F] \land \tau[G])$,
  - $\tau[(F; G)] = (\tau[F] \lor \tau[G])$.

The translation of a logic program Π is $\tau[\Pi] = \{\tau[r] \mid r \in \Pi\}$.

Given a logic program Π and a set X of atoms,
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- The translation, $\tau[(F \leftarrow G)]$, of a (nested) rule $(F \leftarrow G)$ is defined recursively as follows:
  - $\tau[(F \leftarrow G)] = (\tau[G] \rightarrow \tau[F])$,
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Logic programs as propositional theories

- The normal logic program $\Pi = \{p \leftarrow not\ q,\ q \leftarrow not\ p\}$ corresponds to $\tau[\Pi] = \{\sim q \rightarrow p,\ \sim p \rightarrow q\}$.
  - ➥ Answer sets: $\{p\}$ and $\{q\}$

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Classical Negation: Overview

Overview

Syntax

Status quo

- In logic programs *not* (or $\sim$) denotes default negation.

Generalization

- We allow classical negation for atoms (only!).
  - ➥ Logic programs in "negation normal form."
- Given an alphabet $\mathcal{A}$ of atoms, let $\overline{\mathcal{A}} = \{\neg A \mid A \in \mathcal{A}\}$.
  - ☞ We assume $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$.
- The atoms A and $\neg A$ are complementary.
  - ➥ $\neg A$ is the classical negation of A, and vice versa.

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Semantics

- A set X of atoms is consistent, if $X \cap \{\neg A \mid A \in (\mathcal{A} \cap X)\} = \emptyset$, and inconsistent, otherwise.
- A set X of atoms is an answer set of a logic program Π over $\mathcal{A} \cup \overline{\mathcal{A}}$ if X is an answer set of $\Pi \cup \{B \leftarrow A, \neg A \mid A \in \mathcal{A}, B \in (\mathcal{A} \cup \overline{\mathcal{A}})\}$
  - ➥ The only inconsistent answer set (candidate) is $X = \mathcal{A} \cup \overline{\mathcal{A}}$.
- For a normal or disjunctive logic program Π over $\mathcal{A} \cup \overline{\mathcal{A}}$, exactly one of the following two cases applies:
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To cross or not to cross...?

- $\Pi_1 = \{cross \leftarrow not\ train\}$
  - Answer set: $\{cross\}$
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- $\Pi_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$
  - Answer set: $\{cross, \neg train\}$
- $\Pi_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$
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Example

- $\Pi = \{p \leftarrow,\ \neg p \leftarrow,\ q \leftarrow not\ r\}$
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Let A be an atom and X be a set of atoms.

- For a positive normal logic program Π:
  - Deciding whether X is the answer set of Π is **P**-complete.
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Complexity (ctd)

- For a positive disjunctive logic program Π:
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  - Deciding whether A is in an answer set of Π is $\mathbf{NP}^{\mathbf{NP}}$-complete.
- For a nested logic program Π:
  - Deciding whether X is an answer set of Π is **co-NP**-complete.
  - Deciding whether A is in an answer set of Π is $\mathbf{NP}^{\mathbf{NP}}$-complete.
- For a propositional theory $\mathcal{F}$:
  - Deciding whether X is an answer set of $\mathcal{F}$ is **co-NP**-complete.
  - Deciding whether A is in an answer set of $\mathcal{F}$ is $\mathbf{NP}^{\mathbf{NP}}$-complete.

Language Extensions: Overview

Overview

Language extensions

- The expressiveness of a language can be enhanced by introducing new constructs.
- To this end, we must address the following issues:
  - What is the syntax of the new language construct?
  - What is the semantics of the new language construct?
  - How to implement the new language construct?
- A way of providing semantics is to furnish a translation removing the new constructs, eg. classical negation.
- This translation might also be used for implementing the language extension. When is this feasible?

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Overview

Integrity Constraints

- Purpose Integrity constraints eliminate unwanted solution candidates
- Syntax An integrity constraint is of the form

$$\leftarrow A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n,$$

where $n \geq m \geq 1$, and each A_i $(1 \leq i \leq n)$ is a atom.

- Example `:- edge(X,Y), color(X,C), color(Y,C).`
- Implementation For a new symbol x, map

$$\begin{array}{rrcl} & & \leftarrow & A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n \\ \mapsto & x & \leftarrow & A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n, not\ x \end{array}$$

- Another example $\Pi = \{p \leftarrow not\ q,\ q \leftarrow not\ p\}$
  versus $\Pi' = \Pi \cup \{\leftarrow p\}$ and $\Pi'' = \Pi \cup \{\leftarrow not\ p\}$

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Overview

Choice rules

- Idea Choices over subsets.
- Syntax

$$\{A_1, \dots, A_m\} \leftarrow A_{m+1}, \dots, A_n, \textit{not } A_{n+1}, \dots, \textit{not } A_o,$$

- Informal meaning If the body is satisfied in an answer set, then any subset of $\{A_1, \dots, A_m\}$ can be included in the answer set.
- Example `1 {color(X,C) : col(C)} 1 :- node(X).`
- Another Example The program $\Pi = \{\ \{a\} \leftarrow b,\ b \leftarrow\}$ has two answer sets: $\{b\}$ and $\{a, b\}$.
- Implementation lparse/gringo + smodels/cmodels/clasp

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Embedding in normal logic programs

- A choice rule of form

$$\{A_1, \dots, A_m\} \leftarrow A_{m+1}, \dots, A_n, not\ A_{n+1}, \dots, not\ A_o$$

can be translated into $2m + 1$ rules

$$\begin{array}{rcl} A & \leftarrow & A_{m+1}, \dots, A_n, not\ A_{n+1}, \dots, not\ A_o \\ A_1 & \leftarrow & A, not\ \overline{A_1} \quad \dots \quad A_m \ \leftarrow \ A, not\ \overline{A_m} \\ \overline{A_1} & \leftarrow & not\ A_1 \qquad\ \dots \quad \overline{A_m} \ \leftarrow \ not\ A_m \end{array}$$

by introducing new atoms $A, \overline{A_1}, \dots, \overline{A_m}$.

Overview

Cardinality constraints

- Syntax A (positive) cardinality constraint is of the form
  $l \ \{A_1, \dots, A_m\} \ u$
- Informal meaning A cardinality constraint is satisfied in an answer set X, if the number of atoms from $\{A_1, \dots, A_m\}$ satisfied in X is between l and u (inclusive).
  More formally, if $l \leq |\{A_1, \dots, A_m\} \cap X| \leq u$.
- Conditions $l \ \{A_1 : B_1, \dots, A_m : B_m\} \ u$
  where $B_1, \dots, B_m$ are used for restricting instantiations of variables occurring in $A_1, \dots, A_m$.
- Example 2 {hd(a),...,hd(m)} 4
- Implementation lparse/gringo + smodels/cmodels/clasp

Overview

Cardinality rules

- Idea Control cardinality of subsets.
- Syntax

$$A_0 \leftarrow l\ \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\}$$

- Informal meaning If at least l elements of the "body" are true in an answer set, then add A_0 to the answer set.
  - ➥ l is a lower bound on the "body"
- Example The program $\Pi = \{\ a \leftarrow 1\{b, c\},\ \ b \leftarrow\}$ has one answer set: $\{a, b\}$.
- Implementation lparse/gringo + smodels/cmodels/clasp
  ☞ gringo distinguishes sets and multi-sets!

Embedding in normal logic programs (ctd)

- Replace each cardinality rule

$$A_0 \leftarrow l\ \{A_1, \ldots, A_m\} \qquad \text{by} \qquad A_0 \leftarrow cc(A_1, l)$$

where atom $cc(A_i, j)$ represents the fact that at least j of the atoms in $\{A_i, \ldots, A_m\}$, that is, of the atoms that have an equal or greater index than i, are in a particular answer set.

- The definition of $cc(A_i, j)$ is given by the rules

$$\begin{array}{rcl} cc(A_i, j{+}1) & \leftarrow & cc(A_{i+1}, j), A_i \\ cc(A_i, j) & \leftarrow & cc(A_{i+1}, j) \\ cc(A_{m+1}, 0) & \leftarrow & \end{array}$$

- What about space complexity?

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- What about space complexity?

... and vice versa

- A normal rule

$$A_0 \leftarrow A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n,$$

can be represented by the cardinality rule

$$A_0 \leftarrow n{+}m\ \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\}.$$

Cardinality rules with upper bounds

- A rule of the form

$$A_0 \leftarrow l \; \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\} \; u$$

  stands for

$$\begin{array}{rcl} A_0 & \leftarrow & B, not\ C \\ B & \leftarrow & l \; \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\} \\ C & \leftarrow & u{+}1 \; \{A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\} \end{array}$$

Cardinality constraints as heads

- A rule of the form

$$l\ \{A_1, \dots, A_m\}\ u \leftarrow A_{m+1}, \dots, A_n, not\ A_{n+1}, \dots, not\ A_o,$$

stands for

$$\begin{array}{rcl} B & \leftarrow & A_{m+1}, \dots, A_n, not\ A_{n+1}, \dots, not\ A_o \\ \{A_1, \dots, A_m\} & \leftarrow & B \\ C & \leftarrow & l\ \{A_1, \dots, A_m\}\ u \\ & \leftarrow & B, not\ C \end{array}$$

Full-fledged cardinality rules

- A rule of the form

$$l_0 \ S_0 \ u_0 \leftarrow l_1 \ S_1 \ u_1, \ldots, l_n \ S_n \ u_n$$

stands for $0 \leq i \leq n$

$$\begin{array}{rcl}
B_i & \leftarrow & l_i \ S_i \\
C_i & \leftarrow & u_i{+}1 \ S_i \\
A & \leftarrow & B_1, \ldots, B_n, not \ C_1, \ldots, not \ C_n \\
 & \leftarrow & A, not \ B_0 \\
 & \leftarrow & A, C_0 \\
S_0 \cap \mathcal{A} & \leftarrow & A
\end{array}$$

where $\mathcal{A}$ is the underlying alphabet.

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Overview

Weight constraints

- Syntax $l\ [A_1 = w_1, \dots, A_m = w_m,$
  $\quad not\ A_{m+1} = w_{m+1}, \dots, not\ A_n = w_n]\ u$
- Informal meaning A weight constraint is satisfied in an answer set X, if

$$l \leq \left(\sum_{1 \leq i \leq m, A_i \in X} w_i + \sum_{m < i \leq n, A_i \notin X} w_i \right) \leq u\ .$$

➥ Generalization of cardinality constraints.

- Example 80 [hd(a)=50,...,hd(m)=100] 400
- Implementation lparse/gringo + smodels/cmodels/clasp
  ☞ gringo distinguishes sets and multi-sets!

Optimization statements

- Idea Compute optimal answer sets by minimizing or maximizing a weighted sum of given elements, respectively.
- Syntax
  - $\#minimize\ [A_1 = w_1, \dots, A_m = w_m,$
    $not\ A_{m+1} = w_{m+1}, \dots, not\ A_n = w_n]$
  - $\#maximize\ [A_1 = w_1, \dots, A_m = w_m,$
    $not\ A_{m+1} = w_{m+1}, \dots, not\ A_n = w_n]$
- Several optimization statements are interpreted lexicographically.
- Example
  - #minimize [hd(a)=30,. . . ,hd(m)=50]
  - #minimize [road(X,Y) : length(X,Y,L) = L]
- Implementation lparse/gringo + smodels/clasp

Weak integrity constraints

- Syntax $:\sim\ A_1, \dots, A_m, not\ A_{m+1}, \dots, not\ A_n\ [w : l]$
- Informal meaning
  1. minimize the sum of weights of violated constraints in the highest level;
  2. minimize the sum of weights of violated constraints in the next lower level;
  3. etc
- Implementation dlv

Overview

Conditional literals in lparse and gringo

- We often want to encode the contents of a (multi-)set rather than enumerating each of the elements.
- To support this, lparse and gringo allow for conditional literals.

- Syntax
  $A_0 : A_1 : \ldots : A_m : \mathit{not}\ A_{m+1} : \ldots : \mathit{not}\ A_n$
- Informal meaning
  List all ground instances of A_0 such that corresponding instances of $A_1, \ldots, A_m, \mathit{not}\ A_{m+1}, \ldots, \mathit{not}\ A_n$ are true.
- Example gringo instantiates the program:

```
p(1). p(2). p(3).  q(2).      {r(X) : p(X) : not q(X)}.
```

to:

```
p(1). p(2). p(3).  q(2).      {r(1), r(3)}.
```

Domain predicates in lparse and gringo

- The predicates of literals on the right-hand side of a colon (:) must be defined from facts without any negative recursion.
- Such domain predicates are fully evaluated by `lparse` and `gringo`.

- Example

```
p(1). p(2).
q(X) :- p(X), not p(X+1).
q(X) :- p(X), q(X+1).
r(X) :- p(X), not r(X+1).
```

  - `p/1` and `q/1` are domain predicates because none of them negatively depends on itself.
  - `r/1` is not a domain predicate because it is defined in terms of `not r(X+1)`.

- See `gringo` documentation for further details.

Normal form in `lparse` and `gringo`

- Consider a logic program consisting of
  - normal rules
  - choice rules
  - cardinality rules
  - weight rules
  - optimization statements
- Such a format is obtained by `lparse` or `gringo`

  and directly implemented by `smodels` and `clasp`.

Aggregates: Overview

Overview

37 Motivation

38 Syntax

39 Semantics

Motivation

- Aggregates provide a general way to obtain a single value from a collection of input values given as a set, a bag, or a list.
- Popular aggregate (functions):
  - Average
  - Count
  - Maximum
  - Minimum
  - Sum
- Cardinality and Weight constraints rely on Count and Sum aggregates.

Overview

Syntax

- An aggregate has the form:

$$F \ \langle A_1 = w_1, \dots, A_m = w_m, \mathit{not}\ A_{m+1} = w_{m+1}, \dots, \mathit{not}\ A_n = w_n \rangle \prec k$$

  where

  - F stands for a function mapping multi-sets of $\mathbb{Z}$ to $\mathbb{Z} \cup \{+\infty, -\infty\}$,
  - $\prec$ stands for a relation between $\mathbb{Z} \cup \{+\infty, -\infty\}$ and $\mathbb{Z}$,
  - k an integer,
  - A_i is an atom, and
  - w_i are integers

  for $1 \leq i \leq n$.

- For instance, $\mathit{sum}\ \langle hd(a) = 30, \dots, hd(m) = 50 \rangle \leq 300$

Overview

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Semantics

- A (positive) aggregate $F\ \langle A_1 = w_1, \ldots, A_n = w_n \rangle \prec k$ can be represented by the formula:

$$\bigwedge_{I \subseteq \{1,\ldots,n\}, F\langle w_i | i \in I \rangle \not\prec k} \left(\bigwedge_{i \in I} A_i \rightarrow \bigvee_{i \in \overline{I}} A_i \right)$$

  where $\overline{I} = \{1, \ldots, n\} \setminus I$ and $\not\prec$ is the complement of $\prec$.
- Then, $F\ \langle A_1 = w_1, \ldots, A_n = w_n \rangle \prec k$ is true in X iff the above formula is true in X.

An example

- Consider $sum\langle p = 1, q = 1\rangle \neq 1$
  ➥ that is, $A_1 = p$, $A_2 = q$ and $w_1 = 1$, $w_2 = 1$
- Calculemus!

I	$\langle w_i \mid i \in I\rangle$	$\sum\langle w_i \mid i \in I\rangle$	$\sum\langle w_i \mid i \in I\rangle = 1$
$\emptyset$	$\langle\rangle$	0	*false*
$\{1\}$	$\langle 1\rangle$	1	*true*
$\{2\}$	$\langle 1\rangle$	1	*true*
$\{1,2\}$	$\langle 1,1\rangle$	2	*false*

- We get $(p \rightarrow q) \wedge (q \rightarrow p)$
- Analogously, we obtain $(p \vee q) \wedge \neg(p \wedge q)$ for $sum\langle p = 1, q = 1\rangle = 1$.

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- We get $(p \to q) \land (q \to p)$
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Monotonicity

- Monotone aggregates
  - For instance,
    - $body^+(r)$
    - $sum\langle p = 1, q = 1\rangle > 1$ amounts to $p \wedge q$
  - We get a simpler characterization: $\bigwedge_{I \subseteq \{1,\dots,n\}, F\langle w_i \mid i \in I\rangle \not\prec k} \bigvee_{i \in \bar{I}} A_i$
- Anti-monotone aggregates
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The smodels approach: Overview

Overview

(Towards) the `smodels` approach

- Wanted:
  - An efficient procedure to compute answer sets
- The `smodels` approach:
  - Backtracking search building a binary search tree
  - A node in the search tree corresponds to a 3-valued interpretation
  - The search space is pruned by
    - deriving deterministic consequences and detecting conflicts (**expand**)
    - making one choice at a time by appeal to a heuristic (**select**)
  - ☞ Heuristic choices are made on atoms

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Overview

Approximating answer sets

First Idea Approximate an answer set X by two sets of atoms L and U such that $L \subseteq X \subseteq U$.

- L and U constitute lower and upper bounds on X.
- L and $(\mathcal{A} \setminus U)$ describe a 3-valued model of the program.

Observation

$$X \subseteq Y \text{ implies } \Pi^Y \subseteq \Pi^X \text{ implies } Cn(\Pi^Y) \subseteq Cn(\Pi^X)$$

Properties Let X be an answer set of normal logic program Π.

- If $L \subseteq X$, then $X \subseteq Cn(\Pi^L)$.
- If $X \subseteq U$, then $Cn(\Pi^U) \subseteq X$.
- If $L \subseteq X \subseteq U$, then $L \cup Cn(\Pi^U) \subseteq X \subseteq U \cap Cn(\Pi^L)$.

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- If $X \subseteq U$, then $Cn(\Pi^U) \subseteq X$.
- If $L \subseteq X \subseteq U$, then $L \cup Cn(\Pi^U) \subseteq X \subseteq U \cap Cn(\Pi^L)$.

Approximating answer sets

First Idea Approximate an answer set X by two sets of atoms L and U such that $L \subseteq X \subseteq U$.

- ➥ L and U constitute lower and upper bounds on X.
- ➥ L and $(\mathcal{A} \setminus U)$ describe a 3-valued model of the program.

Observation

$$X \subseteq Y \text{ implies } \Pi^Y \subseteq \Pi^X \text{ implies } Cn(\Pi^Y) \subseteq Cn(\Pi^X)$$

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Approximating answer sets (ctd)

Second Idea

Iterate

- Replace L by $L \cup Cn(\Pi^U)$
- Replace U by $U \cap Cn(\Pi^L)$

until L and U do not change anymore.

Observations

- At each iteration step
  - L becomes larger (or equal)
  - U becomes smaller (or equal)
- $L \subseteq X \subseteq U$ is invariant for every answer set X of Π
- If $L \not\subseteq U$, then Π has no answer set!
- If $L = U$, then L is an answer set of Π.

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- If $L \not\subseteq U$, then Π has no answer set!
- If $L = U$, then L is an answer set of Π.

The simplistic expand algorithm

expand(L, U)

repeat

$L' \leftarrow L$
$U' \leftarrow U$
$L \leftarrow L' \cup Cn(\Pi^{U'})$
$U \leftarrow U' \cap Cn(\Pi^{L'})$
if $L \not\subseteq U$ **then return**

until $L = L'$ and $U = U'$

☞ Π is a global parameter!

Let's expand!

$$\Pi = \left\{ \begin{array}{l} a \leftarrow \\ b \leftarrow a, not\ c \\ d \leftarrow b, not\ e \\ e \leftarrow\ \ not\ d \end{array} \right\}$$

	L'	$Cn(\Pi^{U'})$	L	U'	$Cn(\Pi^{L'})$	U
1	$\emptyset$	$\{a\}$	$\{a\}$	$\{a, b, c, d, e\}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$
2	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$
3	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$

➥ We have $\{a, b\} \subseteq X$ and
$(\mathcal{A} \setminus \{a, b, d, e\}) \cap X = (\{c\} \cap X) = \emptyset$
for every answer set X of Π.

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The simplistic expand algorithm (ctd)

expand

- tightens the approximation on answer sets
- is answer set preserving

Let's expand with d !

$$\Pi = \left\{ \begin{array}{l} a \leftarrow \\ b \leftarrow a, not\ c \\ d \leftarrow b, not\ e \\ e \leftarrow\ \ not\ d \end{array} \right\}$$

	L'	$Cn(\Pi^{U'})$	L	U'	$Cn(\Pi^{L'})$	U
1	$\{d\}$	$\{a\}$	$\{a, d\}$	$\{a, b, c, d, e\}$	$\{a, b, d\}$	$\{a, b, d\}$
2	$\{a, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$
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Let's expand with "*not d*" !

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Overview

Interlude: Partial interpretations

or: 3-valued interpretations

A partial interpretation of a logic program Π maps atoms on truth values: $\{true, false, unknown\}$.

Representation $\langle T, F\rangle$, where

- T is the set of all *true* atoms and
- F is the set of all *false* atoms.
- Truth of atoms in $atom(\Pi) \setminus (T \cup F)$ is *unknown*.

☞ By $atom(\Pi)$, we denote the set of atoms occuring in Π.

Properties
- $\langle T, F\rangle$ is conflicting iff $T \cap F \neq \emptyset$.
- $\langle T, F\rangle$ is total iff $T \cup F = atom(\Pi)$ and $T \cap F = \emptyset$.

Definition For $\langle T_1, F_1\rangle$ and $\langle T_2, F_2\rangle$, define:
- $\langle T_1, F_1\rangle \sqsubseteq \langle T_2, F_2\rangle$ iff $T_1 \subseteq T_2$ and $F_1 \subseteq F_2$
- $\langle T_1, F_1\rangle \sqcup \langle T_2, F_2\rangle = \langle T_1 \cup T_2, F_1 \cup F_2\rangle$

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Overview

The **smodels** (decision) algorithm

Global: Normal logic program Π

smodels$(\langle T, F\rangle)$

$\langle T, F\rangle \leftarrow$ **expand**$(\langle T, F\rangle)$
if $\langle T, F\rangle$ is conflicting **then return**
else if $\langle T, F\rangle$ is total **then exit with** T
else

$A \leftarrow$ **select**$(atom(\Pi) \setminus (T \cup F))$
smodels$(\langle T \cup \{A\}, F\rangle)$
smodels$(\langle T, F \cup \{A\}\rangle)$

Call: **smodels**$(\langle \emptyset, \emptyset\rangle)$

The **smodels** (decision) algorithm

Global: Normal logic program Π

smodels($\langle T, F\rangle$)

> $\langle T, F\rangle \leftarrow$ **expand**($\langle T, F\rangle$)
> **if** $\langle T, F\rangle$ is conflicting **then return**
> **else if** $\langle T, F\rangle$ is total **then exit with** T
> **else**
>
> > $A \leftarrow$ **select**($atom(\Pi) \setminus (T \cup F)$)
> > **smodels**($\langle T \cup \{A\}, F\rangle$)
> > **smodels**($\langle T, F \cup \{A\}\rangle$)

Call: **smodels**($\langle \emptyset, \emptyset\rangle$)

Deterministic consequences via **expand**

Global: Normal logic program Π

expand$(\langle T, F\rangle)$

repeat

$\langle T, F\rangle \leftarrow$ **atleast**$(\langle T, F\rangle)$

if $\langle T, F\rangle$ is conflicting **then return** $\langle T, F\rangle$

else

$F' \leftarrow F$

$F \leftarrow F \cup$ **atmost**$(\langle T, F\rangle)$

until $F = F'$

return $\langle T, F\rangle$

- ☞ **atleast**$(\langle T, F\rangle)$ derives deterministic consequences from Clark's completion
- ☞ **atmost**$(\langle T, F\rangle)$ derives deterministic consequences from unfounded sets

Deterministic consequences via **expand**

Global: Normal logic program Π

expand($\langle T, F\rangle$)

repeat
  $\langle T, F\rangle \leftarrow$ **atleast**($\langle T, F\rangle$)
  if $\langle T, F\rangle$ is conflicting **then return** $\langle T, F\rangle$
  else
    $F' \leftarrow F$
    $F \leftarrow F \cup$ **atmost**($\langle T, F\rangle$)
until $F = F'$
return $\langle T, F\rangle$

- ☞ **atleast**($\langle T, F\rangle$) derives deterministic consequences from Clark's completion
- ☞ **atmost**($\langle T, F\rangle$) derives deterministic consequences from unfounded sets

A glimpse at **atleast**($\langle T, F\rangle$)

repeat

  if $\langle T, F\rangle$ is conflicting **then return** $\langle T, F\rangle$

  $\langle T', F'\rangle \leftarrow \langle T, F\rangle$

  case of

  $r \in \Pi$ **such that** $head(r) \notin T$ **and**
  $body^+(r) \subseteq T, body^-(r) \subseteq F$**:**

    $T \leftarrow T \cup \{head(r)\}$

  $A \in (atom(\Pi) \setminus F)$ **such that for all** $r \in \Pi$:
  $head(r) \neq A$ **or** $(body^+(r) \cap F) \cup (body^-(r) \cap T) \neq \emptyset$**:**

    $F \leftarrow F \cup \{A\}$

  $head(r) \in F, r \in \Pi$ **such that** $body^+(r) \cap body^-(r) = \emptyset$ **and**
  $(body^+(r) \setminus T) \cup (body^-(r) \setminus F) = \{A\}$**:**

    if $A \in body^+(r)$ **then** $F \leftarrow F \cup \{A\}$ **else** $T \leftarrow T \cup \{A\}$

  $(A = head(r)) \in T, r \in \Pi$ **such that** $body^+(r) \not\subseteq T$ **or** $body^-(r) \not\subseteq F$ **and**
  for all $r' \in \Pi \setminus \{r\}$: $head(r') \neq A$ **or** $(body^+(r') \cap F) \cup (body^-(r') \cap T) \neq \emptyset$**:**

    $T \leftarrow T \cup body^+(r)$
    $F \leftarrow F \cup body^-(r)$

until $\langle T, F\rangle = \langle T', F'\rangle$

return $\langle T, F\rangle$

A glimpse at **atleast**($\langle T, F\rangle$)

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    $T \leftarrow T \cup body^+(r)$

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until $\langle T, F\rangle = \langle T', F'\rangle$

return $\langle T, F\rangle$

A glimpse at **atmost**($\langle T, F \rangle$)

return $\mathbf{U}_\Pi\langle T, F\rangle$

A glimpse at **atmost**$(\langle T, F\rangle)$

return $\mathbf{U}_{\Pi}\langle T, F\rangle$

Completion: Overview

Overview

Completion

Let Π be a normal logic program.
The completion of Π is defined as follows:

$$Comp(body(r)) = \bigwedge_{A \in body^+(r)} A \wedge \bigwedge_{A \in body^-(r)} \neg A$$

$$Comp(\Pi) = \{A \leftrightarrow \bigvee_{r \in \Pi, head(r)=A} Comp(body(r)) \mid A \in atom(\Pi)\}$$

Every answer set of Π is a model of $Comp(\Pi)$, but not vice versa.
Models of $Comp(\Pi)$ are called the supported models of Π.

In other words, every answer set of Π is a supported model of Π.
By definition, every supported model of Π is also a model of Π.

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Completion

Let Π be a normal logic program.
The completion of Π is defined as follows:

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\begin{aligned}
Comp(body(r)) &= \bigwedge_{A \in body^+(r)} A \wedge \bigwedge_{A \in body^-(r)} \neg A \\
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A first example

$$\Pi = \left\{ \begin{array}{lcl} a & \leftarrow & \\ b & \leftarrow & a \\ c & \leftarrow & b \\ c & \leftarrow & d \\ d & \leftarrow & c, e \end{array} \right\} \qquad Comp(\Pi) = \left\{ \begin{array}{lcl} a & \leftrightarrow & \top \\ b & \leftrightarrow & a \\ c & \leftrightarrow & (b \vee d) \\ d & \leftrightarrow & (c \wedge e) \\ e & \leftrightarrow & \bot \end{array} \right\}$$

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$$\Pi = \left\{ \begin{array}{lcl} q & \leftarrow & \mathit{not}\ p \\ p & \leftarrow & \mathit{not}\ q, \mathit{not}\ x \end{array} \right\} \quad \mathit{Comp}(\Pi) = \left\{ \begin{array}{lcl} q & \leftrightarrow & \neg p \\ p & \leftrightarrow & (\neg q \wedge \neg x) \\ x & \leftrightarrow & \bot \end{array} \right\}$$

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A third example

$\Pi = \{ p \leftarrow p \}$ $\quad Comp(\Pi) = \{ p \leftrightarrow p \}$

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Overview

Fitting operator: Basic idea

Idea Extend T_Π to normal logic programs.

Logical background Completion

- The head atom of a rule must be *true* if the rule's body is *true*.
- An atom must be *false* if the body of each rule having it as head is *false*.

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Fitting operator: Definition

Let Π be a normal logic program.

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$$\mathbf{\Phi}_\Pi\langle T,F\rangle = \langle \mathbf{T}_\Pi\langle T,F\rangle, \mathbf{F}_\Pi\langle T,F\rangle\rangle$$

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$$\mathbf{T}_\Pi\langle T,F\rangle = \{head(r) \mid r \in \Pi, body^+(r) \subseteq T, body^-(r) \subseteq F\}$$
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Fitting semantics

Define the iterative variant of $\mathbf{\Phi}_\Pi$ analogously to T_Π:

$$\mathbf{\Phi}^0_\Pi\langle T, F\rangle = \langle T, F\rangle \qquad \mathbf{\Phi}^{i+1}_\Pi\langle T, F\rangle = \mathbf{\Phi}_\Pi\mathbf{\Phi}^i_\Pi\langle T, F\rangle$$

Define the Fitting semantics of a normal logic program Π as the partial interpretation:

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Fitting semantics: Properties

Let Π be a normal logic program.

- $\mathbf{\Phi}_\Pi\langle\emptyset,\emptyset\rangle$ is monotonic.
  That is, $\mathbf{\Phi}^i_\Pi\langle\emptyset,\emptyset\rangle \sqsubseteq \mathbf{\Phi}^{i+1}_\Pi\langle\emptyset,\emptyset\rangle$.
- The Fitting semantics of Π is
  - not conflicting,
  - and generally not total.

Fitting fixpoints

Let Π be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation.

Define $\langle T, F\rangle$ as a Fitting fixpoint of Π if $\mathbf{\Phi}_\Pi\langle T, F\rangle = \langle T, F\rangle$.

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Π_1 has three total Fitting fixpoints:

1. $\langle \{a, c\}, \{b, d, e\} \rangle$
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Properties of Fitting operator

Let Π be a normal logic program,
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- Let $\mathbf{\Phi}_\Pi \langle T, F \rangle = \langle T', F' \rangle$.
  If X is an answer set of Π such that $T \subseteq X$ and $X \cap F = \emptyset$,
  then $T' \subseteq X$ and $X \cap F' = \emptyset$.
- That is, $\mathbf{\Phi}_\Pi$ is answer set preserving.
  - $\mathbf{\Phi}_\Pi$ can be used for approximating answer sets and so for propagation in ASP-solvers.

However, $\mathbf{\Phi}_\Pi$ is still insufficient, because total fixpoints correspond to supported models, not necessarily answer sets.

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The missing piece is non-circularity of derivations !

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$$\Pi = \left\{ \begin{array}{lcl} a & \leftarrow & b \\ b & \leftarrow & a \end{array} \right\} \qquad \begin{array}{lcl} \Phi^0_\Pi\langle\emptyset,\emptyset\rangle & = & \langle\emptyset,\emptyset\rangle \\ \Phi^1_\Pi\langle\emptyset,\emptyset\rangle & = & \langle\emptyset,\emptyset\rangle \end{array}$$

That is, Fitting semantics cannot assign *false* to a and b, although they can never become *true* !

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Overview

Rebuilding **atleast**($\langle T, F\rangle$)

from Fitting operator

repeat

  if $\langle T, F\rangle$ is conflicting **then return** $\langle T, F\rangle$

  $\langle T', F'\rangle \leftarrow \langle T, F\rangle$

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Relationship with Fitting semantics

Let Π be a normal logic program.

- $\mathbf{atleast}(\langle\emptyset,\emptyset\rangle) = \bigsqcup_{i\geq 0}\mathbf{\Phi}^i_\Pi\langle\emptyset,\emptyset\rangle$

What about supported models?
Consider:

$$\Pi = \left\{ \begin{array}{lll} a \leftarrow b & b \leftarrow \mathit{not}\ c & c \leftarrow \mathit{not}\ b \\ d \leftarrow e & e \leftarrow \mathit{not}\ f & f \leftarrow \mathit{not}\ e \end{array} \right\}$$

$\mathbf{atleast}(\langle\{a\},\{d\}\rangle) = \langle\{a\},\{d\}\rangle$
The only supported model X of Π such that $a \in X$ and $d \notin X$ is $\{a, b, f\}$!

We can enhance $\mathbf{atleast}(\langle T, F\rangle)$ by backward propagation !

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Let Π be a normal logic program and $\langle T, F\rangle$ a total interpretation.

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$$\Pi_1 = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, \mathit{not}\ d & e \leftarrow b \\ b \leftarrow \mathit{not}\ a & d \leftarrow \mathit{not}\ c, \mathit{not}\ e & e \leftarrow e \end{array} \right\}$$

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Overview

(Non-)cyclic derivations

- Cyclic derivations are causing the mismatch between supported models and answer sets.
- Atoms in an answer set can be "derived" from a program in a finite number of steps.
- Atoms in a cycle (not being "supported from outside the cycle") cannot be "derived" from a program in a finite number of steps.
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Non-cyclic derivations

Let X be an answer set of normal logic program Π.

- For every atom $A \in X$, there is a finite sequence of positive rules

  $$\langle r_1, \dots, r_n \rangle$$

  such that

  1. $head(r_1) = A$,
  2. $body^+(r_i) \subseteq \{head(r_j) \mid i < j \leq n\}$ for $1 \leq i \leq n$,
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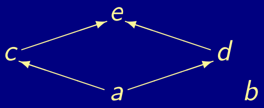
Positive atom dependency graph

Let Π be a normal logic program.
The positive atom dependency graph of Π is a directed graph $G(\Pi) = (V, E)$ such that

1. $V = atom(\Pi)$ and
2. $E = \{(p, q) \mid r \in \Pi, p \in body^+(r), head(r) = q\}$.

Examples

$$\Pi_2 = \left\{ \begin{array}{ll} a \leftarrow \mathit{not}\ b & b \leftarrow \mathit{not}\ a \\ c \leftarrow a, \mathit{not}\ d & d \leftarrow a, \mathit{not}\ c \\ e \leftarrow c, \mathit{not}\ a & e \leftarrow d, \mathit{not}\ b \end{array} \right\}$$



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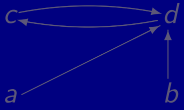

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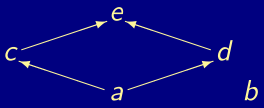

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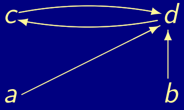

Tight programs

- A normal logic program Π is tight iff $G(\Pi)$ is acyclic.
- For example, Π_2 is tight, whereas Π_3 is not.
- If a normal logic program Π is tight, then

  X is an answer set of Π iff X is a model of $Comp(\Pi)$.

  That is, for tight programs, answer sets and supported models coincide.
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Answer sets: $\{\{a,c\},\{a,d,e\},\{b\}\}$

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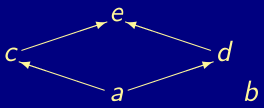

Answer sets: $\{\{a, c\}, \{a, d, e\}, \{b\}\}$

Supported models: $\{\{a, c\}, \{a, d, e\}, \{b\}\}$

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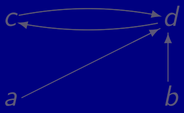

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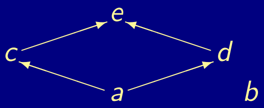

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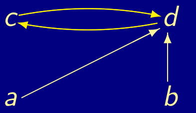

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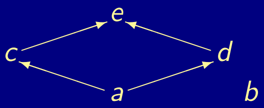

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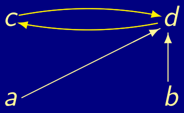

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Unfounded Sets: Overview

Overview

Unfounded sets

Let Π be a normal logic program,
and let $\langle T, F \rangle$ be a partial interpretation.

A set $U \subseteq atom(\Pi)$ is an unfounded set of Π with respect to $\langle T, F \rangle$ if,
for each rule $r \in \Pi$, we have

1. $head(r) \notin U$,
2. $body^+(r) \cap F \neq \emptyset$ or $body^-(r) \cap T \neq \emptyset$, or
3. $body^+(r) \cap U \neq \emptyset$.

- Intuitively, $\langle T, F \rangle$ is what we already know about Π.
- Rules satisfying Condition 1 or 2 are not usable for further derivations.
- Condition 3 is the unfounded set condition treating cyclic derivations: All rules still being usable to derive an atom in U require an(other) atom in U to be true.

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$$\Pi = \left\{ \begin{array}{ccc} a & \leftarrow & b \\ b & \leftarrow & a \end{array} \right\}$$

- $\emptyset$ is an unfounded set (by definition).
- $\{a\}$ is not an unfounded set of Π wrt $\langle \emptyset, \emptyset \rangle$.
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Greatest unfounded sets

Observation The union of two unfounded sets is an unfounded set.

Let Π be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation.
The greatest unfounded set of Π with respect to $\langle T, F\rangle$, denoted by $\mathbf{U}_\Pi\langle T, F\rangle$, is the union of all unfounded sets of Π with respect to $\langle T, F\rangle$.

- Alternatively, we may define

$$\mathbf{U}_\Pi\langle T, F\rangle = atom(\Pi) \setminus Cn(\{r \in \Pi \mid body^+(r) \cap F = \emptyset\}^T).$$

- Observe that $Cn(\{r \in \Pi \mid body^+(r) \cap F = \emptyset\}^T)$ contains all non-circularly derivable atoms from Π wrt $\langle T, F\rangle$.

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Overview

Well-founded operator

Let Π be a normal logic program,
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Observation Condition 2 (in the definition of an unfounded set) corresponds to set $\mathbf{F}_\Pi\langle T, F\rangle$ of Fitting's $\mathbf{\Phi}_\Pi\langle T, F\rangle$.

Idea Extend (negative part of) Fitting's operator $\mathbf{\Phi}_\Pi$.
That is,

- keep definition of $\mathbf{T}_\Pi\langle T, F\rangle$ from $\mathbf{\Phi}_\Pi\langle T, F\rangle$ and
- replace $\mathbf{F}_\Pi\langle T, F\rangle$ from $\mathbf{\Phi}_\Pi\langle T, F\rangle$ by $\mathbf{U}_\Pi\langle T, F\rangle$.

In words, an atom must be *false*
if it belongs to the greatest unfounded set.

Definition $\mathbf{\Omega}_\Pi\langle T, F\rangle = \langle \mathbf{T}_\Pi\langle T, F\rangle, \mathbf{U}_\Pi\langle T, F\rangle\rangle$

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Well-founded operator: Example

$$\Pi_1 = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, \mathit{not}\ d & e \leftarrow b \\ b \leftarrow \mathit{not}\ a & d \leftarrow \mathit{not}\ c, \mathit{not}\ e & e \leftarrow e \end{array} \right\}$$

Let's iterate $\mathbf{\Omega}_{\Pi_1}$ on $\langle \{c\}, \emptyset \rangle$:

$$\begin{array}{rcl} \mathbf{\Omega}_{\Pi_1}\langle \{c\}, \emptyset \rangle & = & \langle \{a\}, \{d\} \rangle \\ \mathbf{\Omega}_{\Pi_1}\langle \{a\}, \{d\} \rangle & = & \langle \{a, c\}, \{b, e\} \rangle \\ \mathbf{\Omega}_{\Pi_1}\langle \{a, c\}, \{b, e\} \rangle & = & \langle \{a\}, \{b, d, e\} \rangle \\ \mathbf{\Omega}_{\Pi_1}\langle \{a\}, \{b, d, e\} \rangle & = & \langle \{a, c\}, \{b, e\} \rangle \\ & \vdots & \end{array}$$

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$$\begin{array}{rcl} \mathbf{\Omega}_{\Pi_1} \langle \{c\}, \emptyset \rangle & = & \langle \{a\}, \{d\} \rangle \\ \mathbf{\Omega}_{\Pi_1} \langle \{a\}, \{d\} \rangle & = & \langle \{a, c\}, \{b, e\} \rangle \\ \mathbf{\Omega}_{\Pi_1} \langle \{a, c\}, \{b, e\} \rangle & = & \langle \{a\}, \{b, d, e\} \rangle \\ \mathbf{\Omega}_{\Pi_1} \langle \{a\}, \{b, d, e\} \rangle & = & \langle \{a, c\}, \{b, e\} \rangle \\ & \vdots & \end{array}$$

Well-founded semantics

Define the iterative variant of $\mathbf{\Omega}_\Pi$ analogously to $\mathbf{\Phi}_\Pi$:

$$\mathbf{\Omega}^0_\Pi\langle T,F\rangle = \langle T,F\rangle \qquad \mathbf{\Omega}^{i+1}_\Pi\langle T,F\rangle = \mathbf{\Omega}_\Pi\mathbf{\Omega}^i_\Pi\langle T,F\rangle$$

Define the well-founded semantics of a normal logic program Π as the partial interpretation:

$$\textstyle\bigsqcup_{i\geq 0}\mathbf{\Omega}^i_\Pi\langle\emptyset,\emptyset\rangle$$

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Well-founded semantics: Properties

Let Π be a normal logic program.

- $\mathbf{\Omega}_\Pi\langle\emptyset,\emptyset\rangle$ is monotonic.
  That is, $\mathbf{\Omega}^i_\Pi\langle\emptyset,\emptyset\rangle \sqsubseteq \mathbf{\Omega}^{i+1}_\Pi\langle\emptyset,\emptyset\rangle$.
- The well-founded semantics of Π is
  - not conflicting,
  - and generally not total.
- We have $\bigsqcup_{i\geq 0}\mathbf{\Phi}^i_\Pi\langle\emptyset,\emptyset\rangle \sqsubseteq \bigsqcup_{i\geq 0}\mathbf{\Omega}^i_\Pi\langle\emptyset,\emptyset\rangle$.

Well-founded fixpoints

Let Π be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation.
Define $\langle T, F\rangle$ as a well-founded fixpoint of Π if $\mathbf{\Omega}_\Pi\langle T, F\rangle = \langle T, F\rangle$.

- The well-founded semantics is the $\sqsubseteq$-least well-founded fixpoint of Π.
- Any other well-founded fixpoint extends the well-founded semantics.
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Π_1 has two total well-founded fixpoints:

1. $\langle \{a, c\}, \{b, d, e\} \rangle$
2. $\langle \{a, d\}, \{b, c, e\} \rangle$

Both of them represent answer sets.

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Properties of well-founded operator

Let Π be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation.

- Let $\mathbf{\Omega}_\Pi\langle T, F\rangle = \langle T', F'\rangle$.
  If X is an answer set of Π such that $T \subseteq X$ and $X \cap F = \emptyset$,
  then $T' \subseteq X$ and $X \cap F' = \emptyset$.
- That is, $\mathbf{\Omega}_\Pi$ is answer set preserving.
  - $\mathbf{\Omega}_\Pi$ can be used for approximating answer sets and so for propagation in ASP-solvers.

Unlike $\mathbf{\Phi}_\Pi$, operator $\mathbf{\Omega}_\Pi$ is sufficient for propagation because total fixpoints correspond to answer sets.

☞ In addition to $\mathbf{\Omega}_\Pi$, most ASP-solvers apply backward propagation (cf. Page 488), originating from program completion (although this is unnecessary from a formal point of view).

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Overview

Rebuilding **atmost**$(\langle T, F\rangle)$

from (greatest) unfounded sets

return $\mathbf{U}_{\Pi}\langle T, F\rangle$

Rebuilding $\mathbf{atmost}(\langle T, F\rangle)$

from (greatest) unfounded sets

return $\mathbf{U}_\Pi\langle T, F\rangle$

Recalling **expand**

Global: Normal logic program Π

expand($\langle T, F\rangle$)

repeat

$\quad\langle T, F\rangle \leftarrow$ **atleast**($\langle T, F\rangle$)

$\quad$**if** $\langle T, F\rangle$ is conflicting **then return** $\langle T, F\rangle$

$\quad$**else**

$\quad\quad F' \leftarrow F$

$\quad\quad F \leftarrow F \cup$ **atmost**($\langle T, F\rangle$)

until $F = F'$

return $\langle T, F\rangle$

- ☞ **atleast**($\langle T, F\rangle$) derives deterministic consequences from Clark's completion
- ☞ **atmost**($\langle T, F\rangle$) derives deterministic consequences from unfounded sets

Relationship with well-founded semantics

Let Π be a normal logic program.

- **expand**$(\langle\emptyset,\emptyset\rangle) = \bigsqcup_{i\geq 0}\mathbf{\Omega}^i_\Pi\langle\emptyset,\emptyset\rangle$

☞ That is, **expand** is basically an implementation of well-founded semantics !

☞ Additional backward propagation in **atleast** prunes the search space further !

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Relationship with answer sets

Let Π be a normal logic program and $\langle T, F\rangle$ a total interpretation.

- $\mathbf{expand}(\langle T, F\rangle) = \langle T, F\rangle$ iff T is an answer set of Π

Given $\mathbf{atmost}(\langle T, F\rangle) = \mathbf{U}_\Pi\langle T, F\rangle$,
we can apply **smodels** to compute answer sets !
Reconsider:

$$\Pi_1 = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, not\ d & e \leftarrow b \\ b \leftarrow not\ a & d \leftarrow not\ c, not\ e & e \leftarrow e \end{array} \right\}$$

Call	Interpretation	Result
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Additional remarks on `smodels`

The `smodels` implementation also features:

- Extended rules
  - Cardinality constraints
  - Weight constraints
- Optimiziation via *minimize* and *maximize*
- Efficient counter-based propagation
- Lazy implementation of **atmost** based on “source pointers”
- Failed-literal detection, also called lookahead, for stronger propagation

Overview

Characterizing non-cyclic derivations

An alternative approach

Question Is there a propositional formula $F(\Pi)$ such that the models of $F(\Pi)$ correspond to the answer sets of Π ?

☞ If we consider the completion of a program, $Comp(\Pi)$, then the problem boils down to eliminating the circular support of atoms that are true in the supported models of Π.

Idea Add formulas to $Comp(\Pi)$ that prohibit circular support of sets of atoms.

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Loops

Let Π be a normal logic program, and
let $G(\Pi) = (atom(\Pi), E)$ be the positive atom dependency graph of Π.

- A set $\emptyset \subset L \subseteq atom(\Pi)$ is a loop of Π if it induces a non-trivial strongly connected subgraph of $G(\Pi)$.
- That is, each pair of atoms in L is connected by a path of non-zero length in $(L, E \cap (L \times L))$.
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Let Π be a normal logic program.

- For $L \subseteq atom(\Pi)$, define the external supports of L for Π as

$$ES_\Pi(L) = \{ \, r \in \Pi \mid head(r) \in L, body^+(r) \cap L = \emptyset \, \}.$$

- The (disjunctive) loop formula of L for Π is

$$\begin{aligned} LF_\Pi(L) &= (\textstyle\bigvee_{A \in L} A) \rightarrow (\textstyle\bigvee_{r \in ES_\Pi(L)} Comp(body(r))) \\ &\equiv (\textstyle\bigwedge_{r \in ES_\Pi(L)} \neg Comp(body(r))) \rightarrow (\textstyle\bigwedge_{A \in L} \neg A). \end{aligned}$$

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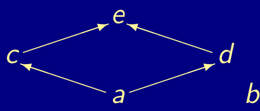
Lin-Zhao Theorem

Theorem

Let Π *be a normal logic program and* $X \subseteq atom(\Pi)$.
Then, X *is an answer set of* Π *iff* $X \models Comp(\Pi) \cup LF(\Pi)$.

Loops and loop formulas: Examples

$$\Pi_2 = \left\{ \begin{array}{ll} a \leftarrow not\ b & b \leftarrow not\ a \\ c \leftarrow a, not\ d & d \leftarrow a, not\ c \\ e \leftarrow c, not\ a & e \leftarrow d, not\ b \end{array} \right\}$$



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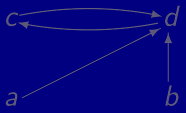

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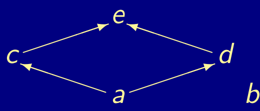

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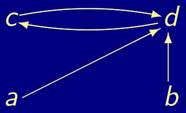

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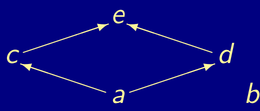

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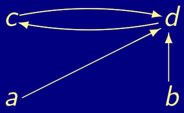

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Let X be a supported model of normal logic program Π.

Then, X is an answer set of Π iff

- $X \models \{ LF_\Pi(U) \mid U \subseteq atom(\Pi) \}$;
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Loops and loop formulas: Properties (ctd)

If $\mathcal{P} \not\subseteq \mathcal{NC}^1/poly$,[1] then there is no translation $\mathcal{T}$ from logic programs to propositional formulas such that, for each normal logic program Π, both of the following conditions hold:

1. The propositional variables in $\mathcal{T}[\Pi]$ are a subset of $atom(\Pi)$.
2. The size of $\mathcal{T}[\Pi]$ is polynomial in the size of Π.

☞ Every vocabulary-preserving translation from normal logic programs to propositional formulas must be exponential (in the worst case).

Observations

- Translation $Comp(\Pi) \cup LF(\Pi)$ preserves the vocabulary of Π.
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Tableau Calculi: Overview

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Goal Analyze computations in ASP-solvers

Wanted A declarative and fine-grained instrument for characterizing operations as well as strategies of ASP-solvers

Idea View answer set computations as derivations in an inference system

➥ Tableau-based proof system for analyzing ASP-solving

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Tableau calculi

- Traditionally, tableau calculi are used for
  - automated theorem proving and
  - proof theoretical analysis

  in classical as well as non-classical logics.
- **General idea:** Given an input, prove some property by decomposition. Decomposition is done by applying deduction rules.
- For details, see [17].

Overview

Tableau calculi: General definitions

- A tableau is a (mostly binary) tree.
- A branch in a tableau is a path from the root to a leaf.
- A branch containing $\gamma_1, \dots, \gamma_m$ can be extended by applying tableau rules of form:

$$\frac{\gamma_1, \dots, \gamma_m}{\begin{array}{c}\alpha_1 \\ \vdots \\ \alpha_n\end{array}} \qquad\qquad \frac{\gamma_1, \dots, \gamma_m}{\beta_1 \mid \dots \mid \beta_n}$$

- Rules of the former format append entries $\alpha_1, \dots, \alpha_n$ to the branch.
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Tableau calculus: Example

A simple tableau calculus for proving unsatisfiability of propositional formulas, composed from $\neg$, $\wedge$, and $\vee$, consists of rules:

$$\frac{\neg\neg\alpha}{\alpha} \qquad \frac{\alpha_1 \wedge \alpha_2}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta_1 \vee \beta_2}{\beta_1 \mid \beta_2}$$

- All rules are semantically valid, interpreting entries in a branch as connected via "and" and distinct (sub-)branches as connected via "or".
- A propositional formula φ (composed from $\neg$, $\wedge$, and $\vee$) is unsatisfiable iff there is a tableau with φ as the root node such that
  1. all other entries can be produced by tableau rules and
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All three branches of the tableau are contradictory (cf. 2, 5, 7, 8, 10).

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(2)	a	[1]
(3)	$(\neg b \wedge (\neg a \vee b)) \vee \neg\neg\neg a$	[1]

(4)	$\neg b \wedge (\neg a \vee b)$	[3]	**(9)**	$\neg\neg\neg a$	[3]
(5)	$\neg b$	[4]	**(10)**	$\neg a$	[9]
(6)	$\neg a \vee b$	[4]			

(7)	$\neg a$	[6]	**(8)**	b	[6]

All three branches of the tableau are contradictory (cf. 2, 5, 7, 8, 10).

➥ $a \wedge ((\neg b \wedge (\neg a \vee b)) \vee \neg\neg\neg a)$ is unsatisfiable.

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Overview

Tableaux and ASP: The idea

- A tableau rule captures an elementary inference scheme in an ASP-solver.
- A branch in a tableau corresponds to a successful or unsuccessful computation of an answer set.
- An entire tableau represents a traversal of the search space.

Tableaux and ASP: Specific definitions

- A (signed) tableau for a logic program Π is a binary tree such that
  - the root node of the tree consists of the rules in Π;
  - the other nodes in the tree are entries of the form $\mathbf{T}v$ or $\mathbf{F}v$, called signed literals, where v is a variable,
  - generated by extending a tableau using deduction rules (given below).
- An entry $\mathbf{T}v$ ($\mathbf{F}v$) reflects that variable v is *true* (*false*) in a corresponding variable assignment.
  - ➥ A set of signed literals constitutes a partial assignment.
- For a normal logic program Π,
  - atoms of Π in $atom(\Pi)$ and
  - bodies of Π in $body(\Pi) = \{body(r) \mid r \in \Pi\}$

  can occur as variables in signed literals.

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Tableau rules for ASP at a glance [43]

$$\text{(FTB)}\quad \frac{p \leftarrow l_1, \dots, l_n \quad \mathbf{t}l_1, \dots, \mathbf{t}l_n}{\mathbf{T}\{l_1, \dots, l_n\}} \qquad \text{(BFB)}\quad \frac{\mathbf{F}\{l_1, \dots, l_i, \dots, l_n\} \quad \mathbf{t}l_1, \dots, \mathbf{t}l_{i-1}, \mathbf{t}l_{i+1}, \dots, \mathbf{t}l_n}{\mathbf{f}l_i}$$

$$\text{(FTA)}\quad \frac{p \leftarrow l_1, \dots, l_n \quad \mathbf{T}\{l_1, \dots, l_n\}}{\mathbf{T}p} \qquad \text{(BFA)}\quad \frac{p \leftarrow l_1, \dots, l_n \quad \mathbf{F}p}{\mathbf{F}\{l_1, \dots, l_n\}}$$

$$\text{(FFB)}\quad \frac{p \leftarrow l_1, \dots, l_i, \dots, l_n \quad \mathbf{f}l_i}{\mathbf{F}\{l_1, \dots, l_i, \dots, l_n\}} \qquad \text{(BTB)}\quad \frac{\mathbf{T}\{l_1, \dots, l_i, \dots, l_n\}}{\mathbf{t}l_i}$$

$$\text{(FFA)}\quad \frac{\mathbf{F}B_1, \dots, \mathbf{F}B_m}{\mathbf{F}p}\ (\S) \qquad \text{(BTA)}\quad \frac{\mathbf{T}p \quad \mathbf{F}B_1, \dots, \mathbf{F}B_{i-1}, \mathbf{F}B_{i+1}, \dots, \mathbf{F}B_m}{\mathbf{T}B_i}$$

$$\text{(WFN)}\quad \frac{\mathbf{F}B_1, \dots, \mathbf{F}B_m}{\mathbf{F}p}\ (\dagger) \qquad \text{(WFJ)}\quad \frac{\mathbf{T}p \quad \mathbf{F}B_1, \dots, \mathbf{F}B_{i-1}, \mathbf{F}B_{i+1}, \dots, \mathbf{F}B_m}{\mathbf{T}B_i}$$

$$\text{(FL)}\quad \frac{\mathbf{F}B_1, \dots, \mathbf{F}B_m}{\mathbf{F}p}\ (\ddagger) \qquad \text{(BL)}\quad \frac{\mathbf{T}p \quad \mathbf{F}B_1, \dots, \mathbf{F}B_{i-1}, \mathbf{F}B_{i+1}, \dots, \mathbf{F}B_m}{\mathbf{T}B_i}$$

$$\text{(Cut}[X])\quad \frac{}{\mathbf{T}v \mid \mathbf{F}v}\ (\sharp[X])$$

More concepts

- A **tableau calculus** is a set of tableau rules.
- A branch in a tableau is conflicting,
  if it contains both $\mathbf{T}v$ and $\mathbf{F}v$ for some variable v.
- A branch in a tableau is total for a program Π,
  if it contains either $\mathbf{T}v$ or $\mathbf{F}v$ for each $v \in \mathit{atom}(\Pi) \cup \mathit{body}(\Pi)$.
- A branch in a tableau of some calculus $\mathcal{T}$ is closed,
  if no rule in $\mathcal{T}$ other than *Cut* can produce any new entries.
- A branch in a tableau is complete,
  if it is either conflicting or both total and closed.
- A tableau is complete,
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Example

Consider the program

$$\Pi = \left\{ \begin{array}{l} a \leftarrow \\ c \leftarrow not\ b, not\ d \\ d \leftarrow a, not\ c \end{array} \right\}$$

having two answer sets $\{a, c\}$ and $\{a, d\}$.

(Previewed) Example

$$
\begin{array}{rcccc}
 & & a \leftarrow & & \\
 & & c \leftarrow \mathit{not}\ b, \mathit{not}\ d & & \\
 & & d \leftarrow a, \mathit{not}\ c & & \\
(FTB) & & \mathbf{T}\emptyset & & \\
(FTA) & & \mathbf{T}a & & \\
(FFA) & & \mathbf{F}b & & \\
(Cut[\mathit{atom}(\Pi)]) & \mathbf{T}c & & & \mathbf{F}c \\
(BTA) & \mathbf{T}\{\mathit{not}\ b, \mathit{not}\ d\} & & (BFA) & \mathbf{F}\{\mathit{not}\ b, \mathit{not}\ d\} \\
(BTB) & \mathbf{F}d & & (BFB) & \mathbf{T}d \\
(FFB) & \mathbf{F}\{a, \mathit{not}\ c\} & & (FTB) & \mathbf{T}\{a, \mathit{not}\ c\}
\end{array}
$$

Recall answer sets $\{a, c\}$ and $\{a, d\}$.

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(Previewed) Example

$$a \leftarrow$$
$$c \leftarrow \mathit{not}\ b, \mathit{not}\ d$$
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(FTB)		$\mathbf{T}\emptyset$		
(FTA)		$\mathbf{T}a$		
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(Cut[$atom(\Pi)$])		$\mathbf{T}c$		$\mathbf{F}c$
	(BTA)	$\mathbf{T}\{\mathit{not}\ b, \mathit{not}\ d\}$	(BFA)	$\mathbf{F}\{\mathit{not}\ b, \mathit{not}\ d\}$
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\begin{array}{rc}
 & a \leftarrow \\
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 & d \leftarrow a, not\ c \\
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\text{(FFA)} & \mathbf{F}b
\end{array}
$$

$$
\begin{array}{rrcrc}
\text{(Cut}[atom(\Pi)]) & & \mathbf{T}c & & \mathbf{F}c \\
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$$
\begin{array}{c}
a \leftarrow \\
c \leftarrow \mathit{not}\ b, \mathit{not}\ d \\
d \leftarrow a, \mathit{not}\ c
\end{array}
$$

(FTB)			$\mathbf{T}\emptyset$		
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 & \mathbf{T}c & & \mathbf{F}c \\
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\end{array}
\end{array}
$$

Recall answer sets $\{a, c\}$ and $\{a, d\}$.

Tableau rules: Auxiliary definitions

- The application of rules makes use of two conjugation functions, **t** and **f**.
- For a literal l, define:

$$\mathbf{t}l = \begin{cases} \mathbf{T}l & \text{if } l \text{ is an atom} \\ \mathbf{F}p & \text{if } l = \mathit{not}\ p \text{ for an atom } p \end{cases}$$

$$\mathbf{f}l = \begin{cases} \mathbf{F}l & \text{if } l \text{ is an atom} \\ \mathbf{T}p & \text{if } l = \mathit{not}\ p \text{ for an atom } p \end{cases}$$

Examples

$$\mathbf{t}p = \mathbf{T}p \qquad \mathbf{f}p = \mathbf{F}p \qquad \mathbf{t}\mathit{not}\ p = \mathbf{F}p \qquad \mathbf{f}\mathit{not}\ p = \mathbf{T}p$$

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Tableau rules: Auxiliary definitions (ctd)

- Some tableau rules require conditions for their application.
  Such conditions are specified as provisos:

$$\frac{\textit{prerequisites}}{\textit{consequence}}\ (\textit{proviso})$$

proviso: some condition(s)

☞ All tableau rules given in the sequel are answer set preserving.

Forward True Body (FTB)

Prerequisites All of a body's literals are *true*.

Consequence The body is *true*.

Tableau Rule FTB

$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_n \\ \mathbf{t}l_1, \dots, \mathbf{t}l_n \end{array}}{\mathbf{T}\{l_1, \dots, l_n\}}$$

Example

$$\frac{\begin{array}{c} a \leftarrow b, not\ c \\ \mathbf{T}b \\ \mathbf{F}c \end{array}}{\mathbf{T}\{b, not\ c\}}$$

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Backward False Body (BFB)

Prerequisites A body is *false*, and all its literals except for one are *true*.

Consequence The residual body literal is *false*.

Tableau Rule BFB

$$\frac{\begin{array}{c}\mathbf{F}\{l_1,\dots,l_i,\dots,l_n\}\\ \mathbf{t}l_1,\dots,\mathbf{t}l_{i-1},\mathbf{t}l_{i+1},\dots,\mathbf{t}l_n\end{array}}{\mathbf{f}l_i}$$

Examples

$$\frac{\begin{array}{c}\mathbf{F}\{b, not\ c\}\\ \mathbf{T}b\end{array}}{\mathbf{T}c} \qquad \frac{\begin{array}{c}\mathbf{F}\{b, not\ c\}\\ \mathbf{F}c\end{array}}{\mathbf{F}b}$$

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Examples

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Forward False Body (FFB)

Prerequisites Some literal of a body is *false*.

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$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_i, \dots, l_n \\ \mathbf{f}\, l_i \end{array}}{\mathbf{F}\{l_1, \dots, l_i, \dots, l_n\}}$$

Examples

$$\frac{\begin{array}{c} a \leftarrow b, \mathit{not}\ c \\ \mathbf{F}b \end{array}}{\mathbf{F}\{b, \mathit{not}\ c\}} \qquad \frac{\begin{array}{c} a \leftarrow b, \mathit{not}\ c \\ \mathbf{T}c \end{array}}{\mathbf{F}\{b, \mathit{not}\ c\}}$$

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Examples

$$\frac{a \leftarrow b, \mathit{not}\ c \qquad \mathbf{F}b}{\mathbf{F}\{b, \mathit{not}\ c\}} \qquad \frac{a \leftarrow b, \mathit{not}\ c \qquad \mathbf{T}c}{\mathbf{F}\{b, \mathit{not}\ c\}}$$

Backward True Body (BTB)

Prerequisites A body is *true*.

Consequence The body's literals are *true*.

Tableau Rule BTB

$$\frac{\mathbf{T}\{l_1, \ldots, l_i, \ldots, l_n\}}{\mathbf{t}l_i}$$

Examples

$$\frac{\mathbf{T}\{b, not\ c\}}{\mathbf{T}b} \qquad \frac{\mathbf{T}\{b, not\ c\}}{\mathbf{F}c}$$

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Reviewing tableau rules for bodies

Consider rule body $B = \{l_1, \dots, l_n\}$.

- Rules FTB and BFB amount to implication:

  $$l_1 \wedge \dots \wedge l_n \rightarrow B$$

- Rules FFB and BTB amount to implication:

  $$B \rightarrow l_1 \wedge \dots \wedge l_n$$

☞ Together they yield:

$$B \equiv l_1 \wedge \dots \wedge l_n$$

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Consider rule body $B = \{l_1, \dots, l_n\}$.

- Rules FTB and BFB amount to implication:

  $$l_1 \wedge \cdots \wedge l_n \rightarrow B$$

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☞ Together they yield:

$$B \equiv l_1 \wedge \cdots \wedge l_n$$

Forward True Atom (FTA)

Prerequisites Some of an atom's bodies is *true*.

Consequence The atom is *true*.

Tableau Rule FTA

$$\frac{\begin{array}{c} p \leftarrow l_1, \ldots, l_n \\ \mathbf{T}\{l_1, \ldots, l_n\} \end{array}}{\mathbf{T}p}$$

Examples

$$\frac{\begin{array}{c} a \leftarrow b, not\ c \\ \mathbf{T}\{b, not\ c\} \end{array}}{\mathbf{T}a} \qquad \frac{\begin{array}{c} a \leftarrow d, not\ e \\ \mathbf{T}\{d, not\ e\} \end{array}}{\mathbf{T}a}$$

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Backward False Atom (BFA)

Prerequisites An atom is *false*.

Consequence The bodies of all rules with the atom as head are *false*.

Tableau Rule BFA

$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_n \\ \mathbf{F}p \end{array}}{\mathbf{F}\{l_1, \dots, l_n\}}$$

Examples

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Forward False Atom (FFA)

Prerequisites: For some atom, the bodies of all rules with the atom as head are *false*.

Consequence: The atom is *false*.

Tableau Rule FFA

$$\frac{\mathbf{F}B_1, \dots, \mathbf{F}B_m}{\mathbf{F}p}\ (body(p) = \{B_1, \dots, B_m\})$$

☞ For an atom p occurring in a logic program Π, we let $body(p) = \{body(r) \mid r \in \Pi, head(r) = p\}$.

Example

$$\frac{\begin{array}{c}\mathbf{F}\{b, not\ c\}\\ \mathbf{F}\{d, not\ e\}\end{array}}{\mathbf{F}a}\ (body(a) = \{\{b, not\ c\}, \{d, not\ e\}\})$$

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Backward True Atom (BTA)

Prerequisites An atom is *true*, and the bodies of all rules with the atom as head except for one are *false*.

Consequence The residual body is *true*.

Tableau Rule BTA

$$\frac{\begin{array}{c}\mathbf{T}p\\ \mathbf{F}B_1,\dots,\mathbf{F}B_{i-1},\mathbf{F}B_{i+1},\dots,\mathbf{F}B_m\end{array}}{\mathbf{T}B_i}\ (body(p)=\{B_1,\dots,B_m\})$$

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Consider an atom p such that $body(p) = \{B_1, \dots, B_m\}$.

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Relationship with Clark's completion

Let Π be a normal logic program.
The eight tableau rules introduced so far essentially provide:

- (straightforward) inferences from $Comp(\Pi)$ (cf. Page 430)
- inferences via **atleast** (cf. Page 488)

Given the same partial assignment (of atoms),

- any literal derived by **atleast** is also derived by tableau rules, while the converse does not hold in general.

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Preliminaries for unfounded sets

Let Π be a normal logic program.

- For $\Pi' \subseteq \Pi$, define the greatest unfounded set, denoted by $GUS(\Pi')$, of Π with respect to Π' as:

  $$GUS(\Pi') = atom(\Pi) \setminus Cn((\Pi')^{\emptyset})$$

- For a loop $L \in Loop(\Pi)$, define

  $$EB(L) = \{body(r) \mid r \in \Pi, head(r) \in L, body^{+}(r) \cap L = \emptyset\}$$

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Well-Founded Negation (WFN)

Prerequisites: An atom is in the greatest unfounded set with respect to rules whose bodies are *false*.

Consequence: The atom is *false*.

Tableau Rule WFN

$$\frac{\mathbf{F}B_1, \ldots, \mathbf{F}B_m}{\mathbf{F}p} \quad (p \in GUS(\{r \in \Pi \mid body(r) \notin \{B_1, \ldots, B_m\}\}))$$

Examples

$$\frac{\begin{array}{c} a \leftarrow not\ b \\ \mathbf{F}\{not\ b\} \end{array}}{\mathbf{F}a} \ (*) \qquad \frac{\begin{array}{c} a \leftarrow a \\ a \leftarrow not\ b \\ \mathbf{F}\{not\ b\} \end{array}}{\mathbf{F}a} \ (*)$$

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Well-Founded Justification (WFJ)

Prerequisites A *true* atom is in the greatest unfounded set with respect to rules whose bodies are *false* if a particular body is made *false*.

Consequence The respective body is *true*.

Tableau Rule WFJ

$$\frac{\begin{array}{c}\mathbf{T}p\\ \mathbf{F}B_1,\dots,\mathbf{F}B_{i-1},\mathbf{F}B_{i+1},\dots,\mathbf{F}B_m\end{array}}{\mathbf{T}B_i}\quad (p \in GUS(\{r \in \Pi \mid body(r) \notin \{B_1,\dots,B_m\}\}))$$

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Reviewing well-founded tableau rules

Tableau rules WFN and WFJ ensure non-circular support for *true* atoms.
Note that

1. WFN subsumes falsifying atoms via FFA,
2. WFJ can be viewed as "backward propagation" for unfounded sets,
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Relationship with well-founded operator

Let Π be a normal logic program, $\langle T, F\rangle$ a partial interpretation, and $\Pi' = \{r \in \Pi \mid body^+(r) \cap F = \emptyset, body^-(r) \cap T = \emptyset\}$.
Then the following conditions are equivalent:

1. $p \in \mathbf{U}_\Pi\langle T, F\rangle$; (cf. Page 530)
2. $p \in \mathbf{atmost}(\langle T, F\rangle)$; (cf. Page 568)
3. $p \in GUS(\Pi')$.
   - ➥ Well-founded operator, **atmost**, and WFN coincide.

- ☞ In contrast to the former, WFN does not necessarily require a rule body to contain a *false* literal for the rule being inapplicable.

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Forward Loop (FL)

Prerequisites The external bodies of a loop are *false*.

Consequence The atoms in the loop are *false*.

Tableau Rule FL

$$\frac{\mathbf{F}B_1, \ldots, \mathbf{F}B_m}{\mathbf{F}p} \; (p \in L, L \in Loop(\Pi), EB(L) = \{B_1, \ldots, B_m\})$$

Example

$$\begin{array}{c} a \leftarrow a \\ a \leftarrow not\ b \\ \hline \end{array}$$

$$\frac{\mathbf{F}\{not\ b\}}{\mathbf{F}a} \; (EB(\{a\}) = \{\{not\ b\}\})$$

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Prerequisites An atom of a loop is *true*, and all external bodies except for one are *false*.

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For a loop L such that $EB(L) = \{B_1, \ldots, B_m\}$,
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Comparison to well-founded tableau rules yields:

- FL (plus FFA and FFB) is equivalent to WFN (plus FFB),
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Relationship with loop formulas

Tableau rules FL and BL essentially provide:

- (straightforward) inferences from loop formulas (cf. Page 589)
  - ☞ But impractical to precompute exponentially many loop formulas !
- an application of the Lin-Zhao Theorem (cf. Page 593)

In practice, ASP-solvers such as `smodels`:

- exploit strongly connected components of positive atom dependency graphs
  - ➥ Can be viewed as an interpolation of FL.
- do not directly implement BL (and neither WFJ)
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Case analysis by *Cut*

Up to now, all tableau rules are deterministic.
That is, rules extend a single branch but cannot create sub-branches.

☞ In general, closing a branch leads to a partial assignment.

Case analysis is done by $Cut[\mathcal{C}]$ where $\mathcal{C} \subseteq atom(\Pi) \cup body(\Pi)$.

Tableau Rule $Cut[\mathcal{C}]$

$$\frac{}{\mathbf{T}v \mid \mathbf{F}v}\ (v \in \mathcal{C})$$

Examples $Cut[\mathcal{C}]$

$$\frac{\begin{array}{c} a \leftarrow not\ b \\ b \leftarrow not\ a \end{array}}{\mathbf{T}a \mid \mathbf{F}a}\ (\mathcal{C} = atom(\Pi)) \qquad \frac{\begin{array}{c} a \leftarrow not\ b \\ b \leftarrow not\ a \end{array}}{\mathbf{T}\{not\ b\} \mid \mathbf{F}\{not\ b\}}\ (\mathcal{C} = body(\Pi))$$

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Well-known tableau calculi

Fitting's operator $\mathbf{\Phi}$ applies forward propagation without sophisticated unfounded set checks. We have:

$$\mathcal{T}_{\mathbf{\Phi}} = \{FTB, FTA, FFB, FFA\}$$

Well-founded operator $\mathbf{\Omega}$ replaces negation of single atoms with negation of unfounded sets. We have:

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"Local" propagation via a program's completion can be determined by elementary inferences on atoms and rule bodies. We have:

$$\mathcal{T}_{completion} = \{FTB, FTA, FFB, FFA, BTB, BTA, BFB, BFA\}$$

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Tableau calculi characterizing ASP-solvers

ASP-solvers combine propagation with case analysis.
We obtain the following tableau calculi characterizing
[4, 63, 51, 77, 57, 54, 2]:

$$
\begin{aligned}
\mathcal{T}_{\mathit{cmodels\text{-}1}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{Cut}[\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)]\} \\
\mathcal{T}_{\mathit{assat}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{FL}\} \cup \{\mathit{Cut}[\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)]\} \\
\mathcal{T}_{\mathit{smodels}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{WFN}\} \cup \{\mathit{Cut}[\mathit{atom}(\Pi)]\} \\
\mathcal{T}_{\mathit{noMoRe}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{WFN}\} \cup \{\mathit{Cut}[\mathit{body}(\Pi)]\} \\
\mathcal{T}_{\mathit{nomore}^{++}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{WFN}\} \cup \{\mathit{Cut}[\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)]\}
\end{aligned}
$$

- SAT-based ASP-solvers, `assat` and `cmodels`, incrementally add loop formulas to a program's completion.
- Genuine ASP-solvers, `smodels`, `dlv`, `noMoRe`, and `nomore++`, essentially differ only in their *Cut* rules.

Tableau calculi characterizing ASP-solvers

ASP-solvers combine propagation with case analysis.
We obtain the following tableau calculi characterizing [4, 63, 51, 77, 57, 54, 2]:

$$
\begin{aligned}
\mathcal{T}_{\mathit{cmodels\text{-}1}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{Cut}[\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)]\} \\
\mathcal{T}_{\mathit{assat}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{FL}\} \cup \{\mathit{Cut}[\mathit{atom}(\Pi) \cup \mathit{body}(\Pi)]\} \\
\mathcal{T}_{\mathit{smodels}} &= \mathcal{T}_{\mathit{completion}} \cup \{\mathit{WFN}\} \cup \{\mathit{Cut}[\mathit{atom}(\Pi)]\} \\
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Proof complexity

The notion of proof complexity is used for describing the relative efficiency of different proof systems.

It compares proof systems based on minimal refutations.

➥ Proof complexity does not depend on heuristics.

A proof system $\mathcal{T}$ polynomially simulates a proof system $\mathcal{T}'$ if every refutation of $\mathcal{T}'$ can be polynomially mapped to a refutation of $\mathcal{T}$.

Otherwise, $\mathcal{T}$ does not polynomially simulate $\mathcal{T}'$.

For showing that proof system $\mathcal{T}$ does not polynomially simulate $\mathcal{T}'$, we have to provide an infinite witnessing family of programs such that minimal refutations of $\mathcal{T}$ asymptotically are exponentially larger than minimal refutations of $\mathcal{T}'$.

The size of tableaux is simply the number of their entries.

☞ We do not need to know the precise number of entries: Counting required *Cut* applications is sufficient !

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$\mathcal{T}_{smodels}$ versus $\mathcal{T}_{noMoRe}$

Recall that $\mathcal{T}_{smodels}$ restricts *Cut* to $atom(\Pi)$ and $\mathcal{T}_{noMoRe}$ to $body(\Pi)$.
Are both approaches similar or is one of them superior to the other?
Let $\{\Pi_a^n\}$, $\{\Pi_b^n\}$, and $\{\Pi_c^n\}$ be infinite families of programs as follows:

$$\Pi_a^n = \left\{ \begin{array}{l} x \leftarrow not\ x \\ x \leftarrow a_1, b_1 \\ \vdots \\ x \leftarrow a_n, b_n \end{array} \right\} \quad \Pi_b^n = \left\{ \begin{array}{ll} \multicolumn{2}{l}{x \leftarrow c_1, \dots, c_n, not\ x} \\ c_1 \leftarrow a_1 & c_1 \leftarrow b_1 \\ \vdots & \vdots \\ c_n \leftarrow a_n & c_n \leftarrow b_n \end{array} \right\} \quad \Pi_c^n = \left\{ \begin{array}{l} a_1 \leftarrow not\ b \\ b_1 \leftarrow not\ a \\ \vdots \\ a_n \leftarrow not\ b \\ b_n \leftarrow not\ a \end{array} \right.$$

In minimal refutations for $\Pi_a^n \cup \Pi_c^n$, the number of applications of $Cut[body(\Pi_a^n \cup \Pi_c^n)]$ with $\mathcal{T}_{noMoRe}$ is linear in n, whereas $\mathcal{T}_{smodels}$ requires exponentially many applications of $Cut[atom(\Pi_a^n \cup \Pi_c^n)]$.
Vice versa, minimal refutations for $\Pi_b^n \cup \Pi_c^n$ require linearly many applications of $Cut[atom(\Pi_b^n \cup \Pi_c^n)]$ with $\mathcal{T}_{smodels}$ and exponentially many applications of $Cut[body(\Pi_b^n \cup \Pi_c^n)]$ with $\mathcal{T}_{noMoRe}$.

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Relative efficiency

As witnessed by $\{\Pi_a^n \cup \Pi_c^n\}$ and $\{\Pi_b^n \cup \Pi_c^n\}$, respectively, $\mathcal{T}_{smodels}$ and $\mathcal{T}_{noMoRe}$ do not polynomially simulate one another.
Any refutation of $\mathcal{T}_{smodels}$ or $\mathcal{T}_{noMoRe}$ is a refutation of $\mathcal{T}_{nomore^{++}}$ (but not vice versa).

It follows that

- both $\mathcal{T}_{smodels}$ and $\mathcal{T}_{noMoRe}$ are polynomially simulated by $\mathcal{T}_{nomore^{++}}$ and
- $\mathcal{T}_{nomore^{++}}$ is polynomially simulated by neither $\mathcal{T}_{smodels}$ nor $\mathcal{T}_{noMoRe}$.

➥ The proof system obtained with $Cut[atom(\Pi) \cup body(\Pi)]$ is exponentially stronger than the ones with either $Cut[atom(\Pi)]$ or $Cut[body(\Pi)]$!

☞ Case analyses (at least) on atoms and bodies are mandatory in powerful ASP-solvers.

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$\mathcal{T}_{smodels}$: Example tableau

(r_1)	$a \leftarrow not\ b$	(r_2)	$b \leftarrow d, not\ a$	(r_3)	$c \leftarrow b, d$
(r_4)	$c \leftarrow g$	(r_5)	$d \leftarrow c$	(r_6)	$d \leftarrow g$
(r_7)	$e \leftarrow f, not\ c$	(r_8)	$f \leftarrow not\ g$	(r_9)	$g \leftarrow not\ a, not\ f$

(1)	$\mathbf{T}a$	[Cut]
(2)	$\mathbf{T}\{not\ b\}$	[BTA: r_1, 1]
(3)	$\mathbf{F}b$	[BTB: 2]
(4)	$\mathbf{F}\{d, not\ a\}$	[BFA: r_2, 3]
(5)	$\mathbf{F}\{not\ a, not\ f\}$	[FFB: r_9, 1]
(6)	$\mathbf{F}g$	[FFA: r_9, 5]
(7)	$\mathbf{T}\{not\ g\}$	[FTB: r_8, 6]
(8)	$\mathbf{T}f$	[FTA: r_8, 7]
(9)	$\mathbf{F}\{b, d\}$	[FFB: r_3, 3]
(10)	$\mathbf{F}\{g\}$	[FFB: r_4, r_6, 6]
(11)	$\mathbf{F}c$	[FFA: r_3, r_4, 9, 10]
(12)	$\mathbf{F}\{c\}$	[FFB: r_5, 11]
(13)	$\mathbf{F}d$	[FFA: r_5, r_6, 10, 12]
(14)	$\mathbf{T}\{f, not\ c\}$	[FTB: r_7, 8, 11]
(15)	$\mathbf{T}e$	[FTA: r_7, 14]

(16)	$\mathbf{F}a$	[Cut]
(17)	$\mathbf{F}\{not\ b\}$	[BFA: r_1, 16]
(18)	$\mathbf{T}b$	[BFB: 17]
(19)	$\mathbf{T}\{d, not\ a\}$	[BTA: r_2, 18]
(20)	$\mathbf{T}d$	[BTB: 19]
(21)	$\mathbf{T}\{b, d\}$	[FTB: r_3, 18, 20]
(22)	$\mathbf{T}c$	[FTA: r_3, 21]
(23)	$\mathbf{F}\{f, not\ c\}$	[FFB: r_7, 22]
(24)	$\mathbf{F}e$	[FFA: r_7, 23]
(25)	$\mathbf{T}\{c\}$	[FTB: r_5, 22]

(26)	$\mathbf{T}f$	[Cut]
(27)	$\mathbf{F}\{not\ a, not\ f\}$	[FFB: r_9, 26]
(28)	$\mathbf{F}c$	[WFN: 27]

(29)	$\mathbf{F}f$	[Cut]
(30)	$\mathbf{T}\{not\ a, not\ f\}$	[FTB: r_9,
(31)	$\mathbf{T}g$	[FTA: r_9, 3
(32)	$\mathbf{T}\{g\}$	[FTB: r_4,
(33)	$\mathbf{F}\{not\ g\}$	[FFB: r_8, 3

$\mathcal{T}_{noMoRe}$: Example tableau

(r_1)	$a \leftarrow not\ b$	(r_2)	$b \leftarrow d, not\ a$	(r_3)	$c \leftarrow b, d$
(r_4)	$c \leftarrow g$	(r_5)	$d \leftarrow c$	(r_6)	$d \leftarrow g$
(r_7)	$e \leftarrow f, not\ c$	(r_8)	$f \leftarrow not\ g$	(r_9)	$g \leftarrow not\ a, not\ f$

(1)	$\mathbf{T}\{not\ b\}$	$[Cut]$
(2)	$\mathbf{T}a$	$[FTA$: $r_1, 1]$
(3)	$\mathbf{F}b$	$[BTB$: $1]$
(4)	$\mathbf{F}\{d, not\ a\}$	$[BFA$: $r_2, 3]$
(5)	$\mathbf{F}\{not\ a, not\ f\}$	$[FFB$: $r_9, 2]$
(6)	$\mathbf{F}g$	$[FFA$: $r_9, 5]$
(7)	$\mathbf{T}\{not\ g\}$	$[FTB$: $r_8, 6]$
(8)	$\mathbf{T}f$	$[FTA$: $r_8, 7]$
(9)	$\mathbf{F}\{b, d\}$	$[FFB$: $r_3, 3]$
(10)	$\mathbf{F}\{g\}$	$[FFB$: $r_4, r_6, 6]$
(11)	$\mathbf{F}c$	$[FFA$: $r_3, r_4, 9, 10]$
(12)	$\mathbf{F}\{c\}$	$[FFB$: $r_5, 11]$
(13)	$\mathbf{F}d$	$[FFA$: $r_5, r_6, 10, 12]$
(14)	$\mathbf{T}\{f, not\ c\}$	$[FTB$: $r_7, 8, 11]$
(15)	$\mathbf{T}e$	$[FTA$: $r_7, 14]$

(16)	$\mathbf{F}\{not\ b\}$	$[Cut]$
(17)	$\mathbf{F}a$	$[FFA$: $r_1, 16]$
(18)	$\mathbf{T}b$	$[BFB$: $16]$
(19)	$\mathbf{T}\{d, not\ a\}$	$[BTA$: $r_2, 18]$
(20)	$\mathbf{T}d$	$[BTB$: $19]$
(21)	$\mathbf{T}\{b, d\}$	$[FTB$: $r_3, 18, 20]$
(22)	$\mathbf{T}c$	$[FTA$: $r_3, 21]$
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(26)	$\mathbf{T}\{not\ g\}$	$[Cut]$
(27)	$\mathbf{F}g$	$[BTB$: $26]$
(28)	$\mathbf{F}\{g\}$	$[FFB$: $r_4, r_6, 27]$
(29)	$\mathbf{F}c$	$[WFN$: $28]$

(30)	$\mathbf{F}\{not\ g\}$	$[Cut]$
(31)	$\mathbf{T}g$	$[BFB$: $30]$
(32)	$\mathbf{T}\{g\}$	$[FTB$: $r_4,$
(33)	$\mathbf{F}f$	$[FFA$: $r_8, 3$
(34)	$\mathbf{T}\{not\ a, not\ f\}$	$[FTB$: $r_9,$

$\mathcal{T}_{nomore^{++}}$: Example tableau

(r_1)	$a \leftarrow not\ b$	(r_2)	$b \leftarrow d, not\ a$	(r_3)	$c \leftarrow b, d$
(r_4)	$c \leftarrow g$	(r_5)	$d \leftarrow c$	(r_6)	$d \leftarrow g$
(r_7)	$e \leftarrow f, not\ c$	(r_8)	$f \leftarrow not\ g$	(r_9)	$g \leftarrow not\ a, not\ f$

(1)	$\mathbf{T}a$	$[Cut]$
(2)	$\mathbf{T}\{not\ b\}$	$[BTA\colon r_1, 1]$
(3)	$\mathbf{F}b$	$[BTB\colon 2]$
(4)	$\mathbf{F}\{d, not\ a\}$	$[BFA\colon r_2, 3]$
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(6)	$\mathbf{F}g$	$[FFA\colon r_9, 5]$
(7)	$\mathbf{T}\{not\ g\}$	$[FTB\colon r_8, 6]$
(8)	$\mathbf{T}f$	$[FTA\colon r_8, 7]$
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(12)	$\mathbf{F}\{c\}$	$[FFB\colon r_5, 11]$
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(14)	$\mathbf{T}\{f, not\ c\}$	$[FTB\colon r_7, 8, 11]$
(15)	$\mathbf{T}e$	$[FTA\colon r_7, 14]$

(16)	$\mathbf{F}a$	$[Cut]$
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(19)	$\mathbf{T}\{d, not\ a\}$	$[BTA\colon r_2, 18]$
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(25)	$\mathbf{T}\{c\}$	$[FTB\colon r_5, 22]$

(26)	$\mathbf{T}\{not\ g\}$	$[Cut]$
(27)	$\mathbf{F}g$	$[BTB\colon 26]$
(28)	$\mathbf{F}\{g\}$	$[FFB\colon r_4, r_6, 27]$
(29)	$\mathbf{F}c$	$[WFN\colon 28]$

(30)	$\mathbf{F}\{not\ g\}$	$[Cut]$
(31)	$\mathbf{T}g$	$[BFB\colon 30]$
(32)	$\mathbf{T}\{g\}$	$[FTB\colon r_4,$
(33)	$\mathbf{F}f$	$[FFA\colon r_8, 3$
(34)	$\mathbf{T}\{not\ a, not\ f\}$	$[FTB\colon r_9, 1$

Conflict-Driven Answer Set Solving: Overview

Overview

Motivation

Goal New approach to computing answer sets of logic programs, based on concepts from

- Constraint Processing (CSP) and
- Satisfiability Checking (SAT)

Idea View inferences in Answer Set Programming (ASP) as unit propagation on nogoods.

Benefits

- A uniform constraint-based framework for different kinds of inferences in ASP
- Advanced techniques from the areas of CSP and SAT
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Overview

Assignments

- An assignment A over $dom(A) = atom(\Pi) \cup body(\Pi)$ is a sequence

  $$(\sigma_1, \dots, \sigma_n)$$

  of signed literals σ_i of form $\mathbf{T}p$ or $\mathbf{F}p$ for $p \in dom(A)$ and $1 \leq i \leq n$.
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- The complement, $\overline{\sigma}$, of a literal σ is defined as $\overline{\mathbf{T}p} = \mathbf{F}p$ and $\overline{\mathbf{F}p} = \mathbf{T}p$.
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Nogoods, Solutions, and Unit Propagation

- A nogood is a set $\{\sigma_1, \dots, \sigma_n\}$ of signed literals, expressing a constraint violated by any assignment containing $\sigma_1, \dots, \sigma_n$.
- An assignment A such that $A^{\mathbf{T}} \cup A^{\mathbf{F}} = dom(A)$ and $A^{\mathbf{T}} \cap A^{\mathbf{F}} = \emptyset$ is a solution for a set Δ of nogoods, if $\delta \not\subseteq A$ for all $\delta \in \Delta$.
- For a nogood δ, a literal $\sigma \in \delta$, and an assignment A, we say that $\overline{\sigma}$ is unit-resulting for δ wrt A, if
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  2. $\overline{\sigma} \notin A$.
- For a set Δ of nogoods and an assignment A, unit propagation is the iterated process of extending A with unit-resulting literals until no further literal is unit-resulting for any nogood in Δ.

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Overview

Nogoods from logic programs

via Clark's completion

The completion of a logic program Π can be defined as follows:

$$
\begin{aligned}
& \{p_\beta \leftrightarrow p_1 \wedge \cdots \wedge p_m \wedge \neg p_{m+1} \wedge \cdots \wedge \neg p_n \mid \\
& \qquad \beta \in \mathit{body}(\Pi), \beta = \{p_1, \ldots, p_m, \mathit{not}\ p_{m+1}, \ldots, \mathit{not}\ p_n\}\} \\
\cup \quad & \{p \leftrightarrow p_{\beta_1} \vee \cdots \vee p_{\beta_k} \mid \\
& \qquad p \in \mathit{atom}(\Pi), \mathit{body}(p) = \{\beta_1, \ldots, \beta_k\}\}\ ,
\end{aligned}
$$

where $\mathit{body}(p) = \{\mathit{body}(r) \mid r \in \Pi, \mathit{head}(r) = p\}$.

Nogoods from logic programs (ctd)

via Clark's completion

Let $\beta = \{p_1, \dots, p_m, not\ p_{m+1}, \dots, not\ p_n\}$ be a body.
The equivalence

$$p_\beta \leftrightarrow p_1 \wedge \dots \wedge p_m \wedge \neg p_{m+1} \wedge \dots \wedge \neg p_n$$

can be decomposed into two implications.

1. We get

$$p_\beta \rightarrow p_1 \wedge \dots \wedge p_m \wedge \neg p_{m+1} \wedge \dots \wedge \neg p_n \ ,$$

which is equivalent to the conjunction of

$$\neg p_\beta \vee p_1, \ \dots, \ \neg p_\beta \vee p_m, \ \neg p_\beta \vee \neg p_{m+1}, \ \dots, \ \neg p_\beta \vee \neg p_n \ .$$

This set of clauses expresses the following set of nogoods:

$$\Delta(\beta) = \{ \{\mathbf{T}\beta, \mathbf{F}p_1\}, \dots, \{\mathbf{T}\beta, \mathbf{F}p_m\}, \{\mathbf{T}\beta, \mathbf{T}p_{m+1}\}, \dots, \{\mathbf{T}\beta, \mathbf{T}p_n\} \} \ .$$

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2. The converse of the previous implication, viz.

$$p_1 \wedge \dots \wedge p_m \wedge \neg p_{m+1} \wedge \dots \wedge \neg p_n \rightarrow p_\beta \ ,$$

gives rise to the nogood

$$\delta(\beta) = \{\mathbf{F}\beta, \mathbf{T}p_1, \dots, \mathbf{T}p_m, \mathbf{F}p_{m+1}, \dots, \mathbf{F}p_n\} \ .$$

Intuitively, $\delta(\beta)$ is a constraint enforcing the truth of body β, or the falsity of a contained literal.

Nogoods from logic programs (ctd)

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Nogoods from logic programs (ctd)

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Proceeding analogously with the atom-based equivalences, viz.

$$p \leftrightarrow p_{\beta_1} \vee \cdots \vee p_{\beta_k}$$

we obtain for an atom $p \in atom(\Pi)$ along with its bodies $body(p) = \{\beta_1, \ldots, \beta_k\}$ the nogoods

$$\Delta(p) = \{\, \{\mathbf{F}p, \mathbf{T}\beta_1\}, \ldots, \{\mathbf{F}p, \mathbf{T}\beta_k\} \,\} \quad \text{and}$$

$$\delta(p) = \{\mathbf{T}p, \mathbf{F}\beta_1, \ldots, \mathbf{F}\beta_k\} \; .$$

Nogoods from logic programs

atom-oriented nogoods

For an atom p where $body(p) = \{\beta_1, \dots, \beta_k\}$, recall that

$$\begin{array}{rcl} \delta(p) & = & \{\mathbf{T}p, \mathbf{F}\beta_1, \dots, \mathbf{F}\beta_k\} \\ \Delta(p) & = & \{\{\mathbf{F}p, \mathbf{T}\beta_1\}, \dots, \{\mathbf{F}p, \mathbf{T}\beta_k\}\}\ . \end{array}$$

For example, for atom x with $body(x) = \{\{y\}, \{not\ z\}\}$, we obtain

$$\boxed{\begin{array}{lcl} x & \leftarrow & y \\ x & \leftarrow & not\ z \end{array}} \qquad \begin{array}{rcl} \delta(x) & = & \{\mathbf{T}x, \mathbf{F}\{y\}, \mathbf{F}\{not\ z\}\} \\ \Delta(x) & = & \{\{\mathbf{F}x, \mathbf{T}\{y\}\}, \{\mathbf{F}x, \mathbf{T}\{not\ z\}\}\} \end{array}$$

For nogood $\delta(x) = \{\mathbf{T}x, \mathbf{F}\{y\}, \mathbf{F}\{not\ z\}\}$, the signed literal

- $\mathbf{F}x$ is unit-resulting wrt assignment $(\mathbf{F}\{y\}, \mathbf{F}\{not\ z\})$ and
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Nogoods from logic programs

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For nogood $\delta(x) = \{\mathbf{T}x, \mathbf{F}\{y\}, \mathbf{F}\{not\ z\}\}$, the signed literal

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Nogoods from logic programs

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Nogoods from logic programs

body-oriented nogoods

For a body $\beta = \{p_1, \dots, p_m, \mathit{not}\ p_{m+1}, \dots, \mathit{not}\ p_n\}$, recall that

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$$\boxed{\begin{array}{c}\dots \leftarrow x, \mathit{not}\ y\\ \vdots\\ \dots \leftarrow x, \mathit{not}\ y\end{array}}$$

$$\begin{aligned}\delta(\{x, \mathit{not}\ y\}) &= \{\mathbf{F}\{x, \mathit{not}\ y\}, \mathbf{T}x, \mathbf{F}y\}\\ \Delta(\{x, \mathit{not}\ y\}) &= \{\,\{\mathbf{T}\{x, \mathit{not}\ y\}, \mathbf{F}x\}, \{\mathbf{T}\{x, \mathit{not}\ y\}, \mathbf{T}\end{aligned}$$

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- $\mathbf{T}\{x, \mathit{not}\ y\}$ is unit-resulting wrt assignment $(\mathbf{T}x, \mathbf{F}y)$ and
- $\mathbf{T}y$ is unit-resulting wrt assignment $(\mathbf{F}\{x, \mathit{not}\ y\}, \mathbf{T}x)$.

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For a body $\beta = \{p_1, \dots, p_m, not\ p_{m+1}, \dots, not\ p_n\}$, recall that

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$$\begin{aligned}\delta(\{x, \mathit{not}\ y\}) &= \{\mathbf{F}\{x, \mathit{not}\ y\}, \mathbf{T}x, \mathbf{F}y\} \\ \Delta(\{x, \mathit{not}\ y\}) &= \{\,\{\mathbf{T}\{x, \mathit{not}\ y\}, \mathbf{F}x\}, \{\mathbf{T}\{x, \mathit{not}\ y\}, \mathbf{T}\end{aligned}$$

For nogood $\delta(\{x, \mathit{not}\ y\}) = \{\mathbf{F}\{x, \mathit{not}\ y\}, \mathbf{T}x, \mathbf{F}y\}$, the signed literal

- $\mathbf{T}\{x, \mathit{not}\ y\}$ is unit-resulting wrt assignment $(\mathbf{T}x, \mathbf{F}y)$ and
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Nogoods from logic programs

body-oriented nogoods

For a body $\beta = \{p_1, \dots, p_m, \mathit{not}\ p_{m+1}, \dots, \mathit{not}\ p_n\}$, recall that

$$\begin{aligned}\delta(\beta) &= \{\mathbf{F}\beta, \mathbf{T}p_1, \dots, \mathbf{T}p_m, \mathbf{F}p_{m+1}, \dots, \mathbf{F}p_n\}\\ \Delta(\beta) &= \{\, \{\mathbf{T}\beta, \mathbf{F}p_1\}, \dots, \{\mathbf{T}\beta, \mathbf{F}p_m\}, \{\mathbf{T}\beta, \mathbf{T}p_{m+1}\}, \dots, \{\mathbf{T}\beta, \mathbf{T}p_n\}\,\}\end{aligned}$$

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Characterization of answer sets

for tight logic programs

Let Π be a logic program and

$$\begin{aligned} \Delta_\Pi &= \{\delta(p) \mid p \in atom(\Pi)\} \cup \{\delta \in \Delta(p) \mid p \in atom(\Pi)\} \\ &\cup \{\delta(\beta) \mid \beta \in body(\Pi)\} \cup \{\delta \in \Delta(\beta) \mid \beta \in body(\Pi)\}\ . \end{aligned}$$

Theorem

Let Π be a tight logic program. Then,
$X \subseteq atom(\Pi)$ is an answer set of Π iff
$X = A^{\mathbf{T}} \cap atom(\Pi)$ for a (unique) solution A for Δ_Π.

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Atom-oriented nogoods and tableau rules

- Tableau rules FTA, BFA, FFA, and BTA are atom-oriented.
- For an atom p such that $body(p) = \{\beta_1, \dots, \beta_k\}$, consider the equivalence: $p \leftrightarrow p_{\beta_1} \vee \cdots \vee p_{\beta_k}$
- Inferences from nogoods $\Delta(p) = \{\, \{\mathbf{F}p, \mathbf{T}\beta_1\}, \dots, \{\mathbf{F}p, \mathbf{T}\beta_k\} \,\}$ correspond to those from tableau rules FTA and BFA:

$$\frac{\begin{array}{c} p \leftarrow \beta \\ \mathbf{T}\beta \end{array}}{\mathbf{T}p} \qquad \frac{\begin{array}{c} p \leftarrow \beta \\ \mathbf{F}p \end{array}}{\mathbf{F}\beta}$$

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$$\frac{p \leftarrow l_1, \dots, l_n \qquad \mathbf{t}l_1, \dots, \mathbf{t}l_n}{\mathbf{T}\{l_1, \dots, l_n\}} \qquad\qquad \frac{\mathbf{F}\{l_1, \dots, l_n\} \qquad \mathbf{t}l_1, \dots, \mathbf{t}l_{i-1}, \mathbf{t}l_{i+1}, \dots, \mathbf{t}l_n}{\mathbf{f}l_i}$$

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Nogoods from logic programs

via loop formulas (cf. Page 589)

Let Π be a normal logic program and recall that:

- For $L \subseteq atom(\Pi)$, the external supports of L for Π are

$$ES_\Pi(L) = \{r \in \Pi \mid head(r) \in L, body^+(r) \cap L = \emptyset\}.$$

- The (disjunctive) loop formula of L for Π is

$$\begin{aligned} LF_\Pi(L) &= \left(\bigvee_{A \in L} A\right) \rightarrow \left(\bigvee_{r \in ES_\Pi(L)} Comp(body(r))\right) \\ &\equiv \left(\bigwedge_{r \in ES_\Pi(L)} \neg Comp(body(r))\right) \rightarrow \left(\bigwedge_{A \in L} \neg A\right). \end{aligned}$$

  ☞ The loop formula of L enforces all atoms in L to be *false* whenever L is not externally supported.

- The external bodies of L for Π are

$$\begin{aligned} EB(L) &= \{body(r) \mid r \in \Pi, head(r) \in L, body^+(r) \cap L = \emptyset\} \\ &= \{body(r) \mid r \in ES_\Pi(L)\}. \end{aligned}$$

Nogoods from logic programs

loop nogoods

For a logic program Π and some $\emptyset \subset U \subseteq atom(\Pi)$, define the loop nogood of an atom $p \in U$ as

$$\lambda(p, U) \quad = \quad \{\mathbf{T}p, \mathbf{F}\beta_1, \ldots, \mathbf{F}\beta_k\}$$

where $EB(U) = \{\beta_1, \ldots, \beta_k\}$.

In all, we get the following set of loop nogoods for Π:

$$\Lambda_\Pi \quad = \quad \bigcup_{\emptyset \subset U \subseteq atom(\Pi)} \{\lambda(p, U) \mid p \in U\}$$

☞ The set Λ_Π of loop nogoods denies cyclic support among *true* atoms.

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Example

Consider

$$\Pi = \left\{ \begin{array}{ll} x \leftarrow not\ y & u \leftarrow x \\ & u \leftarrow v \\ y \leftarrow not\ x & v \leftarrow u, y \end{array} \right\}$$

For u in the set $\{u, v\}$, we obtain the loop nogood:

$$\lambda(u, \{u, v\}) \quad = \quad \{\mathbf{T}u, \mathbf{F}\{x\}\}$$

Similarly for v in $\{u, v\}$, we get:

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Characterization of answer sets

For a logic program Π,
let Δ_Π and Λ_Π as defined on Page 755 and Page 767, respectively.

Theorem

Let Π be a logic program. Then,
$X \subseteq atom(\Pi)$ is an answer set of Π iff
$X = A^{\mathbf{T}} \cap atom(\Pi)$ for a (unique) solution A for $\Delta_\Pi \cup \Lambda_\Pi$.

Some remarks

- Nogoods in Λ_Π augment Δ_Π with conditions checking for unfounded sets, in particular, those being loops. While $|\Delta_\Pi|$ is linear in the size of Π, Λ_Π may contain exponentially many (non-redundant) loop nogoods !

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- Nogoods in Λ_Π augment Δ_Π with conditions checking for unfounded sets, in particular, those being loops.
- While $|\Delta_\Pi|$ is linear in the size of Π, Λ_Π may contain exponentially many (non-redundant) loop nogoods !

Overview

Conflict-driven search

Boolean constraint solving algorithms pioneered for SAT led to:

Traditional approach

- (Unit) propagation
- Exhaustive (chronological) backtracking

☞ DPLL [20, 19]

State of the art

- (Unit) propagation
- Conflict analysis (via resolution)
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Outline of CDNL-ASP algorithm

[38]

- Keep track of deterministic consequences by unit propagation on:
  - Clark's completion $[\Delta_\Pi]$
  - Loop nogoods, determined and recorded on demand $[\Lambda_\Pi]$
    - ☞ Dedicated unfounded set detection !
  - Dynamic nogoods, derived from conflicts and unfounded sets $[\nabla]$
- When a nogood in $\Delta_\Pi \cup \nabla$ becomes violated:
  - Analyze the conflict by resolution until reaching the First Unique Implication Point (First-UIP) [68]
  - Learn the derived conflict nogood δ
  - Backjump to the earliest (heuristic) choice such that the complement of the First-UIP is unit-resulting for δ
  - Assert the complement of the First-UIP and proceed (by unit propagation)
- Terminate when either:
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Algorithm 1: CDNL-ASP

Input : A logic program Π.
Output : An answer set of Π or "no answer set".

$A \leftarrow \emptyset$ // *assignment over* $atom(\Pi) \cup body(\Pi)$
$\nabla \leftarrow \emptyset$ // *set of (dynamic) nogoods*
$dl \leftarrow 0$ // *decision level*
loop
  $(A, \nabla) \leftarrow \textsc{NogoodPropagation}(\Pi, \nabla, A)$
  if $\varepsilon \subseteq A$ for some $\varepsilon \in \Delta_\Pi \cup \nabla$ **then**
    if $dl = 0$ **then return** no answer set
    $(\delta, k) \leftarrow \textsc{ConflictAnalysis}(\varepsilon, \Pi, \nabla, A)$
    $\nabla \leftarrow \nabla \cup \{\delta\}$ // *learning*
    $A \leftarrow (A \setminus \{\sigma \in A \mid k < dl(\sigma)\})$ // *backjumping*
    $dl \leftarrow k$
  else if $A^{\mathbf{T}} \cup A^{\mathbf{F}} = atom(\Pi) \cup body(\Pi)$ **then**
    return $A^{\mathbf{T}} \cap atom(\Pi)$ // *answer set*
  else
    $\sigma_d \leftarrow \textsc{Select}(\Pi, \nabla, A)$ // *heuristic choice of* $\sigma_d \notin A$
    $dl \leftarrow dl + 1$
    $A \leftarrow A \circ (\sigma_d)$ // $dl(\sigma_d) = dl$

Observations

- Decision level dl, initially set to 0, is used to count the number of heuristically chosen literals in assignment A.
- For a heuristically chosen literal $\sigma_d = \mathbf{T}p$ or $\sigma_d = \mathbf{F}p$, respectively, we require $p \in (atom(\Pi) \cup body(\Pi)) \setminus (A^{\mathbf{T}} \cup A^{\mathbf{F}})$.
- For any literal $\sigma \in A$, $dl(\sigma)$ denotes the decision level of σ, viz. the value dl had when σ was assigned.
- A conflict is detected from violation of a nogood $\varepsilon \subseteq \Delta_\Pi \cup \nabla$.
- A conflict at decision level 0 (where A contains no heuristically chosen literals) indicates non-existence of answer sets.
- A nogood δ derived by conflict analysis is asserting, that is, some literal is unit-resulting for δ at a decision level $k < dl$.
  - ➥ After learning δ and backjumping to decision level k, at least one literal is newly derivable by unit propagation.
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Example: CDNL-ASP

Consider

$$\Pi = \left\{ \begin{array}{llll} x \leftarrow \mathit{not}\ y & u \leftarrow x, y & v \leftarrow x & w \leftarrow \mathit{not}\ x, \mathit{not}\ y \\ y \leftarrow \mathit{not}\ x & u \leftarrow v & v \leftarrow u, y & \end{array} \right\}$$

dl	σ_d	$\overline{\sigma}$	δ	
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2	$\mathbf{F}\{\mathit{not}\ x, \mathit{not}\ y\}$			
		$\mathbf{F}w$	$\{\mathbf{T}w, \mathbf{F}\{\mathit{not}\ x, \mathit{not}\ y\}\} = \delta(w)$	
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- Derive deterministic consequences via:
  - Unit propagation on Δ_Π and ∇;
  - Unfounded sets $U \subseteq atom(\Pi)$.
- Note that U is unfounded if $EB(U) \subseteq A^{\mathbf{F}}$.
  - ☞ For any $p \in U$, we have $(\lambda(p,U) \setminus \{\mathbf{T}p\}) \subseteq A$.
- An "interesting" unfounded set U satisfies:

$$\emptyset \subset U \subseteq (atom(\Pi) \setminus A^{\mathbf{F}}) \ .$$

- Wrt a fixpoint of unit propagation,
  such an unfounded set contains some loop of Π.
  - ➥ Tight programs do not yield "interesting" unfounded sets !
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Algorithm 2: NogoodPropagation

Input : A logic program Π, a set ∇ of nogoods, and an assignment A.
Output : An extended assignment and set of nogoods.

$U \leftarrow \emptyset$ *// set of unfounded atoms*
loop
  repeat
    if $\delta \subseteq A$ for some $\delta \in \Delta_\Pi \cup \nabla$ **then return** (A, ∇) *// conflict*
    $\Sigma \leftarrow \{\delta \in \Delta_\Pi \cup \nabla \mid (\delta \setminus A) = \{\sigma\}, \overline{\sigma} \notin A\}$ *// unit-resulting nogoods*
    if $\Sigma \neq \emptyset$ **then**
      let $\sigma \in (\delta \setminus A)$ for some $\delta \in \Sigma$ **in**
        $A \leftarrow A \circ (\overline{\sigma})$ *//* $dl(\overline{\sigma}) = max(\{dl(\rho) \mid \rho \in (\delta \setminus \{\sigma\})\} \cup \{0\})$
  until $\Sigma = \emptyset$
  if Π is tight **then return** (A, ∇) *// no unfounded set* $\emptyset \subset U \subseteq (atom(\Pi) \setminus A^{\mathbf{F}})$
  else
    $U \leftarrow (U \setminus A^{\mathbf{F}})$
    if $U = \emptyset$ **then** $U \leftarrow \textsc{UnfoundedSet}(\Pi, A)$
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    let $p \in U$ **in**
      $\nabla \leftarrow \nabla \cup \{\lambda(p, U)\}$ *// record unit-resulting or violated loop nogood*

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- Implementations of UNFOUNDEDSET must guarantee the following for a result U:
  1. $U \subseteq (atom(\Pi) \setminus A^{\mathbf{F}})$;
  2. $EB(U) \subseteq A^{\mathbf{F}}$;
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    Usually, the latter option is implemented in ASP solvers !

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Outline of ConflictAnalysis

- Conflict analysis is triggered whenever some nogood $\delta \in \Delta_\Pi \cup \nabla$ becomes violated, viz. $\delta \subseteq A$, at a decision level $dl > 0$.

☞ Note that all but the first literal assigned at dl have been unit-resulting for nogoods $\varepsilon \in \Delta_\Pi \cup \nabla$.

  ➥ If $\sigma \in \delta$ has been unit-resulting for ε, we obtain a new violated nogood by resolving δ and ε as follows:

$$(\delta \setminus \{\sigma\}) \cup (\varepsilon \setminus \{\overline{\sigma}\}) \; .$$

- Resolution is directed by resolving first over the literal $\sigma \in \delta$ derived last, viz. $(\delta \setminus A[\sigma]) = \{\sigma\}$.

  ☞ Iterated resolution progresses in inverse order of assignment.

- Iterated resolution stops as soon as it generates a nogood δ containing exactly one literal σ assigned at decision level dl.

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Algorithm 3: ConflictAnalysis

Input : A violated nogood δ, a logic program Π, a set ∇ of nogoods, and an assignment A.

Output : A derived nogood and a decision level.

```
loop
    let σ ∈ δ such that (δ \ A[σ]) = {σ} in
        k ← max({dl(ρ) | ρ ∈ δ \ {σ}} ∪ {0})
        if k = dl(σ) then
            let ε ∈ Δ_Π ∪ ∇ such that (ε \ A[σ]) = {σ̄} in
                δ ← (δ \ {σ}) ∪ (ε \ {σ̄})                    // resolution
        else return (δ, k)
```

Example: ConflictAnalysis

Consider

$$\Pi = \left\{ \begin{array}{llll} x \leftarrow not\ y & u \leftarrow x, y & v \leftarrow x & w \leftarrow not\ x, not\ y \\ y \leftarrow not\ x & u \leftarrow v & v \leftarrow u, y & \end{array} \right\}$$

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Remarks

- There always is a First-UIP at which conflict analysis terminates.

☞ In the worst, resolution stops at the heuristically chosen literal assigned at decision level dl.

- The nogood δ containing First-UIP σ is violated by A, viz. $\delta \subseteq A$.
- We have $k = max(\{dl(\rho) \mid \rho \in \delta \setminus \{\sigma\}\} \cup \{0\}) < dl$.
  - ➥ After recording δ in ∇ and backjumping to decision level k, $\overline{\sigma}$ is unit-resulting for δ !
  - ☞ Such a nogood δ is called asserting.

☞ Asserting nogoods direct conflict-driven search into a different region of the search space than traversed before,
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  - ☞ Such a nogood δ is called asserting.

☞ Asserting nogoods direct conflict-driven search into a different region of the search space than traversed before,
without explicitly flipping any heuristically chosen literal !

Overview

The clasp system

[40]

- Native ASP solver combining conflict-driven search with sophisticated reasoning techniques:
  - Advanced preprocessing including, e.g., equivalence reasoning
  - Lookback-based decision heuristics
  - Restart policies
  - Nogood deletion
  - Progress saving
  - Dedicated data structures for binary and ternary nogoods
  - Lazy data structures (watched literals) for long nogoods
  - Dedicated data structures for cardinality and weight constraints
  - Lazy unfounded set checking based on “source pointers”
  - Tight integration of unit propagation and unfounded set checking
  - Reasoning modes
  - ...

☞ Many of these techniques are configurable !

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  - ...

☞ Many of these techniques are configurable !

Reasoning modes of `clasp`

Beyond deciding answer set existence, `clasp` allows for:

- Optimization
- Enumeration [without solution recording]
- Projective Enumeration [without solution recording]
- Brave and Cautious Reasoning determining the
  - union or
  - intersection

  of all answer sets by computing only linearly many of them

☞ Reasoning applicable wrt answer sets as well as supported models

Front-ends also admit `clasp` to solve:

- Propositional CNF formulas
- Pseudo-Boolean formulas

Find `clasp` at: `http://potassco.sourceforge.net`

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Grounding: Overview

Overview

Non-Ground

$q(a, b).$
$q(b, a).$
$q(a, c).$
$p(X, Y) \leftarrow q(X, Y), q(Y, Z).$

Ground

$q(a, b).$ $q(b, a).$ $q(a, c).$

$p(a, a) \leftarrow q(a, a), q(a, a).$	$p(b, a) \leftarrow q(b, a), q(a, a).$	$p(c, a) \leftarrow q(c, a), q(a, a).$
$p(a, a) \leftarrow q(a, a), q(a, b).$	$p(b, a) \leftarrow q(b, a), q(a, b).$	$p(c, a) \leftarrow q(c, a), q(a, b).$
$p(a, a) \leftarrow q(a, a), q(a, c).$	$p(b, a) \leftarrow q(b, a), q(a, c).$	$p(c, a) \leftarrow q(c, a), q(a, c).$
$p(a, b) \leftarrow q(a, b), q(b, a).$	$p(b, b) \leftarrow q(b, b), q(b, a).$	$p(c, b) \leftarrow q(c, b), q(b, a).$
$p(a, b) \leftarrow q(a, b), q(b, b).$	$p(b, b) \leftarrow q(b, b), q(b, b).$	$p(c, b) \leftarrow q(c, b), q(b, b).$
$p(a, b) \leftarrow q(a, b), q(b, c).$	$p(b, b) \leftarrow q(b, b), q(b, c).$	$p(c, b) \leftarrow q(c, b), q(b, c).$
$p(a, c) \leftarrow q(a, c), q(c, a).$	$p(b, c) \leftarrow q(b, c), q(c, a).$	$p(c, c) \leftarrow q(c, c), q(c, a).$
$p(a, c) \leftarrow q(a, c), q(c, b).$	$p(b, c) \leftarrow q(b, c), q(c, b).$	$p(c, c) \leftarrow q(c, c), q(c, b).$
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☞ Only a small part of the program is relevant !

Non-Ground

$q(a, b).$
$q(b, a).$
$q(a, c).$
$p(X, Y) \leftarrow q(X, Y), q(Y, Z).$

Ground

$q(a, b).\quad q(b, a).\quad q(a, c).$

$p(a, a) \leftarrow q(a, a), q(a, a).$	$p(b, a) \leftarrow q(b, a), q(a, a).$	$p(c, a) \leftarrow q(c, a), q(a, a).$
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Non-Ground

$q(a).$
$q(f(a)).$
$p(X) \leftarrow q(X).$

Ground

$q(a).$
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$p(a) \leftarrow q(a).$
$p(f(a)) \leftarrow q(f(a)).$
$p(f(f(a))) \leftarrow q(f(f(a))).$
$p(f(f(f(a)))) \leftarrow q(f(f(f(a)))).$
$\dots$

- ☞ With functions of non-zero arity, the grounding is infinite !
- ☞ Given a logic program Π, we are interested in a subset Π' of $\mathit{ground}(\Pi)$ s.t. the answer sets of Π' and $\mathit{ground}(\Pi)$ coincide.

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Ground

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- ☞ With functions of non-zero arity, the grounding is infinite !
- ☞ Given a logic program Π, we are interested in a subset Π' of $ground(\Pi)$ s.t. the answer sets of Π' and $ground(\Pi)$ coincide.

Non-Ground

$q(f(a)).$
$p(X) \leftarrow not\ q(X).$

Ground

$q(f(a)).$
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$\dots$

- ☞ All (but one) rules are relevant !
- ☞ The answer set is infinite !
- ☞ For practical reasons, such programs should be rejected.

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Goals

- First Part: What classes of programs yield finite equivalent ground programs?
- Second Part: How to efficiently instantiate a program?

Terminology I

- Variables: $X, Y, Z, \dots$
- Functions: $a/0, f/1, g/2, \dots$ (associated with arities)
- Predicates: $p/0, q/1, r/2, \dots$ (associated with arities)
- Terms: variables or $f(t_1, \dots, t_n)$ s.t. each t_i is a term and f/n is a function
- Atoms: $p(t_1, \dots, t_n)$ s.t. each t_i is a term and p/n is a predicate
  - An atom binds all variables that occur in it.
- Literals: an atom or an atom preceded by *not*
- Ground terms (atoms, literals): terms (atoms, literals) without variables

Terminology II

- Signature σ: a pair of functions and predicates
- Herbrand universe U_σ: the set of all ground terms over functions in σ
- Herbrand base B_σ: the set of all ground atoms over predicates and functions in σ

Example

Given the signature $\sigma = (\{a/0, f/1\}, \{p/1\})$:

- $U_\sigma = \{a, f(a), f(f(a)), f(f(f(a))), \dots\}$
- $B_\sigma = \{p(a), p(f(a)), p(f(f(a))), p(f(f(f(a)))), \dots\}$

In the following, signature σ is often implicitly given by functions and predicates occurring in a logic program.

Terminology III

Let Π be a logic program with signature σ.

- Ground instances of $r \in \Pi$: Set of variable-free rules obtained by replacing all variables in r by elements from U_σ:

$$\mathit{ground}(r) = \{r\theta \mid \theta : \mathrm{vars}(r) \to U_\sigma\}$$

  where
  - $\mathrm{vars}(r)$ stands for the set of all variables occurring in r and
  - θ is a (ground) substitution.

- Ground instantiation of Π:

$$\mathit{ground}(\Pi) = \bigcup_{r \in \Pi} \mathit{ground}(r)$$

- A set $X \subseteq B_\sigma$ is an answer set of Π if $\mathit{Cn}(\mathit{ground}(\Pi)^X) = X$.

Overview

ω-Restricted Programs

Definition

Given a logic program Π:

1. A predicate p/n is a domain predicate if there is a level mapping from predicates to integers s.t., for each rule where p/n occurs in the head, all predicates in the body are domain predicates s.t. their levels are strictly smaller than that of p/n.
2. Π is ω-restricted if, for each rule, every variable occurring in the rule is bound by some atom $p(t_1, \dots, t_n)$ in the positive body s.t. p/n is a domain predicate.

☞ Every ω-restricted program has a finite equivalent ground program.

- Implementation lparse

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- Implementation lparse

Example

Example

$d^0(a)$. $d^0(b)$. $g^0(b)$.

$r^1(X) \leftarrow d^0(X), not\ g^0(X)$.

$p^1(X) \leftarrow q^2(X), d^0(X)$.

$q^2(X) \leftarrow p^1(X), r^1(X)$.

Level mapping

$d/1 \rightarrow 0$

$g/1 \rightarrow 0$

$r/1 \rightarrow 1$

$p/1 \rightarrow 1$

$q/1 \rightarrow 2$

☞ Domain predicates: $d/1, g/1, r/1$.

☞ The program is ω-restricted.

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λ-Restricted Programs

Definition

A logic program is λ-restricted if there is a level mapping from predicates to integers s.t., for each rule, every variable occurring in the rule is bound by some atom in the positive body whose predicate has a strictly smaller level than the head predicate(s).

- ☞ Every λ-restricted program has a finite equivalent ground program.
- ☞ Every ω-restricted program is also λ-restricted.

- Implementation gringo (below version 3.0.0)

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Example

Example

$d^0(a).\ \ d^0(b).\ \ g^0(b).$

$p^1(X) \leftarrow q^2(X), d^0(X).$

$q^2(X) \leftarrow p^1(X).$

$r^3 \leftarrow q^2(X), \mathit{not}\ g^0(X), \mathit{not}\ r^3.$

☞ The program is λ-restricted.

☞ The program is not ω-restricted.

Level mapping

$d/1 \rightarrow 0$

$g/1 \rightarrow 0$

$p/1 \rightarrow 1$

$q/1 \rightarrow 2$

$r/0 \rightarrow 3$

Example

Example

$d^0(a)$. $d^0(b)$. $g^0(b)$.

$p^1(X) \leftarrow q^2(X), d^0(X)$.

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Level mapping

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Safe Programs

Definition

A logic program is safe if, for each rule, every variable occurring in the rule is bound by some atom in the positive body.

- ☞ Every safe program (without functions of non-zero arity) has a finite equivalent ground program.
- ☞ Every λ-restricted program is also safe.

- Implementation dlv & gringo (from version 3.0.0)

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- Implementation dlv & gringo (from version 3.0.0)

Example

Example I

$d(a). \quad d(b). \quad g(b).$

$p(X) \leftarrow q(X).$

$q(X) \leftarrow p(X).$

$r \leftarrow q(X), not\ g(X), not\ r.$

- ☞ The program is safe.
- ☞ The program is not λ-restricted.

Example II

$p(a).$

$p(f(X)) \leftarrow p(X).$

- The grounding is infinite !

Example

Example I

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$p(X) \leftarrow q(X).$

$q(X) \leftarrow p(X).$

$r \leftarrow q(X), \mathit{not}\ g(X), \mathit{not}\ r.$

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$p(a).$

$p(f(X)) \leftarrow p(X).$

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Example

Example I

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$r \leftarrow q(X), \mathit{not}\ g(X), \mathit{not}\ r.$

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Example II

$p(a).$

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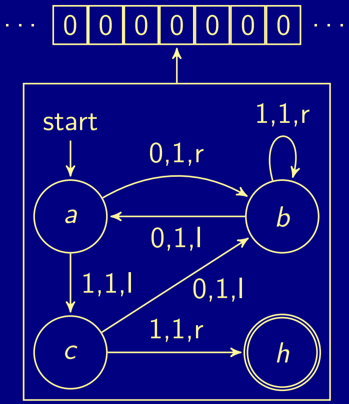
Encoding a 3-State Busy Beaver Machine




```
$ cat beaver.lp
start(a).
blank(0).
tape(n,0,n).

trans(a,0,1,b,r).
trans(a,1,1,c,l).
trans(b,0,1,a,l).
trans(b,1,1,b,r).
trans(c,0,1,b,l).
trans(c,1,1,h,r).
```

Encoding a Universal Turing Machine

```
$ cat turing.lp
conf(S,L,A,R) :- start(S), tape(L,A,R).

conf(SN,l(L,AN),AR,R) :- conf(S,L,A,r(AR,R)),
                         trans(S,A,AN,SN,r).
conf(SN,l(L,AN),AR,n) :- conf(S,L,A,n), blank(AR),
                         trans(S,A,AN,SN,r).
conf(SN,L,AL,r(AN,R)) :- conf(S,l(L,AL),A,R),
                         trans(S,A,AN,SN,l).
conf(SN,n,AL,r(AN,R)) :- conf(S,n,A,R), blank(AL),
                         trans(S,A,AN,SN,l).
```

Running the Turing Machine

```
$ gringo -t beaver.lp turing.lp
...
conf(a,n,0,n).
conf(b,l(n,1),0,n).
conf(a,n,1,r(1,n)).
...
conf(a,l(l(l(l(n,1),1),1),1),1,r(1,n)).
conf(c,l(l(l(n,1),1),1),1,r(1,r(1,n))).
conf(h,l(l(l(l(n,1),1),1),1),1,r(1,n)).
```

- Halts if Turing machine halts
- Finiteness check for safe programs is undecidable

Running the Turing Machine

```
$ gringo -t beaver.lp turing.lp
...
conf(a,n,0,n).
conf(b,l(n,1),0,n).
conf(a,n,1,r(1,n)).
...
conf(a,l(l(l(l(n,1),1),1),1),1,r(1,n)).
conf(c,l(l(l(n,1),1),1),1,r(1,r(1,n))).
conf(h,l(l(l(l(n,1),1),1),1),1,r(1,n)).
```

- Halts if Turing machine halts
- Finiteness check for safe programs is undecidable

Overview

Definition

Given a set P of atoms, a signature σ, and a domain $D \subseteq B_\sigma$:

$$\mathrm{inst}(P, D) = \{\theta : \mathrm{vars}(P) \to U_\sigma \mid A\theta \in D \text{ for all } A \in P\}$$

Algorithm: $\mathrm{instantiate}_\omega(\Pi)$

Input : An ω-restricted program Π with level mapping λ
Output : A ground program G

$X \leftarrow$ set of predicates occurring in Π
$D \leftarrow \emptyset$
$G \leftarrow \emptyset$
while $X \neq \emptyset$ **do**
  remove a predicate p/n with smallest level $\lambda(p/n)$ from X
  foreach *rule* $r \in \Pi$ *with* p/n *in the head* **do**
    $P \leftarrow \{A \in \mathrm{body}^+(r) \mid$ the predicate of A is a domain predicate$\}$
    foreach $\theta \in \mathrm{inst}(P, D)$ **do**
      $D \leftarrow D \cup \{\mathrm{head}(r)\theta\}$
      $G \leftarrow G \cup \{r\theta\}$

Definition

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Algorithm: $\mathrm{instantiate}_\omega(\Pi)$

Input : An ω-restricted program Π with level mapping λ

Output : A ground program G

```
X ← set of predicates occurring in Π
D ← ∅
G ← ∅
while X ≠ ∅ do
    remove a predicate p/n with smallest level λ(p/n) from X
    foreach rule r ∈ Π with p/n in the head do
        P ← {A ∈ body⁺(r) | the predicate of A is a domain predicate}
        foreach θ ∈ inst(P, D) do
            D ← D ∪ {head(r)θ}
            G ← G ∪ {rθ}
```

Instantiating λ-Restricted Programs

Algorithm: $\mathsf{instantiate}_\lambda(\Pi)$

Input : A λ-restricted program Π with level mapping λ
Output : A ground program G

$X \leftarrow$ set of predicates occurring in Π
$D \leftarrow \emptyset$
$G \leftarrow \emptyset$
while $X \neq \emptyset$ **do**
  remove a predicate p/n with smallest level $\lambda(p/n)$ from X
  foreach *rule* $r \in \Pi$ *with* p/n *in the head* **do**
    $P \leftarrow \{A \in \mathit{body}^+(r) \mid \lambda(p/n)$ is greater than the level of the predicate of $A\}$
    foreach $\theta \in \mathit{inst}(P, D)$ **do**
      $D \leftarrow D \cup \{\mathit{head}(r)\theta\}$
      $G \leftarrow G \cup \{r\theta\}$

- ☞ More predicates to instantiate with !
- ☞ Possibly smaller grounding.

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  foreach *rule* $r \in \Pi$ *with* p/n *in the head* **do**
    $P \leftarrow \{A \in \text{body}^+(r) \mid \lambda(p/n)$ is greater than the level of the predicate of $A\}$
    foreach $\theta \in \text{inst}(P, D)$ **do**
      $D \leftarrow D \cup \{\text{head}(r)\theta\}$
      $G \leftarrow G \cup \{r\theta\}$

- More predicates to instantiate with !
- Possibly smaller grounding.

Example

$d^0(a).\quad d^0(b).\quad g^0(b).\quad p^1(X) \leftarrow q^2(X), d^0(X).$

$q^2(X) \leftarrow p^1(X).\qquad r^3 \leftarrow q^2(X), \mathit{not}\ g^0(X), \mathit{not}\ r^3.$

p	P	$\mathrm{inst}(P, D)$	D	G
$d/1$	$\emptyset$	$\{\emptyset\}$	$\cup\{d(a)\}$	$\cup\{d(a).\}$
	$\emptyset$	$\{\emptyset\}$	$\cup\{d(b)\}$	$\cup\{d(b).\}$
$g/1$	$\emptyset$	$\{\emptyset\}$	$\cup\{g(b)\}$	$\cup\{g(b).\}$
$p/1$	$\{d(X)\}$	$\{\{X \to a\},$	$\cup\{p(a)\}$	$\cup\{p(a) \leftarrow q(a), d(a).\}$
		$\{X \to b\}\}$	$\cup\{p(b)\}$	$\cup\{p(b) \leftarrow q(b), d(b).\}$
$q/1$	$\{p(X)\}$	$\{\{X \to a\},$	$\cup\{q(a)\}$	$\cup\{q(a) \leftarrow p(a).\}$
		$\{X \to b\}\}$	$\cup\{q(b)\}$	$\cup\{q(b) \leftarrow p(b).\}$
$r/0$	$\{q(X)\}$	$\{\{X \to a\},$	$\cup\{r\}$	$\cup\{r \leftarrow q(a), \mathit{not}\ g(a), \mathit{not}\ r.\}$
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$d^0(a).\quad d^0(b).\quad g^0(b).\quad p^1(X) \leftarrow q^2(X), d^0(X).$

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$d/1$	$\emptyset$	$\{\emptyset\}$	$\cup\{d(a)\}$	$\cup\{d(a).\}$
	$\emptyset$	$\{\emptyset\}$	$\cup\{d(b)\}$	$\cup\{d(b).\}$
$g/1$	$\emptyset$	$\{\emptyset\}$	$\cup\{g(b)\}$	$\cup\{g(b).\}$
$p/1$	$\{d(X)\}$	$\{\{X \rightarrow a\},$	$\cup\{p(a)\}$	$\cup\{p(a) \leftarrow q(a), d(a).\}$
		$\{X \rightarrow b\}\}$	$\cup\{p(b)\}$	$\cup\{p(b) \leftarrow q(b), d(b).\}$
$q/1$	$\{p(X)\}$	$\{\{X \rightarrow a\},$	$\cup\{q(a)\}$	$\cup\{q(a) \leftarrow p(a).\}$
		$\{X \rightarrow b\}\}$	$\cup\{q(b)\}$	$\cup\{q(b) \leftarrow p(b).\}$
$r/0$	$\{q(X)\}$	$\{\{X \rightarrow a\},$	$\cup\{r\}$	$\cup\{r \leftarrow q(a), not\ g(a), not\ r.\}$
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$g/1$	$\emptyset$	$\{\emptyset\}$	$\cup \{g(b)\}$	$\cup \{g(b).\}$
$p/1$	$\{d(X)\}$	$\{\{X \to a\},$	$\cup \{p(a)\}$	$\cup \{p(a) \leftarrow q(a), d(a).\}$
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Instantiating Safe Programs

Algorithm: $\text{instantiate}_{\text{safe}}(\Pi)$

Input : A safe program Π
Output : A ground program G

$D \leftarrow \emptyset$
$G \leftarrow \emptyset$
repeat
  $D' \leftarrow D$
  foreach $r \in \Pi$ **do**
    $P \leftarrow \text{body}^{+}(r)$
    foreach $\theta \in \text{inst}(P, D)$ **do**
      $D \leftarrow D \cup \{\text{head}(r)\theta\}$
      $G \leftarrow G \cup \{r\theta\}$
until $D = D'$

- ☞ Possibly generates fewer rules than $\text{instantiate}_{\omega}$ and $\text{instantiate}_{\lambda}$.
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Example

$p(a, b).$

$p(b, c).$

$p(c, d).$

$p(X, Z) \leftarrow p(X, Y), p(Y, Z).$

$\text{inst}(P, D)$	D	G
$\{\emptyset\}$	$\cup \{p(a, b)\}$	$\cup \{p(a, b).\}$
$\{\emptyset\}$	$\cup \{p(b, c)\}$	$\cup \{p(b, c).\}$
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$\{\{X \to a, Y \to b, Z \to c\},$	$\cup \{p(a, c)\}$	$\cup \{p(a, c) \leftarrow p(a, b), p(b, c).\}$
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Optimizations

- Remove facts from rule bodies:
  1. $r(c)$. has already been found
  2. $p(a) \leftarrow q(b), r(c)$. is found

  ☞ Simplify ground rule to $p(a) \leftarrow q(b)$.
- Skip rules that contain false literals:
  1. $r(c)$. has already been found
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  ☞ Skip the ground rule.

☞ Allows for finitely grounding larger class of programs:
  - Consider $\Pi = \{\, p(a). \quad q(f(f(a))). \quad p(f(X)) \leftarrow p(X), not\ q(X).\}$

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  ☞ Skip the ground rule.

☞ Allows for finitely grounding larger class of programs:

- Consider $\Pi = \{\, p(a). \quad q(f(f(a))). \quad p(f(X)) \leftarrow p(X), \mathit{not}\ q(X). \}$

  ☞ $\mathit{instantiate}_{\mathsf{safe}}(\Pi)$ will terminate !

Overview

Predicate-Rule Dependency Graph

Definition

Let Π be a logic program.

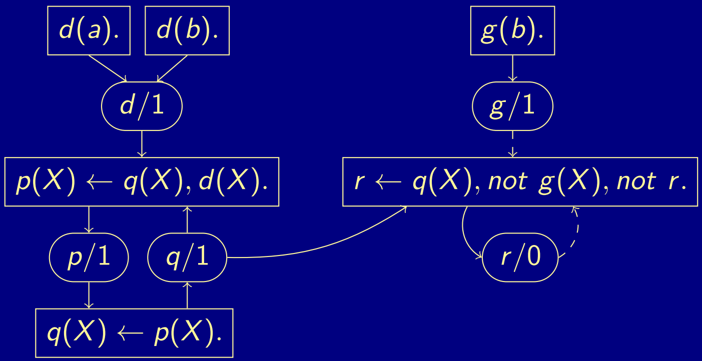
1. The predicate-rule dependency graph $G_\Pi = (V, E)$ of Π is a directed graph s.t.:
   - V is the set of predicates and rules of Π
   - $(p/n, r) \in E$ if predicate p/n occurs in the body of rule r
   - $(r, p/n) \in E$ if predicate p/n occurs in the head of rule r
2. $(p/n, r) \in E$ is negative if predicate p/n occurs in the negative body of rule r

☞ More fine-grained static program analysis.

Example

$d(a).\quad d(b).\quad g(b).\quad p(X) \leftarrow q(X), d(X).$

$q(X) \leftarrow p(X).\qquad\qquad r \leftarrow q(X), \mathit{not}\ g(X), \mathit{not}\ r.$



Strongly Connected Components I

A graph is strongly connected if all vertices pairwise reach each other via some path.

Definition

Let $G = (V, E)$ be a graph.

1. A set $C \subseteq V$ of vertices belonging to a maximal strongly connected subgraph of G is called a strongly connected component (SCC) of G.
2. An SCC A depends on an SCC B if $(B \times A) \cap E \neq \emptyset$.

☞ Dependencies among SCCs are acyclic.

☞ The SCCs of a predicate-rule dependency graph can be used to partition a logic program.

Strongly Connected Components II

Definition

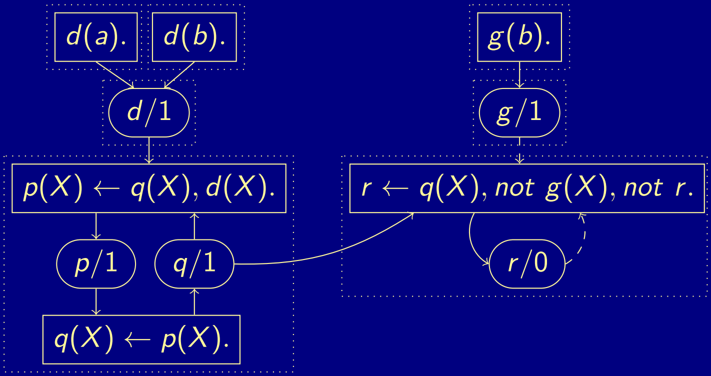
Given a logic program Π, an SCC of G_Π is

- normal if it contains a negative edge or depends on a normal SCC,
- basic if it is not normal and contains at least one edge,
- fact otherwise.

☞ A program is λ-restricted if its components are λ-restricted.

☞ Basic and fact components do not involve "choices".

☞ SCCs can be grounded in topological order.

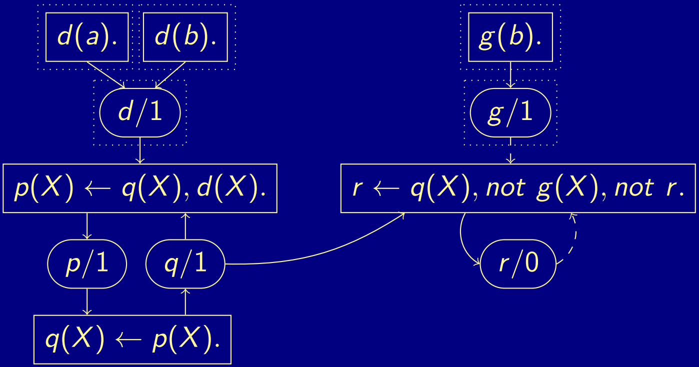


fact $\{d(a).\}$, $\{d(b).\}$, $\{g(b).\}$, $\{d/1\}$, $\{g/1\}$

basic $\{p(X) \leftarrow q(X), d(X). , q(X) \leftarrow p(X). , p/1 , q/1\}$

normal $\{r \leftarrow q(X), not\ g(X), not\ r. , r/0\}$

A topological order



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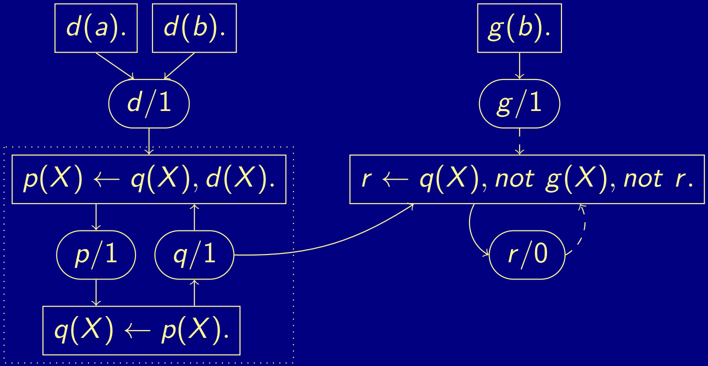

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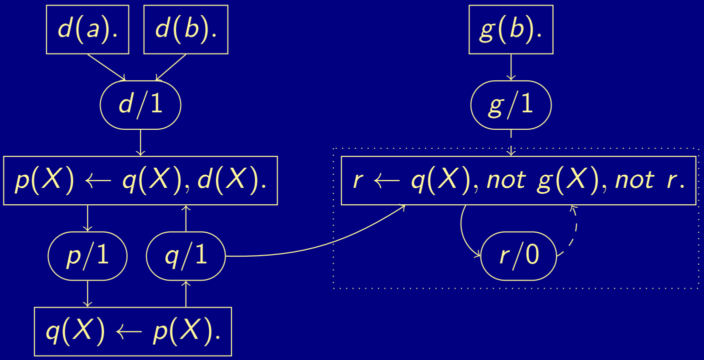

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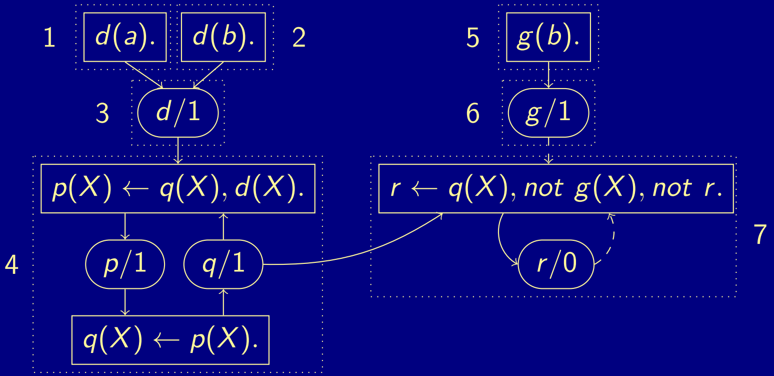

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A topological order

Overview

The Backtracking Instantiator

Definition

Given a signature σ, a substitution θ, an atom A, and a domain D:

$$\mathsf{match}(\theta, A, D) = \{\theta \cup \theta' \mid \theta' : \mathsf{vars}(A\theta) \rightarrow U_\sigma, (A\theta)\theta' \in D\}$$

Algorithm: $\mathsf{instantiate}_{\mathsf{bt}}(\theta, P)$

Input : A substitution θ and a list P of atoms
Output : Set of (ground) substitutions
Global : Domain D

if $P = []$ **then** **return** $\{\theta\}$
else
  $S \leftarrow \emptyset$
  foreach $\theta' \in \mathsf{match}(\theta, \mathsf{first}(P), D)$ **do**
    $S \leftarrow S \cup \mathsf{instantiate}_{\mathsf{bt}}(\theta', \mathsf{tail}(P))$
  return S

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```
if P = [] then return {θ}
else
    S ← ∅
    foreach θ' ∈ match(θ, first(P), D) do
        S ← S ∪ instantiate_bt(θ', tail(P))
    return S
```

Example

Example

$P = [p(X, Y), q(Y, Z), r(Z)]$

p(a,b)	q(b,c)	r(c)
p(b,a)	q(b,a)	
	q(a,c)	

- $\theta = \{X \rightarrow a, Y \rightarrow b\}$
- $S = \emptyset$

Example

Example

$P = [p(X, Y), q(Y, Z), r(Z)]$

$$\begin{array}{lll} \text{p(a,b)} \longrightarrow & \text{q(b,c)} & \text{r(c)} \\ \text{p(b,a)} & \text{q(b,a)} & \\ & \text{q(a,c)} & \end{array}$$

- $\theta = \{X \to a, Y \to b, Z \to c\}$
- $S = \emptyset$

Example

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$P = [p(X, Y), q(Y, Z), r(Z)]$

$$\mathrm{p(a,b)} \longrightarrow \mathrm{q(b,c)} \longrightarrow \mathrm{r(c)}$$

p(b,a) q(b,a)

q(a,c)

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Example

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$P = [p(X,Y), q(Y,Z), r(Z)]$

$$\begin{array}{ccc} p(a,b) & q(b,c) & r(c) \\ p(b,a) & q(b,a) & \\ & q(a,c) & \end{array}$$

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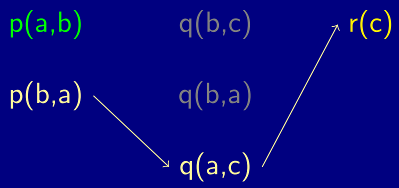

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p(a,b) q(b,c) r(c)

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q(a,c)

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Example

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p(a,b)	q(b,c)	r(c)
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- $\theta = \{\}$
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The Backjumping Instantiator

Algorithm: $\text{instantiate}_{\text{bj}}(\theta, P)$

Input : A substitution θ and a list P of atoms
Output : Set of (ground) substitutions and variables to bind
Global : Output variables O and domain D

if $P = []$ **then** **return** $(\{\theta\}, O)$
else
    $A \leftarrow \text{first}(P)$
    $M \leftarrow \text{match}(\theta, A, D)$
    if $M = \emptyset$ **then**
        return $(\emptyset, \text{vars}(A))$
    else
        $S \leftarrow \emptyset$
        $B \leftarrow \emptyset$
        foreach $\theta' \in M$ **do**
            $(S, B) \leftarrow (S, B) \sqcup \text{instantiate}_{\text{bj}}(\theta', \text{tail}(P))$
            if $\text{vars}(A\theta) \cap B = \emptyset$ **then** **return** (S, B)
        return $(S, B \cup \text{vars}(A))$

Advanced Modeling: Overview

Overview

Motivation

- Many problems can nicely be encoded using ASP
  - There are often many degrees of freedom to encode a problem
  - Even worse, different encodings may lead to drastically different solving times
  - ☞ We will try to find some hints on how to efficiently encode problems using ASP
- Some problems can, due to increased complexity, no longer be (polynomially) represented using normal logic programs
  - ☞ We will take a look on how disjunctive rules can be used to overcome this situation

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Solving a Problem Using ASP

- My (Roland) steps to solve a problem using ASP
  1. Create a small test instance
  2. Come up with a quick solution
  3. Debug this solution using the test instance
     - Use ASPViz or write some small scripts
  4. Switch to larger instances
  5. Analyze the flaws of the quick solution
     - Size of the grounding
     - Time needed to solve the problem
  6. Incrementally refine the solution
     - The quick solution serves as cross-check
  7. Throw away everything and try something different

☞ Basically it is a Trial and Error process

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Overview

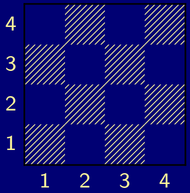
N-Queens Problem

Problem Specification

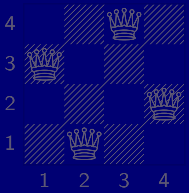
Given an $N \times N$ chessboard,
place N queens such that they do not attack each other.

$N = 4$

Chessboard



Placement



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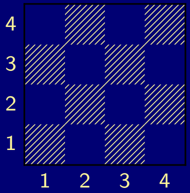

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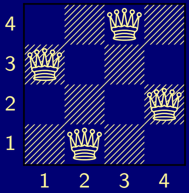

A First Encoding

1. Each square may host a queen.
2. No row, column, or diagonal hosts two queens.
3. A placement is given by instances of `queen` in an answer set.

queens_0.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- queen(X1,Y1), queen(X1,Y2), Y1 < Y2.

% DISPLAY
#hide. #show queen/2.
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Anything missing?

A First Encoding

1. Each square may host a queen.
2. No row, column, or diagonal hosts two queens.
3. A placement is given by instances of `queen` in an answer set.
4. We have to place (at least) N queens.

queens_0.lp

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% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
[...]
:- not n #count{ queen(X,Y) }.

% DISPLAY
#hide. #show queen/2.
```

A First Encoding

Let's Place 8 Queens!

```
gringo -c n=8 queens_0.lp | clasp --stats
```

```
Answer: 1
queen(1,6) queen(2,3) queen(3,1) queen(4,7)
queen(5,5) queen(6,8) queen(7,2) queen(8,4)
SATISFIABLE

Models      : 1+
Time        : 0.006s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
Choices     : 18
Conflicts   : 13
Restarts    : 0

Variables   : 793
Constraints : 729
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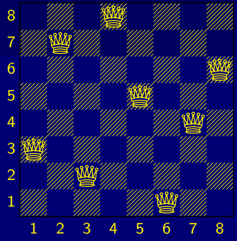

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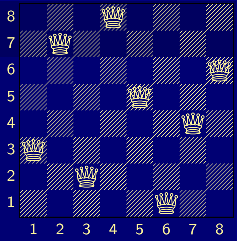


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```

A First Encoding

Let's Place 22 Queens!

```
gringo -c n=22 queens_0.lp | clasp --stats
```

```
Answer: 1
queen(1,10) queen(2,6) queen(3,16) queen(4,14) queen(5,8) ...
SATISFIABLE

Models      : 1+
Time        : 150.531s (Solving: 150.37s 1st Model: 150.34s Unsat: 0.00s)
CPU Time    : 147.480s
Choices     : 594960
Conflicts   : 574565
Restarts    : 19

Variables   : 17271
Constraints : 16787
```

A First Encoding

Let's Place 22 Queens!

```
gringo -c n=22 queens_0.lp | clasp --stats
```

```
Answer: 1
queen(1,10) queen(2,6) queen(3,16) queen(4,14) queen(5,8) ...
SATISFIABLE

Models      : 1+
Time        : 150.531s (Solving: 150.37s 1st Model: 150.34s Unsat: 0.00s)
CPU Time    : 147.480s
Choices     : 594960
Conflicts   : 574565
Restarts    : 19

Variables   : 17271
Constraints : 16787
```

A First Refinement

At least N queens? Exactly one queen per row and column!

queens_0.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- queen(X1,Y1), queen(X1,Y2), Y1 < Y2.
:- queen(X1,Y1), queen(X2,Y1), X1 < X2.
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.
:- not n #count{ queen(X,Y) }.

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

At least N queens?

Exactly one queen per row and column!

queens_0.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- queen(X1,Y1), queen(X2,Y1), X1 < X2.
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.
:- not n #count{ queen(X,Y) }.

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

At least N queens? Exactly one queen per row and column!

queens_0.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.
:- not n #count{ queen(X,Y) }.

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

At least N queens? Exactly one queen per row and column!

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.


% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Let's Place 22 Queens!

```
gringo -c n=22 queens_1.lp | clasp --stats
```

```
Answer: 1
queen(1,18) queen(2,10) queen(3,21) queen(4,3) queen(5,5) ...
SATISFIABLE

Models      : 1+
Time        : 0.113s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.020s
Choices     : 132
Conflicts   : 105
Restarts    : 1

Variables   : 7238
Constraints : 6710
```

A First Refinement

Let's Place 22 Queens!

```
gringo -c n=22 queens_1.lp | clasp --stats
```

```
Answer: 1
queen(1,18) queen(2,10) queen(3,21) queen(4,3) queen(5,5) ...
SATISFIABLE

Models      : 1+
Time        : 0.113s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.020s
Choices     : 132
Conflicts   : 105
Restarts    : 1

Variables   : 7238
Constraints : 6710
```

A First Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_1.lp | clasp --stats
```

```
Answer: 1
queen(1,24) queen(2,52) queen(3,37) queen(4,60) queen(5,76) ...
SATISFIABLE

Models      : 1+
Time        : 79.475s (Solving: 1.06s 1st Model: 1.06s Unsat: 0.00s)
CPU Time    : 6.930s
Choices     : 1373
Conflicts   : 845
Restarts    : 4

Variables   : 1211338
Constraints : 1196210
```

A First Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_1.lp | clasp --stats
```

```
Answer: 1
queen(1,24) queen(2,52) queen(3,37) queen(4,60) queen(5,76) ...
SATISFIABLE

Models      : 1+
Time        : 79.475s (Solving: 1.06s 1st Model: 1.06s Unsat: 0.00s)
CPU Time    : 6.930s
Choices     : 1373
Conflicts   : 845
Restarts    : 4

Variables   : 1211338
Constraints : 1196210
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A First Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_1.lp | clasp --stats
```

```
Answer: 1
queen(1,24) queen(2,52) queen(3,37) queen(4,60) queen(5,76) ...
SATISFIABLE

Models      : 1+
Time        : 79.475s (Solving: 1.06s 1st Model: 1.06s Unsat: 0.00s)
CPU Time    : 6.930s
Choices     : 1373
Conflicts   : 845
Restarts    : 4

Variables   : 1211338
Constraints : 1196210
```

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | clasp --stats

1241358 7402724 24950848

real 1m15.468s
user 1m15.980s
sys  0m0.090s
```

☞ Grounding makes the problem!

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | wc)

1241358 7402724 24950848

real 1m15.468s
user 1m15.980s
sys  0m0.090s
```

☞ Grounding makes the problem!

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | wc)

1241358 7402724 24950848

real 1m15.468s
user 1m15.980s
sys  0m0.090s
```

Just kidding :-)

☞ Grounding makes the problem!

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | wc)
```

```
1241358 7402724 24950848

real  1m15.468s
user  1m15.980s
sys   0m0.090s
```

☞ Grounding makes the problem!

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | wc)
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```
1241358 7402724 24950848

real 1m15.468s
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☞ Grounding makes the problem!

A First Refinement

Where Time Has Gone

```
time(gringo -c n=122 queens_1.lp | wc)
```

```
1241358 7402724 24950848

real 1m15.468s
user 1m15.980s
sys  0m0.090s
```

☞ Grounding makes the problem!

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                  O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                        O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                      O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                      O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.       O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                         O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                               O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                             O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                             O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.         O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                    O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                          O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.         O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                    O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                          O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.         O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                      O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                            O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                          O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                          O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.           O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time $\sim$ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                      O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                            O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                          O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                          O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.           O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                    O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                          O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                        O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.         O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                   O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                         O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                       O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                       O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.        O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time $\sim$ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                  O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                       O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                     O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                     O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.      O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time $\sim$ Space

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).                                   O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                         O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                       O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                       O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.        O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

A First Refinement

Grounding Time ∼ Space

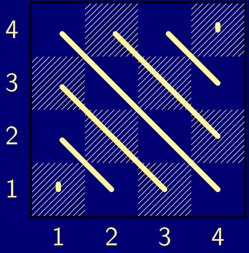
queens_1.lp

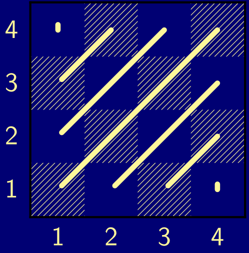
```
% DOMAIN
#const n=4. square(1..n,1..n).                                  O(n×n)

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).                        O(n×n)

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.                      O(n×n)
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.                      O(n×n)
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.   O(n^2×n^2)

% DISPLAY
#hide. #show queen/2.
```

Diagonals make trouble!

A Nomenclature for Diagonals

$N = 4$





$\#\text{diagonal}_1 = (\#\text{row} + \#\text{column}) - 1$

$\#\text{diagonal}_2 = (\#\text{row} - \#\text{column}) + N$

☞ $\#\text{diagonal}_{1/2}$ can be determined in this way for arbitrary N.

A Nomenclature for Diagonals

$N = 4$

4	**4**	**5**	**6**	**7**
3	**3**	**4**	**5**	**6**
2	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**
	1	2	3	4

$\#\text{diagonal}_1 =$
$(\#\text{row} + \#\text{column}) - 1$

4	**7**	**6**	**5**	**4**
3	**6**	**5**	**4**	**3**
2	**5**	**4**	**3**	**2**
1	**4**	**3**	**2**	**1**
	1	2	3	4

$\#\text{diagonal}_2 =$
$(\#\text{row} - \#\text{column}) + N$

☞ $\#\text{diagonal}_{1/2}$ can be determined in this way for arbitrary N.

A Nomenclature for Diagonals

$N = 4$

4	**4**	**5**	**6**	**7**
3	**3**	**4**	**5**	**6**
2	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**
	1	2	3	4

$\#\text{diagonal}_1 =$
$(\#\text{row} + \#\text{column}) - 1$

4	**7**	**6**	**5**	**4**
3	**6**	**5**	**4**	**3**
2	**5**	**4**	**3**	**2**
1	**4**	**3**	**2**	**1**
	1	2	3	4

$\#\text{diagonal}_2 =$
$(\#\text{row} - \#\text{column}) + N$

☞ $\#\text{diagonal}_{1/2}$ can be determined in this way for arbitrary N.

A Nomenclature for Diagonals

$N = 4$

4	**4**	**5**	**6**	**7**
3	**3**	**4**	**5**	**6**
2	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**
	1	2	3	4

$\#\text{diagonal}_1 =$
$(\#\text{row} + \#\text{column}) - 1$

4	**7**	**6**	**5**	**4**
3	**6**	**5**	**4**	**3**
2	**5**	**4**	**3**	**2**
1	**4**	**3**	**2**	**1**
	1	2	3	4

$\#\text{diagonal}_2 =$
$(\#\text{row} - \#\text{column}) + N$

☞ $\#\text{diagonal}_{1/2}$ can be determined in this way for arbitrary N.

A Second Refinement

Let's go for Diagonals!

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- queen(X1,Y1), queen(X2,Y2), X1 < X2, X2-X1 == |Y2-Y1|.


% DISPLAY
#hide. #show queen/2.
```

A Second Refinement

Let's go for Diagonals!

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X+Y)-1 }. % Diagonal 1


% DISPLAY
#hide. #show queen/2.
```

A Second Refinement

Let's go for Diagonals!

queens_1.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X+Y)-1 }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X-Y)+n }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Second Refinement

Let's go for Diagonals!

queens_2.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X+Y)-1 }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X-Y)+n }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Second Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,98) queen(2,54) queen(3,89) queen(4,83) queen(5,59) ...
SATISFIABLE

Models      : 1+
Time        : 2.211s (Solving: 0.13s 1st Model: 0.13s Unsat: 0.00s)
CPU Time    : 0.210s
Choices     : 11036
Conflicts   : 499
Restarts    : 3

Variables   : 16098
Constraints : 970
```

A Second Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,98) queen(2,54) queen(3,89) queen(4,83) queen(5,59) ...
SATISFIABLE

Models      : 1+
Time        : 2.211s (Solving: 0.13s 1st Model: 0.13s Unsat: 0.00s)
CPU Time    : 0.210s
Choices     : 11036
Conflicts   : 499
Restarts    : 3

Variables   : 16098
Constraints : 970
```

A Second Refinement

Let's Place 122 Queens!

```
gringo -c n=122 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,98) queen(2,54) queen(3,89) queen(4,83) queen(5,59) ...
SATISFIABLE

Models      : 1+
Time        : 2.211s (Solving: 0.13s 1st Model: 0.13s Unsat: 0.00s)
CPU Time    : 0.210s
Choices     : 11036
Conflicts   : 499
Restarts    : 3

Variables   : 16098
Constraints : 970
```

A Second Refinement

Let's Place 300 Queens!

```
gringo -c n=300 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,62) queen(2,232) queen(3,176) queen(4,241) queen(5,207) ...
SATISFIABLE

Models      : 1+
Time        : 35.450s (Solving: 6.69s 1st Model: 6.68s Unsat: 0.00s)
CPU Time    : 7.250s
Choices     : 141445
Conflicts   : 7488
Restarts    : 9

Variables   : 92994
Constraints : 2394
```

A Second Refinement

Let's Place 300 Queens!

```
gringo -c n=300 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,62) queen(2,232) queen(3,176) queen(4,241) queen(5,207) ...
SATISFIABLE

Models      : 1+
Time        : 35.450s (Solving: 6.69s 1st Model: 6.68s Unsat: 0.00s)
CPU Time    : 7.250s
Choices     : 141445
Conflicts   : 7488
Restarts    : 9

Variables   : 92994
Constraints : 2394
```

A Second Refinement

Let's Place 300 Queens!

```
gringo -c n=300 queens_2.lp | clasp --stats
```

```
Answer: 1
queen(1,62) queen(2,232) queen(3,176) queen(4,241) queen(5,207) ...
SATISFIABLE

Models      : 1+
Time        : 35.450s (Solving: 6.69s 1st Model: 6.68s Unsat: 0.00s)
CPU Time    : 7.250s
Choices     : 141445
Conflicts   : 7488
Restarts    : 9

Variables   : 92994
Constraints : 2394
```

A Third Refinement

Let's Precompute Diagonals!

queens_2.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).
diag1(X,Y,(X+Y)-1) :- square(X,Y). diag2(X,Y,(X-Y)+n) :- square(X,Y).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X+Y)-1 }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X-Y)+n }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Third Refinement

Let's Precompute Diagonals!

queens_2.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).
diag1(X,Y,(X+Y)-1) :- square(X,Y). diag2(X,Y,(X-Y)+n) :- square(X,Y).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X+Y)-1 }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : D == (X-Y)+n }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Third Refinement

Let's Precompute Diagonals!

queens_2.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).
diag1(X,Y,(X+Y)-1) :- square(X,Y). diag2(X,Y,(X-Y)+n) :- square(X,Y).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : diag1(X,Y,D) }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : diag2(X,Y,D) }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Third Refinement

Let's Precompute Diagonals!

queens_3.lp

```
% DOMAIN
#const n=4. square(1..n,1..n).
diag1(X,Y,(X+Y)-1) :- square(X,Y). diag2(X,Y,(X-Y)+n) :- square(X,Y).

% GENERATE
0 #count{ queen(X,Y) } 1 :- square(X,Y).

% TEST
:- X = 1..n, not 1 #count{ queen(X,Y) } 1.
:- Y = 1..n, not 1 #count{ queen(X,Y) } 1.
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : diag1(X,Y,D) }. % Diagonal 1
:- D = 1..2*n-1, 2 #count{ queen(X,Y) : diag2(X,Y,D) }. % Diagonal 2

% DISPLAY
#hide. #show queen/2.
```

A Third Refinement

Let's Place 300 Queens!

```
gringo -c n=300 queens_3.lp | clasp --stats
```

```
Answer: 1
queen(1,62) queen(2,232) queen(3,176) queen(4,241) queen(5,207) ...
SATISFIABLE

Models      : 1+
Time        : 8.889s (Solving: 6.61s 1st Model: 6.60s Unsat: 0.00s)
CPU Time    : 7.320s
Choices     : 141445
Conflicts   : 7488
Restarts    : 9

Variables   : 92994
Constraints : 2394
```

A Third Refinement

Let's Place 300 Queens!

```
gringo -c n=300 queens_3.lp | clasp --stats
```

```
Answer: 1
queen(1,62) queen(2,232) queen(3,176) queen(4,241) queen(5,207) ...
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Conflicts   : 7488
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Variables   : 92994
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```

```
Answer: 1
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Models      : 1+
Time        : 8.889s (Solving: 6.61s 1st Model: 6.60s Unsat: 0.00s)
CPU Time    : 7.320s
Choices     : 141445
Conflicts   : 7488
Restarts    : 9

Variables   : 92994
Constraints : 2394
```

A Third Refinement

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
```

```
Answer: 1
queen(1,477) queen(2,365) queen(3,455) queen(4,470) queen(5,237) ...
SATISFIABLE

Models      : 1+
Time        : 76.798s (Solving: 65.81s 1st Model: 65.75s Unsat: 0.00s)
CPU Time    : 68.620s
Choices     : 869379
Conflicts   : 25746
Restarts    : 12

Variables   : 365994
Constraints : 4794
```

A Third Refinement

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
```

```
Answer: 1
queen(1,477) queen(2,365) queen(3,455) queen(4,470) queen(5,237) ...
SATISFIABLE

Models      : 1+
Time        : 76.798s (Solving: 65.81s 1st Model: 65.75s Unsat: 0.00s)
CPU Time    : 68.620s
Choices     : 869379
Conflicts   : 25746
Restarts    : 12

Variables   : 365994
Constraints : 4794
```

A Case for Oracles

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
--heuristic=vsids --trans-ext=dynamic
```

```
Answer: 1
queen(1,477) queen(2,365) queen(3,455) queen(4,470) queen(5,237) ...
SATISFIABLE

Models      : 1+
Time        : 76.798s (Solving: 65.81s 1st Model: 65.75s Unsat: 0.00s)
CPU Time    : 68.620s
Choices     : 869379
Conflicts   : 25746
Restarts    : 12

Variables   : 365994
Constraints : 4794
```

A Case for Oracles

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
--heuristic=vsids --trans-ext=dynamic
```

```
Answer: 1
queen(1,477) queen(2,365) queen(3,455) queen(4,470) queen(5,237) ...
SATISFIABLE

Models      : 1+
Time        : 76.798s (Solving: 65.81s 1st Model: 65.75s Unsat: 0.00s)
CPU Time    : 68.620s
Choices     : 869379
Conflicts   : 25746
Restarts    : 12

Variables   : 365994
Constraints : 4794
```

A Case for Oracles

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
--heuristic=vsids --trans-ext=dynamic
```

```
Answer: 1
queen(1,422) queen(2,458) queen(3,224) queen(4,408) queen(5,405) ...
SATISFIABLE

Models      : 1+
Time        : 37.454s (Solving: 26.38s 1st Model: 26.26s Unsat: 0.00s)
CPU Time    : 29.580s
Choices     : 961315
Conflicts   : 3222
Restarts    : 7

Variables   : 365994
Constraints : 4794
```

A Case for Oracles

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
--heuristic=vsids --trans-ext=dynamic
```

```
Answer: 1
queen(1,422) queen(2,458) queen(3,224) queen(4,408) queen(5,405) ...
SATISFIABLE

Models      : 1+
Time        : 37.454s (Solving: 26.38s 1st Model: 26.26s Unsat: 0.00s)
CPU Time    : 29.580s
Choices     : 961315
Conflicts   : 3222
Restarts    : 7

Variables   : 365994
Constraints : 4794
```

A Case for Oracles

Let's Place 600 Queens!

```
gringo -c n=600 queens_3.lp | clasp --stats
--heuristic=vsids --trans-ext=dynamic
```

```
Answer: 1
queen(1,90) queen(2,452) queen(3,494) queen(4,145) queen(5,84) ...
SATISFIABLE

Models      : 1+
Time        : 22.654s (Solving: 10.53s 1st Model: 10.47s Unsat: 0.00s)
CPU Time    : 15.750s
Choices     : 1058729
Conflicts   : 2128
Restarts    : 6

Variables   : 403123
Constraints : 49636
```

Overview

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. check all properties explicitly ... obsolete if properties change
2. use variable-sized conjunction (via ':') ... adapts to changing facts
3. use negation of complement ... adapts to changing facts

Example: vegetables to buy

```
veg(asparagus).         veg(cucumber).
pro(asparagus,cheap).   pro(cucumber,cheap).
pro(asparagus,fresh).   pro(cucumber,fresh).
pro(asparagus,tasty).   pro(cucumber,tasty).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. check all properties explicitly . . . obsolete if properties change ✗
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Example: vegetables to buy

```
veg(asparagus).            veg(cucumber).
pro(asparagus,cheap).  pro(cucumber,cheap).
pro(asparagus,fresh).  pro(cucumber,fresh).
pro(asparagus,tasty).  pro(cucumber,tasty).


buy(X) :- veg(X), pro(X,cheap), pro(X,fresh), pro(X,tasty).
```

Implementing Universal Quantification

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Example: vegetables to buy

```
veg(asparagus).          veg(cucumber).
pro(asparagus,cheap). pro(cucumber,cheap).
pro(asparagus,fresh). pro(cucumber,fresh).
pro(asparagus,tasty). pro(cucumber,tasty).
pro(asparagus,clean).

buy(X) :- veg(X), pro(X,cheap), pro(X,fresh), pro(X,tasty), pro(X,clean).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✓
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Example: vegetables to buy

```
veg(asparagus).        veg(cucumber).
pro(asparagus,cheap).  pro(cucumber,cheap).
pro(asparagus,fresh).  pro(cucumber,fresh).
pro(asparagus,tasty).  pro(cucumber,tasty).
pro(asparagus,clean).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✓
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Example: vegetables to buy

```
veg(asparagus).         veg(cucumber).
pro(asparagus,cheap).  pro(cucumber,cheap).  pre(cheap).
pro(asparagus,fresh).  pro(cucumber,fresh).  pre(fresh).
pro(asparagus,tasty).  pro(cucumber,tasty).  pre(tasty).
pro(asparagus,clean).

buy(X) :- veg(X), pro(X,P) : pre(P).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✓
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Example: vegetables to buy

```
veg(asparagus).         veg(cucumber).
pro(asparagus,cheap).   pro(cucumber,cheap).   pre(cheap).
pro(asparagus,fresh).   pro(cucumber,fresh).   pre(fresh).
pro(asparagus,tasty).   pro(cucumber,tasty).   pre(tasty).
pro(asparagus,clean).                          pre(clean).

buy(X) :- veg(X), pro(X,P) : pre(P).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✔
3. use negation of complement ... adapts to changing facts ✔

Example: vegetables to buy

```
veg(asparagus).        veg(cucumber).
pro(asparagus,cheap).  pro(cucumber,cheap).  pre(cheap).
pro(asparagus,fresh).  pro(cucumber,fresh).  pre(fresh).
pro(asparagus,tasty).  pro(cucumber,tasty).  pre(tasty).
pro(asparagus,clean).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✔
3. use negation of complement ... adapts to changing facts ✔

Example: vegetables to buy

```
veg(asparagus).         veg(cucumber).
pro(asparagus,cheap).   pro(cucumber,cheap).  pre(cheap).
pro(asparagus,fresh).   pro(cucumber,fresh).  pre(fresh).
pro(asparagus,tasty).   pro(cucumber,tasty).  pre(tasty).
pro(asparagus,clean).

buy(X) :- veg(X), not bye(X).     bye(X) :- veg(X), pre(P), not pro(X,P).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

1. ~~check all properties explicitly~~ ... obsolete if properties change ✗
2. use variable-sized conjunction (via ':') ... adapts to changing facts ✔
3. use negation of complement ... adapts to changing facts ✔

Example: vegetables to buy

```
veg(asparagus).         veg(cucumber).
pro(asparagus,cheap).   pro(cucumber,cheap).   pre(cheap).
pro(asparagus,fresh).   pro(cucumber,fresh).   pre(fresh).
pro(asparagus,tasty).   pro(cucumber,tasty).   pre(tasty).
pro(asparagus,clean).                          pre(clean).

buy(X) :- veg(X), not bye(X).    bye(X) :- veg(X), pre(P), not pro(X,P).
```

Implementing Universal Quantification

Goal: identify objects such that ALL properties from a "list" hold

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2. use variable-sized conjunction (via ':') ... adapts to changing facts ✔
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Example: vegetables to buy

```
veg(asparagus).             veg(cucumber).
pro(asparagus,cheap).       pro(cucumber,cheap).  pre(cheap).
pro(asparagus,fresh).       pro(cucumber,fresh).  pre(fresh).
pro(asparagus,tasty).       pro(cucumber,tasty).  pre(tasty).
pro(asparagus,clean).                             pre(clean).

buy(X) :- veg(X), not bye(X).    bye(X) :- veg(X), pre(P), not pro(X,P).
```

Running Example

Latin Square

Problem Specification

Fill an $N \times N$ grid with numbers 1 to N such that each number occurs in every row and column.

$N = 4$

Grid

				1
				2
				3
				4
1	2	3	4	

Placement

1	2	3	4	1
4	1	2	3	2
3	4	1	2	3
2	3	4	1	4
1	2	3	4	

Running Example

Latin Square

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Fill an $N \times N$ grid with numbers 1 to N such that each number occurs in every row and column.

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Grid

				1
				2
				3
				4
1	2	3	4	

Placement

1	2	3	4	1
4	1	2	3	2
3	4	1	2	3
2	3	4	1	4
1	2	3	4	

Projecting Irrelevant Details Out

A Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).



% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- square(X1,Y1), N = 1..n, not num(X1,Y2,N) : square(X1,Y2).
:- square(X1,Y1), N = 1..n, not num(X2,Y1,N) : square(X2,Y1).
```

☞ unreused "singleton variables"

```
gringo latin_0.lp | wc
```

105480 2558984 14005258

```
gringo latin_1.lp | wc
```

42056 273672 1690522

Projecting Irrelevant Details Out

A Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).



% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- square(X1,Y1), N = 1..n, not num(X1,Y2,N) : square(X1,Y2).
:- square(X1,Y1), N = 1..n, not num(X2,Y1,N) : square(X2,Y1).
```

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42056 273672 1690522

Projecting Irrelevant Details Out

A Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).



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% TEST
:- square(X1,Y1), N = 1..n, not num(X1,Y2,N) : square(X1,Y2).
:- square(X1,Y1), N = 1..n, not num(X2,Y1,N) : square(X2,Y1).
```

☞ unreused "singleton variables"

```
gringo latin_0.lp | wc
```

105480 2558984 14005258

```
gringo latin_1.lp | wc
```

42056 273672 1690522

Projecting Irrelevant Details Out

A Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).
squareX(X) :- square(X,Y).      squareY(Y) :- square(X,Y).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- squareX(X1)  , N = 1..n, not num(X1,Y2,N) : square(X1,Y2).
:- squareY(Y1)  , N = 1..n, not num(X2,Y1,N) : square(X2,Y1).
```

☞ unreused "singleton variables"

gringo latin_0.lp | wc	gringo latin_1.lp | wc
105480 2558984 14005258	42056 273672 1690522

Projecting Irrelevant Details Out

A Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).
squareX(X) :- square(X,Y).      squareY(Y) :- square(X,Y).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- squareX(X1)  , N = 1..n, not num(X1,Y2,N) : square(X1,Y2).
:- squareY(Y1)  , N = 1..n, not num(X2,Y1,N) : square(X2,Y1).
```

☞ unreused "singleton variables"

`gringo latin_0.lp | wc`

105480 2558984 14005258

`gringo latin_1.lp | wc`

42056 273672 1690522

Unraveling Symmetric Inequalities

Another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 != Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 != X2.
```

☞ duplicate ground rules (swapping `Y1`/`Y2` and `X1`/`X2` gives the "same")

```
gringo latin_2.lp | wc
```

2071560 12389384 40906946

```
gringo latin_3.lp | wc
```

1055752 6294536 21099558

Unraveling Symmetric Inequalities

Another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 != Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 != X2.
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```
gringo latin_2.lp | wc
```

2071560 12389384 40906946

```
gringo latin_3.lp | wc
```

1055752 6294536 21099558

Unraveling Symmetric Inequalities

Another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

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:- num(X1,Y1,N), num(X1,Y2,N), Y1 != Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 != X2.
```

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```
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```

2071560 12389384 40906946

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gringo latin_3.lp | wc
```

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Unraveling Symmetric Inequalities

Another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 <  Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 <  X2.
```

☞ duplicate ground rules (swapping Y1/Y2 and X1/X2 gives the "same")

```
gringo latin_2.lp | wc
```

2071560 12389384 40906946

```
gringo latin_3.lp | wc
```

1055752 6294536 21099558

Unraveling Symmetric Inequalities

Another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 <  Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 <  X2.
```

☞ duplicate ground rules (swapping Y1/Y2 and X1/X2 gives the "same")

```
gringo latin_2.lp | wc
```

2071560 12389384 40906946

```
gringo latin_3.lp | wc
```

1055752 6294536 21099558

Linearizing Existence Tests

Still another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 < Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 < X2.
```

☞ uniqueness of `N` in a row/column checked by ENUMERATING PAIRS!

```
gringo latin_3.lp | wc
```

```
1055752 6294536 21099558
```

```
gringo latin_4.lp | wc
```

```
228360 1205256 4780744
```

Linearizing Existence Tests

Still another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 < Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 < X2.
```

☞ uniqueness of `N` in a row/column checked by ENUMERATING PAIRS!

```
gringo latin_3.lp | wc
```

1055752 6294536 21099558

```
gringo latin_4.lp | wc
```

228360 1205256 4780744

Linearizing Existence Tests

Still another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- num(X1,Y1,N), num(X1,Y2,N), Y1 < Y2.
:- num(X1,Y1,N), num(X2,Y1,N), X1 < X2.
```

☞ uniqueness of `N` in a row/column checked by ENUMERATING PAIRS!

```
gringo latin_3.lp | wc
```

```
1055752 6294536 21099558
```

```
gringo latin_4.lp | wc
```

```
228360 1205256 4780744
```

Linearizing Existence Tests

Still another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% DEFINE + TEST
gtX(X-1,Y,N) :- num(X,Y,N), 1 < X.      gtY(X,Y-1,N) :- num(X,Y,N), 1 < Y.
gtX(X-1,Y,N) :- gtX(X,Y,N), 1 < X.      gtY(X,Y-1,N) :- gtY(X,Y,N), 1 < Y.
 :- num(X,Y,N), gtX(X,Y,N).              :- num(X,Y,N), gtY(X,Y,N).
```

☞ uniqueness of N in a row/column checked by ENUMERATING PAIRS!

gringo latin_3.lp | wc	gringo latin_4.lp | wc
1055752 6294536 21099558	228360 1205256 4780744

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gringo latin_3.lp | wc	gringo latin_4.lp | wc
1055752 6294536 21099558	228360 1205256 4780744

Assigning Aggregate Values

Yet another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).
sigma(S) :- S = #sum[ square(X,n) = X ].

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% DEFINE + TEST
occX(X,N,C) :- X = 1..n, N = 1..n, C = #count{ num(X,Y,N) }.
occY(Y,N,C) :- Y = 1..n, N = 1..n, C = #count{ num(X,Y,N) }.
:- occX(X,N,C), C != 1.  :- occY(Y,N,C), C != 1.

% DISPLAY
#hide. #show num/3. #show sigma/1.
```

```
gringo latin_5.lp | wc          gringo latin_6.lp | wc
```

Assigning Aggregate Values

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% DOMAIN
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```
gringo latin_5.lp | wc          gringo latin_6.lp | wc
```

Assigning Aggregate Values

Yet another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).
sigma(S) :- S = #sum[ square(X,n) = X ].                    ✔

% GENERATE
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% DEFINE + TEST
occX(X,N,C) :- X = 1..n, N = 1..n, C = #count{ num(X,Y,N) }.
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% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% DEFINE + TEST
occX(X,N,C) :- X = 1..n, N = 1..n, C #count{ num(X,Y,N) } C, C = 0..n.
occY(Y,N,C) :- Y = 1..n, N = 1..n, C #count{ num(X,Y,N) } C, C = 0..n.
:- occX(X,N,C), C != 1.  :- occY(Y,N,C), C != 1.

% DISPLAY
#hide. #show num/3. #show sigma/1.
```

☞ internal transformation by gringo

Assigning Aggregate Values

Yet another Latin square encoding

```
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#const n=32. square(1..n,1..n).
sigma(S) :- S = #sum[ square(X,n) = X ].

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% DEFINE + TEST
occX(X,N,C) :- X = 1..n, N = 1..n, C = #count{ num(X,Y,N) }.      ✗
occY(Y,N,C) :- Y = 1..n, N = 1..n, C = #count{ num(X,Y,N) }.      ✗
:- occX(X,N,C), C != 1.  :- occY(Y,N,C), C != 1.

% DISPLAY
#hide. #show num/3. #show sigma/1.
```

```
gringo latin_5.lp | wc
```

```
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Assigning Aggregate Values

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% DISPLAY
#hide. #show num/3.
```

```
gringo latin_5.lp | wc
```

```
gringo latin_6.lp | wc
```

48136 373768 2185042

Assigning Aggregate Values

Yet another Latin square encoding

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#const n=32. square(1..n,1..n).

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:- occX(X,N,C), C != 1.  :- occY(Y,N,C), C != 1.

% DISPLAY
#hide. #show num/3.
```

```
gringo latin_5.lp | wc
```

304136 5778440 30252505

```
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```

48136 373768 2185042

Assigning Aggregate Values

Yet another Latin square encoding

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.


% DISPLAY
#hide. #show num/3.
```

`gringo latin_5.lp | wc`

304136 5778440 30252505

`gringo latin_6.lp | wc`

Assigning Aggregate Values

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Breaking Symmetries

The ultimate Latin square encoding?

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#hide. #show num/3.
```

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% DISPLAY
#hide. #show num/3.
```

☞ many symmetric solutions (mirroring, rotation, value permutation)

Breaking Symmetries

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% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.


% DISPLAY
#hide. #show num/3.
```

☞ easy and safe to fix a full row/column!

Breaking Symmetries

The ultimate Latin square encoding?

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- square(1,Y), not num(1,Y,Y). % Symmetry Breaking

% DISPLAY
#hide. #show num/3.
```

☞ easy and safe to fix a full row/column!

Breaking Symmetries

The ultimate Latin square encoding?

```
% DOMAIN
#const n=32. square(1..n,1..n).

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1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

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:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- square(1,Y), not num(1,Y,Y). % Symmetry Breaking

% DISPLAY
#hide. #show num/3.
```

☞ Let's compare enumeration speed!

Breaking Symmetries

The ultimate Latin square encoding?

```
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1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.


% DISPLAY
#hide. #show num/3.
```

```
gringo -c n=5 latin_6.lp | clasp -q 0
```

Breaking Symmetries

The ultimate Latin square encoding?

```
% DOMAIN
#const n=32. square(1..n,1..n).

% GENERATE
1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

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:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
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% DISPLAY
#hide. #show num/3.
```

```
gringo -c n=5 latin_6.lp | clasp -q 0
```

```
Models : 161280     Time : 2.078s
```

Breaking Symmetries

The ultimate Latin square encoding?

```
% DOMAIN
#const n=32. square(1..n,1..n).

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1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

% TEST
:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- square(1,Y), not num(1,Y,Y). % Symmetry Breaking

% DISPLAY
#hide. #show num/3.
```

```
gringo -c n=5 latin_7.lp | clasp -q 0
```

```
Models : 161280     Time : 2.078s
```

Breaking Symmetries

The ultimate Latin square encoding?

```
% DOMAIN
#const n=32. square(1..n,1..n).

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1 #count{ num(X,Y,N) : N = 1..n } 1 :- square(X,Y).

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:- X = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- Y = 1..n, N = 1..n, not 1 #count{ num(X,Y,N) } 1.
:- square(1,Y), not num(1,Y,Y). % Symmetry Breaking

% DISPLAY
#hide. #show num/3.
```

```
gringo -c n=5 latin_7.lp | clasp -q 0
```

```
Models : 1344      Time : 0.024s
```

Encode With Care!

1. Create a working encoding
   - Q1: Do you need to modify the encoding if the facts change?
   - Q2: Are all variables significant (or statically functionally dependent)?
   - Q3: Can there be (many) identic ground rules?
   - Q4: Do you enumerate pairs of values (to test uniqueness)?
   - Q5: Do you assign dynamic aggregate values (to check a fixed bound)?
   - Q6: Do you admit (obvious) symmetric solutions?
   - Q7: Do you have additional domain knowledge simplifying the problem?
   - Q8: Are you aware of anything else that, if encoded, would reduce grounding and/or solving efforts?
2. Revise until no "Yes" answer!
   - ☞ If the format of facts makes encoding painful (for instance, abusing grounding for "scientific calculations"), revise the fact format as well.

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Some Hints on (Preventing) Debugging

Kinds of errors

- syntactic . . . follow error messages by the grounder
- semantic . . . (most likely) encoding/intention mismatch

Ways to identify semantic errors (early)

- develop and test incrementally
  - prepare toy instances with "interesting features"
  - build the encoding bottom-up and verify additions (eg. new predicates)
- compare the encoded to the intended meaning
  - check whether the grounding fits (use `gringo -t`)
  - if answer sets are unintended, investigate conditions that fail to hold
  - if answer sets are missing, examine integrity constraints (add heads)
- ask tools (eg. `http://www.kr.tuwien.ac.at/research/projects/mmdasp/`)

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Overcoming Performance Bottlenecks

Grounding

- monitor time spent by and output size of `gringo`
  1. system tools (eg. `time(gringo [...] | wc)`)
  2. profiling info (eg. `gringo --gstats --verbose=3 [...] > /dev/null`)

☞ once identified, reformulate "critical" logic program parts

Solving

- check solving statistics (use `clasp --stats`)

☞ if great search efforts (`Conflicts`/`Choices`/`Restarts`), then
  1. try auto-configuration (offered by `claspfolio`)
  2. try manual fine-tuning (requires expert knowledge!)
  3. if possible, reformulate the problem or add domain knowledge ("redundant" constraints) to help the solver

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  1. try auto-configuration (offered by `claspfolio`)
  2. try manual fine-tuning (requires expert knowledge!)
  3. if possible, reformulate the problem or add domain knowledge ("redundant" constraints) to help the solver

Overcoming Performance Bottlenecks

Grounding

- monitor time spent by and output size of `gringo`
  1. system tools (eg. `time(gringo [...] | wc)`)
  2. profiling info (eg. `gringo --gstats --verbose=3 [...] > /dev/null`)

☞ once identified, reformulate "critical" logic program parts

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Overview

Hitori

A Japanese Grid Puzzle (Beyond Sudoku)

The Puzzle

Given: an $N \times N$ board of numbered squares

4	8	1	6	3	2	5	7
3	6	7	2	1	6	5	4
2	3	4	8	2	8	6	1
4	1	6	5	7	7	3	5
7	2	3	1	8	5	1	2
3	5	6	7	3	1	8	4
6	4	2	3	5	4	7	8
8	7	1	4	2	3	5	6

Wanted: a set of black squares such that

1. no black squares are horizontally or vertically adjacent
2. numbers of white squares are unique for each row and column
3. every pair of white squares is connected via a path (not passing black squares)

	8		6	3	2		7
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Fact and Solution Format

Facts provide instances of `state(X,Y,N)` to express that the square in column `X` and row `Y` contains number `N`.

Example Instance

```
state(1,1,4). state(2,1,8). ... state(8,1,7).
state(1,2,3). state(2,2,6). ... state(8,2,4).
state(1,3,2). state(2,3,3). ... state(8,3,1).
state(1,4,4). state(2,4,1). ... state(8,4,5).
state(1,5,7). state(2,5,2). ... state(8,5,2).
state(1,6,3). state(2,6,5). ... state(8,6,4).
state(1,7,6). state(2,7,4). ... state(8,7,8).
state(1,8,8). state(2,8,7). ... state(8,8,6).
```

4	8	1	6	3	2	5	7
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Example Solution

Black squares given by instances of `blackOut(X,Y)`:

```
blackOut(1,1)  blackOut(2,5)  ...
blackOut(1,3)  ...   blackOut(8,4)
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```

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state(1,5,7). state(2,5,2). ... state(8,5,2).
state(1,6,3). state(2,6,5). ... state(8,6,4).
state(1,7,6). state(2,7,4). ... state(8,7,8).
state(1,8,8). state(2,8,7). ... state(8,8,6).
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A Working Encoding I

Found on the WWW (and Adapted to gringo Syntax)

hitori_0.lp (under GNU GPL: COPYING)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (A) Adjacent grid locations %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Domain predicate (evaluated upon grounding)
adjacent(X,Y,X+1,Y) :- state(X,Y,_), state(X+1,Y,_).
adjacent(X,Y,X,Y+1) :- state(X,Y,_), state(X,Y+1,_).
adjacent(X2,Y2,X1,Y1) :- adjacent(X1,Y1,X2,Y2).

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (B) Generate solution candidate %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Every square is blacked out or normal
1 { blackOut(X,Y), -blackOut(X,Y) } 1 :- state(X,Y,_).
```

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```

A Working Encoding II

Found on the WWW (and Adapted to gringo Syntax)

hitori_0.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.1) Test eliminating adjacent blanks %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Can't have adjacent black squares
 :- adjacent(X1,Y1,X2,Y2), blackOut(X1,Y1), blackOut(X2,Y2).

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.2) Tests eliminating number recurrences %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Can't have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), -blackOut(X1,Y), -blackOut(X2,Y), X1 != X2.

% Can't have the same number twice in the same column
 :- state(X,Y1,N), state(X,Y2,N), -blackOut(X,Y1), -blackOut(X,Y2), Y1 != Y2.
```

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A Working Encoding II

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 :- state(X,Y1,N), state(X,Y2,N), -blackOut(X,Y1), -blackOut(X,Y2), Y1 != Y2.
```

Already spot something?

A Working Encoding III

Found on the WWW (and Adapted to gringo Syntax)

hitori_0.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.3) Test eliminating disconnected numbers %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), -blackOut(X1,Y1),
     -blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).

% Can't have mutually unreachable non-black squares
 :- -blackOut(X1,Y1), -blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
     (X1,Y1) != (X2,Y2).
```

☞ Answer sets (of hitori_0.lp plus instance) match Hitori solutions. ✔

A Working Encoding III

Found on the WWW (and Adapted to gringo Syntax)

hitori_0.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.3) Test eliminating disconnected numbers %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), -blackOut(X1,Y1),
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A Working Encoding

Let's Run it!

```
gringo hitori_0.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 13.485s (Solving: 11.77s 1st Model: 11.77s Unsat: 0.00s)
CPU Time    : 13.290s
Choices     : 458
Conflicts   : 323
Restarts    : 2

Variables   : 260625
Constraints : 1018953
```

4	8	1	6	3	2	5	7
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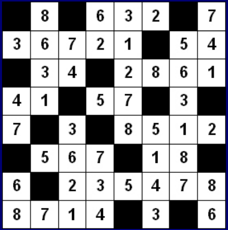

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Why Classical Negation?

hitori_0.lp

```
% Every square is blacked out or normal
1 { blackOut(X,Y), -blackOut(X,Y) } 1 :- state(X,Y,_).

:- blackOut(X,Y), -blackOut(X,Y).
```

☞ no internal transformation by gringo

```
gringo hitori_0.lp instance.lp | wc
```

267534 1608172 5535208

```
gringo hitori_1.lp instance.lp | wc
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267470 1607788 5534184

☞ no noticeable effect on grounding/solving performance

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☞ `blackOut(X,Y)` and `-blackOut(X,Y)` exclusive in view of upper bound!

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gringo hitori_0.lp instance.lp | wc
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gringo hitori_1.lp instance.lp | wc
```

267470 1607788 5534184

☞ no noticeable effect on grounding/solving performance

Why Classical Negation?

hitori_1.lp

```
% Every square is blacked out or normal
1 { blackOut(X,Y), negBlackOut(X,Y) } 1 :- state(X,Y,_).

:- blackOut(X,Y), -blackOut(X,Y).
```

☞ no internal transformation by gringo

```
gringo hitori_0.lp instance.lp | wc
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267534 1608172 5535208

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Why Not Default Negation?

hitori_1.lp

```
% Every square is blacked out or normal
1 { blackOut(X,Y), negBlackOut(X,Y) } 1 :- state(X,Y,_).

% Can't have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), negBlackOut(X1,Y), negBlackOut(X2,Y), X1 != X2.
 ...
```

☞ `blackOut(X,Y)` and `negBlackOut(X,Y)` are two sides of the same coin

Why Not Default Negation?

hitori_1.lp

```
% Every square is blacked out or normal
1 { blackOut(X,Y), negBlackOut(X,Y) } 1 :- state(X,Y,_).

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Why Not Default Negation?

hitori_2.lp

```
% Every square is blacked out or normal
 { blackOut(X,Y) } :- state(X,Y,_).

% Can't have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), not blackOut(X1,Y), not blackOut(X2,Y), X1 != X2.
 ...
```

☞ replace `negBlackOut(X,Y)` by "`not blackOut(X,Y)`"

A First Improvement

```
gringo hitori_1.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 13.485s (Solving: 11.77s 1st Model: 11.77s Unsat: 0.00s)
CPU Time    : 13.290s
Choices     : 458
Conflicts   : 323
Restarts    : 2

Variables   : 260625
Constraints : 1018953
```

A First Improvement

```
gringo hitori_2.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 13.485s (Solving: 11.77s 1st Model: 11.77s Unsat: 0.00s)
CPU Time    : 13.290s
Choices     : 458
Conflicts   : 323
Restarts    : 2

Variables   : 260625
Constraints : 1018953
```

A First Improvement

```
gringo hitori_2.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 10.177s (Solving: 8.42s 1st Model: 8.41s Unsat: 0.00s)
CPU Time    : 9.990s
Choices     : 344
Conflicts   : 264
Restarts    : 2

Variables   : 260433
Constraints : 1018825
```

Remember Symmetric Inequalities

hitori_2.lp

```
% Can’t have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), not blackOut(X1,Y), not blackOut(X2,Y), X1 != X2.

% Can’t have the same number twice in the same column
 :- state(X,Y1,N), state(X,Y2,N), not blackOut(X,Y1), not blackOut(X,Y2), Y1 != Y2.
```

☞ no noticeable effect on grounding/solving performance

Remember Symmetric Inequalities

hitori_3.lp

```
% Can't have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), not blackOut(X1,Y), not blackOut(X2,Y), X1 < X2.

% Can't have the same number twice in the same column
 :- state(X,Y1,N), state(X,Y2,N), not blackOut(X,Y1), not blackOut(X,Y2), Y1 < Y2.
```

☞ no noticeable effect on grounding/solving performance

Remember Symmetric Inequalities

hitori_3.lp

```
% Can’t have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), not blackOut(X1,Y), not blackOut(X2,Y), X1 < X2.

% Can’t have the same number twice in the same column
 :- state(X,Y1,N), state(X,Y2,N), not blackOut(X,Y1), not blackOut(X,Y2), Y1 < Y2.
```

☞ no noticeable effect on grounding/solving performance

Let's Use Counting

hitori_3.lp

```
% Can't have the same number twice in the same row
 :- state(X1,Y,N), state(X2,Y,N), not blackOut(X1,Y), not blackOut(X2,Y), X1 < X2.

% Can't have the same number twice in the same column
 :- state(X,Y1,N), state(X,Y2,N), not blackOut(X,Y1), not blackOut(X,Y2), Y1 < Y2.
```

Let's Use Counting

hitori_4.lp

```
% Can't have the same number twice in the same row or column
 :- state(X1,Y1,N), 2 { not blackOut(X1,Y2) : state(X1,Y2,N) }.
 :- state(X1,Y1,N), 2 { not blackOut(X2,Y1) : state(X2,Y1,N) }.
```

A Second Improvement?

```
gringo hitori_3.lp instance.lp | clasp --stats

Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 10.182s (Solving: 8.47s 1st Model: 8.47s Unsat: 0.00s)
CPU Time    : 10.010s
Choices     : 344
Conflicts   : 264
Restarts    : 2

Variables   : 260433
Constraints : 1018825
```

A Second Improvement?

```
gringo hitori_4.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 10.182s (Solving: 8.47s 1st Model: 8.47s Unsat: 0.00s)
CPU Time    : 10.010s
Choices     : 344
Conflicts   : 264
Restarts    : 2

Variables   : 260433
Constraints : 1018825
```

A Second Improvement?

```
gringo hitori_4.lp instance.lp | clasp --stats

Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 9.781s (Solving: 7.99s 1st Model: 7.99s Unsat: 0.00s)
CPU Time    : 9.610s
Choices     : 278
Conflicts   : 227
Restarts    : 1

Variables   : 260432
Constraints : 1018828
```

Why Double-Check Reachability?

hitori_4.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).




% Can't have mutually unreachable non-black squares
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) != (X2,Y2),  state(X1,Y1,_), state(X2,Y2,_).
```

☞ `reachable(X1,Y1,X2,Y2)` and `reachable(X2,Y2,X1,Y1)` hold jointly

Why Double-Check Reachability?

hitori_4.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).





% Can't have mutually unreachable non-black squares
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) != (X2,Y2),  state(X1,Y1,_), state(X2,Y2,_).
```

☞ `reachable(X1,Y1,X2,Y2)` and `reachable(X2,Y2,X1,Y1)` hold jointly

Why Double-Check Reachability?

hitori_4.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X3,Y3,X2,Y2),
                          (X1,Y1) < (X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3).

% Can't have mutually unreachable non-black squares
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) != (X2,Y2),  state(X1,Y1,_), state(X2,Y2,_).
```

☞ enforce `(X1,Y1) < (X2,Y2)` for instances of `reachable(X1,Y1,X2,Y2)`

Why Double-Check Reachability?

hitori_4.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X3,Y3,X2,Y2),
                          (X1,Y1) < (X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3).

% Can't have mutually unreachable non-black squares
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) != (X2,Y2),  state(X1,Y1,_), state(X2,Y2,_).
```

☞ enforce `(X1,Y1) < (X2,Y2)` for instances of `reachable(X1,Y1,X2,Y2)`

Why Double-Check Reachability?

hitori_5.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X3,Y3,X2,Y2),
                          (X1,Y1) < (X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3).

% Can't have mutually unreachable non-black squares
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) <  (X2,Y2),  state(X1,Y1,_), state(X2,Y2,_).
```

☞ enforce `(X1,Y1) < (X2,Y2)` for instances of `reachable(X1,Y1,X2,Y2)`

A Real Breakthrough?

```
gringo hitori_4.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 9.781s (Solving: 7.99s 1st Model: 7.99s Unsat: 0.00s)
CPU Time    : 9.610s
Choices     : 278
Conflicts   : 227
Restarts    : 1

Variables   : 260432
Constraints : 1018828
```

A Real Breakthrough?

```
gringo hitori_5.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 9.781s (Solving: 7.99s 1st Model: 7.99s Unsat: 0.00s)
CPU Time    : 9.610s
Choices     : 278
Conflicts   : 227
Restarts    : 1

Variables   : 260432
Constraints : 1018828
```

A Real Breakthrough?

```
gringo hitori_5.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 4.054s (Solving: 3.07s 1st Model: 3.07s Unsat: 0.00s)
CPU Time    : 3.810s
Choices     : 438
Conflicts   : 318
Restarts    : 2

Variables   : 129328
Constraints : 504573
```

Two Orders of Magnitude!

hitori_5.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X3,Y3,X2,Y2),
                          (X1,Y1) < (X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3).
```

☞ grounding size: $O(8^6)$

Two Orders of Magnitude!

hitori_5.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X2,Y2,X3,Y3).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X3,Y3,X2,Y2),
                          (X1,Y1) < (X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), reachable(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3).
```

☞ grounding size: $O(8^6)$

Two Orders of Magnitude!

hitori_6.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).

reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X2,Y2,X3,Y3),
                          (X1,Y1) < (X3,Y3), not blackOut(X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3), not blackOut(X3,Y3).
```

☞ grounding size: $O(8^6)$

Two Orders of Magnitude!

hitori_6.lp

```
% Define mutual reachability
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).

reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X2,Y2,X3,Y3),
                          (X1,Y1) < (X3,Y3), not blackOut(X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3), not blackOut(X3,Y3).
```

☞ grounding size: $O(8^4)$

A First Breakthrough

```
gringo hitori_5.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models       : 1+
Time         : 4.054s (Solving: 3.07s 1st Model: 3.07s Unsat: 0.00s)
CPU Time     : 3.810s
Choices      : 438
Conflicts    : 318
Restarts     : 2

Variables    : 129328
Constraints  : 504573
```

A First Breakthrough

```
gringo hitori_6.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 4.054s (Solving: 3.07s 1st Model: 3.07s Unsat: 0.00s)
CPU Time    : 3.810s
Choices     : 438
Conflicts   : 318
Restarts    : 2

Variables   : 129328
Constraints : 504573
```

A First Breakthrough

```
gringo hitori_6.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.093s (Solving: 0.01s 1st Model: 0.01s Unsat: 0.00s)
CPU Time    : 0.040s
Choices     : 64
Conflicts   : 23
Restarts    : 0

Variables   : 11231
Constraints : 32234
```

Let's Think a Bit More

hitori_6.lp

```
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X2,Y2,X3,Y3),
                          (X1,Y1) < (X3,Y3), not blackOut(X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3), not blackOut(X3,Y3).

% Can't have unreachable non-black square
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) < (X2,Y2),   state(X1,Y1,_), state(X2,Y2,_).
```

	8		6	3	2		7
3	6	7	2	1		5	4
	3	4		2	8	6	1
4	1		5	7		3	
7		3		8	5	1	2
	5	6	7		1	8	
6		2	3	5	4	7	8
8	7	1	4		3		6

Q: How many squares adjacent to `(1,1)` can possibly be black?

A:

Let's Think a Bit More

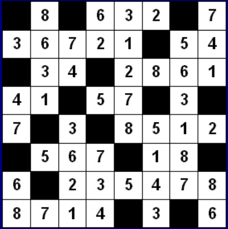
hitori_6.lp

```
reachable(X1,Y1,X2,Y2) :- adjacent(X1,Y1,X2,Y2), (X1,Y1) < (X2,Y2),
                          not blackOut(X1,Y1), not blackOut(X2,Y2).
reachable(X1,Y1,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X2,Y2,X3,Y3),
                          (X1,Y1) < (X3,Y3), not blackOut(X3,Y3).
reachable(X2,Y2,X3,Y3) :- reachable(X1,Y1,X2,Y2), adjacent(X1,Y1,X3,Y3),
                          (X2,Y2) < (X3,Y3), not blackOut(X3,Y3).

% Can't have unreachable non-black square
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) < (X2,Y2),   state(X1,Y1,_), state(X2,Y2,_).
```

Q: How many squares adjacent to (1,1) can possibly be black?

A: At most one!



Let's Think a Bit More

hitori_6.lp

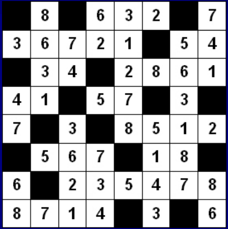
```
reachable(1,1).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2), not blackOut(X2,Y2).




% Can't have unreachable non-black square
 :- not blackOut(X1,Y1), not blackOut(X2,Y2), not reachable(X1,Y1,X2,Y2),
    (X1,Y1) < (X2,Y2),   state(X1,Y1,_), state(X2,Y2,_).
```

Q: How many squares adjacent to `(1,1)` can possibly be black?

A: At most one!



Let's Think a Bit More

hitori_7.lp

```
reachable(1,1).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2), not blackOut(X2,Y2).
```

```
% Can't have unreachable non-black square
 :- state(X,Y,_), not blackOut(X,Y), not reachable(X,Y).
```

Q: How many squares adjacent to `(1,1)` can possibly be black?

A: At most one!

	8		6	3	2		7
3	6	7	2	1		5	4
	3	4		2	8	6	1
4	1		5	7		3	
7		3		8	5	1	2
	5	6	7		1	8	
6		2	3	5	4	7	8
8	7	1	4		3		6

Not That Much Left to Save

```
gringo hitori_6.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.093s (Solving: 0.01s 1st Model: 0.01s Unsat: 0.00s)
CPU Time    : 0.040s
Choices     : 64
Conflicts   : 23
Restarts    : 0

Variables   : 11231
Constraints : 32234
```

Not That Much Left to Save

```
gringo hitori_7.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.093s (Solving: 0.01s 1st Model: 0.01s Unsat: 0.00s)
CPU Time    : 0.040s
Choices     : 64
Conflicts   : 23
Restarts    : 0

Variables   : 11231
Constraints : 32234
```

Not That Much Left to Save

```
gringo hitori_7.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.009s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
Choices     : 77
Conflicts   : 25
Restarts    : 0

Variables   : 539
Constraints : 1137
```

Let's Reach All Squares (Anyway)

hitori_7.lp

```
% Define reachability
reachable(1,1).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2), not blackOut(X2,Y2).

% Can't have unreachable non-black square
 :- state(X,Y,_), not blackOut(X,Y), not reachable(X,Y).
```

☞ require all white squares to be reached

Let's Reach All Squares (Anyway)

hitori_7.lp

```
% Define reachability
reachable(1,1). reachable(1,2).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2), not blackOut(X1,Y1).

% Can't have unreachable non-black square
 :- state(X,Y,_), not blackOut(X,Y), not reachable(X,Y).
```

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Let's Reach All Squares (Anyway)

hitori_8.lp

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% Define reachability
reachable(1,1). reachable(1,2).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2), not blackOut(X1,Y1).

% Can't have unreachable square
 :- state(X,Y,_), not reachable(X,Y).
```

☞ require all white squares to be reached

The Final Result

```
gringo hitori_7.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.009s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
Choices     : 77
Conflicts   : 25
Restarts    : 0

Variables   : 539
Constraints : 1137
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Models      : 1+
Time        : 0.006s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
Choices     : 16
Conflicts   : 5
Restarts    : 0

Variables   : 317
Constraints : 315
```

The Final Encoding (Pretty-Printed) I

hitori_9.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (A) Adjacent grid locations %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Domain predicate (evaluated upon grounding)
adjacent(X,Y,X+1,Y) :- state(X,Y,_;;X+1,Y,_).
adjacent(X,Y,X,Y+1) :- state(X,Y,_;;X,Y+1,_).
adjacent(X2,Y2,X1,Y1) :- adjacent(X1,Y1,X2,Y2).

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (B) Generate solution candidate %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Every square is blacked out or normal
{ blackOut(X,Y) } :- state(X,Y,_).
```

The Final Encoding (Pretty-Printed) II

hitori_9.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.1) Test eliminating adjacent blanks %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Can't have adjacent black squares
 :- adjacent(X1,Y1,X2,Y2), blackOut(X1,Y1;;X2,Y2).

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.2) Tests eliminating number recurrences %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Can't have the same number twice in the same row or column
 :- state(X1,Y1,N), 2 { not blackOut(X1,Y2) : state(X1,Y2,N) }.
 :- state(X1,Y1,N), 2 { not blackOut(X2,Y1) : state(X2,Y1,N) }.
```

The Final Encoding (Pretty-Printed) III

hitori_9.lp

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% (C.3) Test eliminating disconnected numbers %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Define reachability
reachable(1,1).
reachable(1,2).
reachable(X2,Y2) :- reachable(X1,Y1), adjacent(X1,Y1,X2,Y2),
                    not blackOut(X1,Y1).

% Can't have unreachable square
 :- state(X,Y,_), not reachable(X,Y).
```

Recall Where We Started

```
gringo hitori_0.lp instance.lp | clasp --stats

Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 13.485s (Solving: 11.77s 1st Model: 11.77s Unsat: 0.00s)
CPU Time    : 13.290s
Choices     : 458
Conflicts   : 323
Restarts    : 2

Variables   : 260625
Constraints : 1018953
```

And Where We Came

```
gringo hitori_9.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

Models      : 1+
Time        : 0.006s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
CPU Time    : 0.000s
Choices     : 16
Conflicts   : 5
Restarts    : 0

Variables   : 317
Constraints : 315
```

And Where We Came

```
gringo hitori_9.lp instance.lp | clasp --stats
```

```
Answer: 1
blackOut(1,1) blackOut(1,3) blackOut(1,6) blackOut(2,5)
blackOut(2,7) ...           blackOut(8,4) blackOut(8,6)
SATISFIABLE

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Time         : 0.006s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
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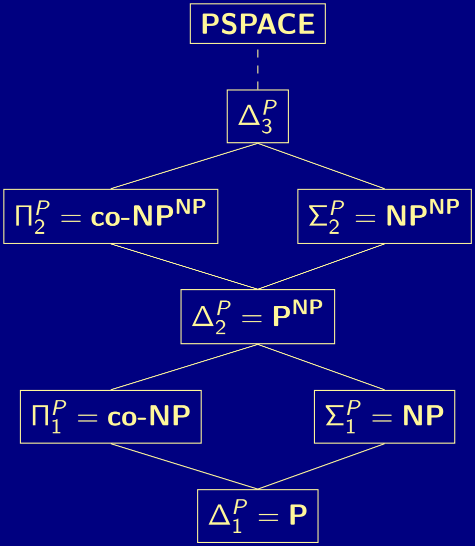
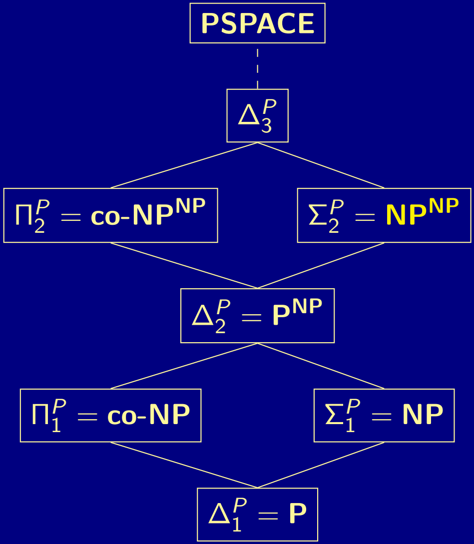
The encoding matters!

Overview

The Polynomial Time Hierarchy



The Polynomial Time Hierarchy



The $\mathbf{NP}^{\mathbf{NP}}$ Class

- What is an $\mathbf{NP}^{\mathbf{NP}}$ problem?
  - ☞ A problem decidable in non-deterministic polynomial time using a (second) **NP** oracle
- How does this relate to disjunctive logic programs?
  1. Guess an answer set candidate for a given disjunctive program
  2. Query the **NP** oracle to verify the guess

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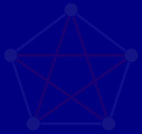
The Ramsey Problem

Theorem

For two numbers r and b, there exists a least number $R(r, b) = n$ s.t. every complete graph with n vertices and edges colored either red or blue contains a complete subgraph (clique) on r vertices whose edges are all red, or a complete subgraph on b vertices whose edges are all blue.



- Contains neither a red nor a blue clique of size 3
- ☞ Shows that $R(3, 3) > 5$
- We will model the problem accordingly
  1. Guess a total edge labeling (ASP as usual)
  2. Verify that the labeling does not admit a clique of size 3 (disjunctive **co-NP** tests)
  - ☞ Satisfiability if $n < R(r, b)$ is not yet a Ramsey number

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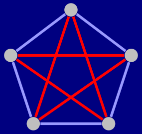

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The Plan

1. Choose some total edge labeling for a complete graph of size n
2. Disjunctive tests to verify that the labeling does not admit a clique
   - Guess subgraphs supposed to form (mono-colored) cliques
   - For each color, derive a special atom if the subgraph is not a clique
   - Derive everything if such a special atom holds
   - Since any answer set is a minimal model of its reduct, some subgraph that is a clique will be chosen whenever possible
     - A special atom will only be derived if there is no clique
   - We may not use default negation/anti-monotone aggregates in the disjunctive part
     - Default negation/anti-monotone aggregates are removed in the reduct
3. Fail whenever some special atom could not be derived
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Modeling

- Helper predicates

```
col(red,3).  col(blue,3).
col(C) :- col(C,N).
```

- Choose a total edge labeling (usual ASP)

```
1 { col(U,V,C) : col(C) } 1 :- U = 1..n, V = (U+1)..n.
```

- Disjunctively guess a clique per color

```
in(U,C) | out(U,C) :- U = 1..n, col(C).
```

- Derive a special bot atom if the guess is invalid or not a clique

```
bot(C) :- col(C,N), (n-N)+1 { out(1..n,C) }.
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Summary

- Disjunctive programs can be used to solve problems beyond **NP**
  - We use claspD for some biological application problems
- Disjunctive program parts are suitable for modeling an additional **co-NP** test per answer set candidate
- It requires some practice to write such programs
  - No default negation/anti-monotone aggregates may be used in the disjunctive part
  - ☞ Instead provide “direct derivations” for conditions that do not hold
- Debugging disjunctive programs is even harder than debugging normal programs
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Equivalence of Logic Programs: Overview

Overview

Motivation

Questions

- How to optimize logic programs?
- How to remove redundancies in automatically generated logic programs?

Difficulty Given that ASP is nonmonotonic, it is difficult to attribute meaning to

- program parts or
- incomplete programs

because the addition of further rules generally changes the overall semantics.

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Notions of Equivalence

Two logic programs Π_1 and Π_2 are

- equivalent ($\Pi_1 \equiv \Pi_2$) if $AS(\Pi_1) = AS(\Pi_2)$.
- strongly equivalent ($\Pi_1 \equiv_s \Pi_2$) if $AS(\Pi_1 \cup \Pi') = AS(\Pi_2 \cup \Pi')$ for any logic program Π'.
- uniformly equivalent ($\Pi_1 \equiv_u \Pi_2$) if $AS(\Pi_1 \cup F) = AS(\Pi_2 \cup F)$ for any set F of facts.

Example $\Pi_1 = \{a \vee b \leftarrow\}$ and $\Pi_2 = \{a \leftarrow not\ b,\ b \leftarrow not\ a\}$

- $\Pi_1 \equiv \Pi_2$ since $AS(\Pi_1) = \{\{a\},\{b\}\} = AS(\Pi_2)$
- $\Pi_1 \not\equiv_s \Pi_2$, e.g. $\Pi' = \{a \leftarrow b,\ b \leftarrow a\}$
- $\Pi_1 \equiv_u \Pi_2$

Implications

- strong equivalence implies uniform equivalence and
- uniform equivalence implies (ordinary) equivalence.

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Implications

- strong equivalence implies uniform equivalence and
- uniform equivalence implies (ordinary) equivalence.

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Two logic programs Π_1 and Π_2 are

- equivalent ($\Pi_1 \equiv \Pi_2$) if $AS(\Pi_1) = AS(\Pi_2)$.
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- Ordinary equivalence in ASP does not allow for substitution of equivalents:

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Strong Equivalence

- Two logic programs Π_1 and Π_2 are strongly equivalent if

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- Strong Equivalence (SE) guarantees substitution of equivalents.
- How to test strong equivalence?
  - ➥ How to avoid testing $AS(\Pi_1 \cup \Pi') = AS(\Pi_2 \cup \Pi')$ for any logic program Π'?

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SE-models

Model-theoretic characterization of Strong Equivalence.

Let Π be a logic program over alphabet $\mathcal{A}$.

- An SE-interpretation over $\mathcal{A}$ is a pair (X, Y) such that $X \subseteq Y \subseteq \mathcal{A}$
- An SE-interpretation (X, Y) is an SE-model of Π if
  1. $Y \models \Pi$
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- $SE(\Pi)$ denotes the set of all SE-models of Π

Theorem $\Pi_1 \equiv_s \Pi_2$ iff $SE(\Pi_1) = SE(\Pi_2)$

Observation If (X, X) is the unique SE-model of Π whose second component is X, then X is an answer set of Π.

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$\Pi_1 = \{a \vee b \leftarrow \}$ and $\Pi_2 = \{a \leftarrow \mathit{not}\ b\ , b \leftarrow \mathit{not}\ a\ \}$

We get the following SE-models over $\{a, b\}$:

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Strong Equivalence

Normal versus Disjunctive logic programs

Reduct-Intersection Let Π be a normal logic program.
If $(U, Y) \in SE(\Pi)$ and $(V, Y) \in SE(\Pi)$, then $(U \cap V, Y) \in SE(\Pi)$.

(☞ Since for any X, Π^X is a Horn program.)

- Reduct-Intersection is not satisfied by disjunctive logic programs.
- If the SE-models of a disjunctive program do not satisfy reduct-intersection, then no strongly equivalent normal programs exists.

Strong Equivalence

Normal versus Disjunctive logic programs

Reduct-Intersection Let Π be a normal logic program.
If $(U, Y) \in SE(\Pi)$ and $(V, Y) \in SE(\Pi)$, then $(U \cap V, Y) \in SE(\Pi)$.

(☞ Since for any X, Π^X is a Horn program.)

- Reduct-Intersection is not satisfied by disjunctive logic programs.
- If the SE-models of a disjunctive program do not satisfy reduct-intersection, then no strongly equivalent normal programs exists.

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Example

- Recall program $\Pi_1 = \{a \vee b \leftarrow\}$ along with

$$\begin{aligned} SE(\Pi_1) = \{&(\{a\},\{a\}), (\{b\},\{b\}), \\ &(\{a\},\{a,b\}), (\{b\},\{a,b\}), (\{a,b\},\{a,b\})\} \end{aligned}$$

- $SE(\Pi_1)$ is not closed under reduct-intersection, since $(\{a\},\{a,b\})$ and $(\{b\},\{a,b\})$ call for $(\emptyset,\{a,b\})$.
  - ➥ No normal logic program is strongly equivalent to $\{a \vee b \leftarrow\}$.

From SE-models to counterexamples

Let Π_1, Π_2 be (disjunctive) logic programs and $(X, Y) \in SE(\Pi_1) \setminus SE(\Pi_2)$.

1. If $(Y, Y) \in SE(\Pi_2)$,
   let $\Pi' = \{A \leftarrow \mid A \in X\} \cup \{A \leftarrow B \mid A, B \in Y \setminus X\}$.
   We get $X \subset Y$ and
   - $X \models (\Pi_1 \cup \Pi')^Y$,
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Overview

UE-models

Model-theoretic characterization of Uniform Equivalence.

Let Π be a logic program over alphabet $\mathcal{A}$.

- An SE-interpretation (X, Y) is a UE-model of Π if
  1. $(X, Y) \in SE(\Pi)$ and
  2. for each Z with $X \subset Z \subset Y$, we have $(Z, Y) \notin SE(\Pi)$.
- $UE(\Pi)$ denotes the set of all UE-models of Π.

Theorem $\Pi_1 \equiv_u \Pi_2$ iff $UE(\Pi_1) = UE(\Pi_2)$

Observation UE-models of a program Π are

- all SE-models (X, X) of Π,
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Example: UE-models

$\Pi_1 = \{a \vee b \leftarrow \}$ and $\Pi_2 = \{a \leftarrow \mathit{not}\ b\ , b \leftarrow \mathit{not}\ a\ \}$

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UE(\Pi_1) &= SE(\Pi_1)\\
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We have

- $UE(\Pi_1) = UE(\Pi_2)$ implies $\Pi_1 \equiv_u \Pi_2$ and $\Pi_1 \equiv \Pi_2$ although $SE(\Pi_1) \neq SE(\Pi_2)$.
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2. If $(Y, Y) \in UE(\Pi_2)$ and $(X', Y) \notin UE(\Pi_2)$ for any $X \subset X' \subset Y$, let $\Pi' = \{A \leftarrow \mid A \in X\}$.
   We get $X \subset Y$ and
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   As with SE-models, we get $Y \in AS(\Pi_1 \cup \Pi') \setminus AS(\Pi_2 \cup \Pi')$.

From UE-models to counterexamples

Let Π_1, Π_2 be (disjunctive) logic programs and $(X, Y) \in UE(\Pi_1) \setminus UE(\Pi_2)$.

1. If $(Y, Y) \in UE(\Pi_2)$ and $(X', Y) \in UE(\Pi_2)$ such that $X \subset X' \subset Y$,
   let $\Pi' = \{A \leftarrow \ \mid A \in X'\}$.
   We get
   - $Y \models (\Pi_1 \cup \Pi')^Y$ but $Z \not\models (\Pi_1 \cup \Pi')^Y$ for any $Z \subset Y$,
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Overview

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WGPPE $r_1 \in \Pi$, $a \in body^+(r_1)$,
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$\{a \leftarrow b, c, not\ d\ , c \leftarrow e, not\ f\ , a \leftarrow b, e, not\ f, not\ d\}$

S-IMP $r_1, r_2 \in \Pi$ such that there exists an $A \subseteq body^-(r_1)$ such that $head(r_2) \subseteq head(r_1) \cup A$, $body^-(r_2) \subseteq body^-(r_1) \setminus A$ and $body^+(r_2) \subseteq body^+(r_1)$,
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