Answer Set Solving in Practice

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Axiomatic Characterization: Overview

1. Completion

2. Tightness

3. Loops and Loop Formulas

Outline

1 Completion

2 Tightness

3 Loops and Loop Formulas

Motivation

- Question Is there a propositional formula $F(P)$ such that the models of $F(P)$ correspond to the stable models of P ?
- Observation Although each atom is defined through a set of rules, each such rule provides only a sufficient condition for its head atom
- Idea The idea of program completion is to turn such implications into a definition by adding the corresponding necessary counterpart

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Program completion

Let P be a normal logic program

- The completion $CF(P)$ of P is defined as follows

$$CF(P) = \left\{ a \leftrightarrow \bigvee_{r \in P, head(r)=a} BF(body(r)) \mid a \in atom(P) \right\}$$

where

$$BF(body(r)) = \bigwedge_{a \in body(r)^+} a \wedge \bigwedge_{a \in body(r)^-} \neg a$$

An example

$$P = \left\{ \begin{array}{l} a \leftarrow \\ b \leftarrow \sim a \\ c \leftarrow a, \sim d \\ d \leftarrow \sim c, \sim e \\ e \leftarrow b, \sim f \\ e \leftarrow e \end{array} \right\} \qquad CF(P) = \left\{ \begin{array}{l} a \leftrightarrow \top \\ b \leftrightarrow \neg a \\ c \leftrightarrow a \wedge \neg d \\ d \leftrightarrow \neg c \wedge \neg e \\ e \leftrightarrow (b \wedge \neg f) \vee e \\ f \leftrightarrow \bot \end{array} \right\}$$

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A closer look

- $CF(P)$ is logically equivalent to $\overleftarrow{CF}(P) \cup \overrightarrow{CF}(P)$, where

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Supported models

- Every stable model of P is a model of $CF(P)$, but not vice versa

  Models of $CF(P)$ are called the supported models of P

  In other words, every stable model of P is a supported model of P

  By definition, every supported model of P is also a model of P

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- P has 21 models, including $\{a, c\}$, $\{a, d\}$, but also $\{a, b, c, d, e, f\}$
- P has 3 supported models, namely $\{a, c\}$, $\{a, d\}$, and $\{a, c, e\}$
- P has 2 stable models, namely $\{a, c\}$ and $\{a, d\}$

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The mismatch

- Question What causes the mismatch between supported models and stable models?
- Hint Consider the unstable yet supported model $\{a, c, e\}$ of P !
- Answer Cyclic derivations are causing the mismatch between supported and stable models

    Atoms in a stable model can be "derived" from a program in a finite number of steps

    Atoms in a cycle (not being "supported from outside the cycle") cannot be "derived" from a program in a finite number of steps

    But such atoms do not contradict the completion of a program and do thus not eliminate an unstable supported model

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Non-cyclic derivations

Let X be a stable model of normal logic program P

- For every atom $A \in X$, there is a finite sequence of positive rules

  $$\langle r_1, \dots, r_n \rangle$$

  such that

  1. $head(r_1) = A$
  2. $body(r_i)^+ \subseteq \{head(r_j) \mid i < j \leq n\}$ for $1 \leq i \leq n$
  3. $r_i \in P^X$ for $1 \leq i \leq n$

- That is, each atom of X has a non-cyclic derivation from P^X

- Example There is no finite sequence of rules providing a derivation for e from $P^{\{a,c,e\}}$

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Positive atom dependency graph

- The origin of (potential) circular derivations can be read off the positive atom dependency graph $G(P)$ of a logic program P given by

$$(atom(P), \{(a, b) \mid r \in P, a \in body(r)^+, head(r) = b\})$$

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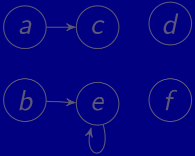

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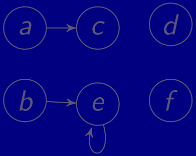

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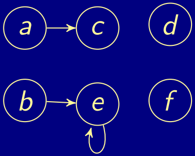

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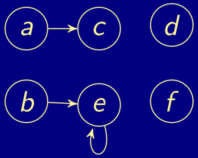

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Tight programs

- A logic program P is called tight, if $G(P)$ is acyclic
- For tight programs, stable and supported models coincide:

Fages' Theorem

Let P be a tight normal logic program and $X \subseteq atom(P)$
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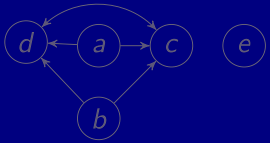

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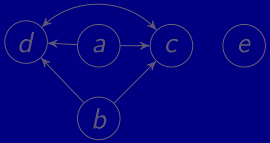

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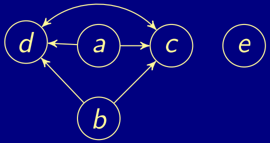

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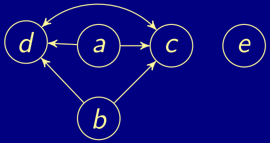

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- Observation Starting from the completion of a program, the problem boils down to eliminating the circular support of atoms holding in the supported models of the program

- Idea Add formulas prohibiting circular support of sets of atoms
- Note Circular support between atoms a and b is possible, if a has a path to b and b has a path to a in the program's positive atom dependency graph

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Loops

Let P be a normal logic program, and
let $G(P) = (atom(P), E)$ be the positive atom dependency graph of P

- A set $\emptyset \subset L \subseteq atom(P)$ is a loop of P
  if it induces a non-trivial strongly connected subgraph of $G(P)$

  That is, each pair of atoms in L is connected by a path of non-zero length in $(L, E \cap (L \times L))$

- We denote the set of all loops of P by $loop(P)$

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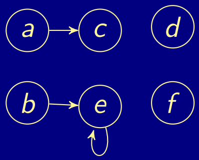

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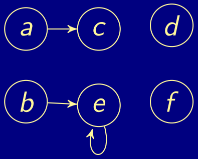

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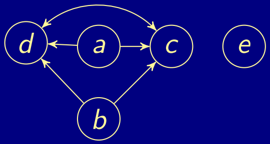

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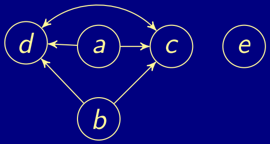

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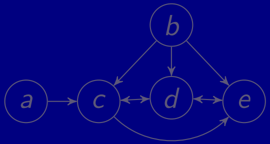

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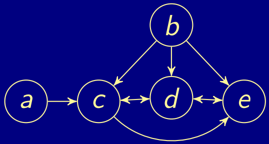

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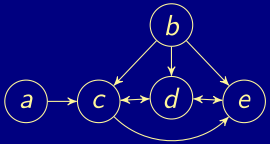

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Loop formulas

Let P be a normal logic program

- For $L \subseteq atom(P)$, define the external supports of L for P as

$$ES_P(L) = \{r \in P \mid head(r) \in L \text{ and } body(r)^+ \cap L = \emptyset\}$$

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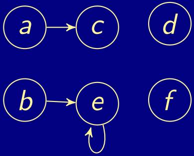

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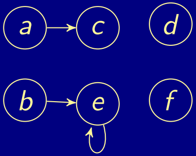

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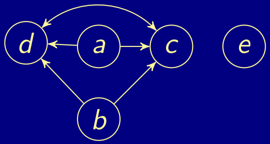

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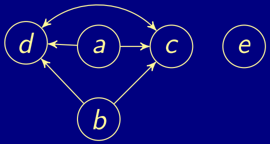

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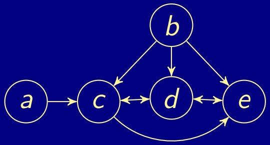

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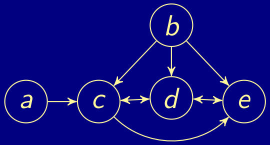

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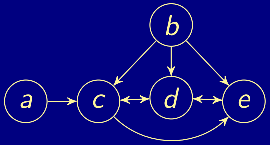

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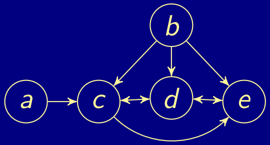

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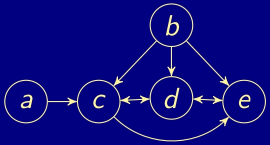

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Lin-Zhao Theorem

Theorem

Let P be a normal logic program and $X \subseteq atom(P)$
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- Result If $\mathcal{P} \not\subseteq \mathcal{NC}^1/poly$,[2] then there is no translation $\mathcal{T}$ from logic programs to propositional formulas such that, for each normal logic program P, both of the following conditions hold:
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- Observations
  - Translation $CF(P) \cup LF(P)$ preserves the vocabulary of P
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Operational Characterization: Overview

4 Partial Interpretations

5 Fitting Operator

6 Unfounded Sets

7 Well-Founded Operator

Outline

4 Partial Interpretations

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7 Well-Founded Operator

Interlude: Partial interpretations

or: 3-valued interpretations

A partial interpretation maps atoms onto truth values *true*, *false*, and *unknown*

- Representation $\langle T, F \rangle$, where
  - T is the set of all *true* atoms and
  - F is the set of all *false* atoms
  - Truth of atoms in $\mathcal{A} \setminus (T \cup F)$ is *unknown*
- Properties
  - $\langle T, F \rangle$ is conflicting if $T \cap F \neq \emptyset$
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- Definition For $\langle T_1, F_1 \rangle$ and $\langle T_2, F_2 \rangle$, define
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Basic idea

- Idea Extend T_P to normal logic programs
- Logical background The idea is to turn a program's completion into an operator such that
  - the head atom of a rule must be *true*
    if the rule's body is *true*
  - an atom must be *false*
    if the body of each rule having it as head is *false*

Definition

- Let P be a normal logic program
- Define

$$\mathbf{\Phi}_P\langle T,F\rangle = \langle \mathbf{T}_P\langle T,F\rangle, \mathbf{F}_P\langle T,F\rangle\rangle$$

where

$$\mathbf{T}_P\langle T,F\rangle = \{head(r) \mid r \in P, body(r)^+ \subseteq T, body(r)^- \subseteq F\}$$
$$\mathbf{F}_P\langle T,F\rangle = \{a \in atom(P) \mid \\ body(r)^+ \cap F \neq \emptyset \text{ or } body(r)^- \cap T \neq \emptyset \\ \text{for each } r \in P \text{ such that } head(r) = a\}$$

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where

$$\begin{aligned}
\mathbf{T}_P\langle T, F\rangle &= \{\mathit{head}(r) \mid r \in P, \mathit{body}(r)^+ \subseteq T, \mathit{body}(r)^- \subseteq F\}\\
\mathbf{F}_P\langle T, F\rangle &= \{a \in \mathit{atom}(P) \mid \\
&\qquad \mathit{body}(r)^+ \cap F \neq \emptyset \text{ or } \mathit{body}(r)^- \cap T \neq \emptyset\\
&\qquad\quad \text{for each } r \in P \text{ such that } \mathit{head}(r) = a\ \}
\end{aligned}$$

Example

$$P = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, \sim d & e \leftarrow b, \sim f \\ b \leftarrow \sim a & d \leftarrow \sim c, \sim e & e \leftarrow e \end{array} \right\}$$

- Let's iterate $\mathbf{\Phi}_P$ on $\langle \{a\}, \{d\} \rangle$:

$$\begin{array}{rcl} \mathbf{\Phi}_P \langle \{a\}, \{d\} \rangle & = & \langle \{a, c\}, \{b, f\} \rangle \\ \mathbf{\Phi}_P \langle \{a, c\}, \{b, f\} \rangle & = & \langle \{a\}, \{b, d, f\} \rangle \\ \mathbf{\Phi}_P \langle \{a\}, \{b, d, f\} \rangle & = & \langle \{a, c\}, \{b, f\} \rangle \\ & \vdots & \end{array}$$

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Fitting semantics

- Define the iterative variant of $\mathbf{\Phi}_P$ analogously to T_P:

$$\mathbf{\Phi}_P^0\langle T, F\rangle = \langle T, F\rangle \qquad \mathbf{\Phi}_P^{i+1}\langle T, F\rangle = \mathbf{\Phi}_P\mathbf{\Phi}_P^i\langle T, F\rangle$$

- Define the Fitting semantics of a normal logic program P as the partial interpretation:

$$\bigsqcup_{i\geq 0}\mathbf{\Phi}_P^i\langle\emptyset,\emptyset\rangle$$

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Properties

Let P be a normal logic program

- $\mathbf{\Phi}_P\langle\emptyset,\emptyset\rangle$ is monotonic
  That is, $\mathbf{\Phi}^i_P\langle\emptyset,\emptyset\rangle \sqsubseteq \mathbf{\Phi}^{i+1}_P\langle\emptyset,\emptyset\rangle$
- The Fitting semantics of P is
  - not conflicting,
  - and generally not total

Fitting fixpoints

Let P be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation

- Define $\langle T, F\rangle$ as a Fitting fixpoint of P if $\mathbf{\Phi}_P\langle T, F\rangle = \langle T, F\rangle$
  - The Fitting semantics is the $\sqsubseteq$-least Fitting fixpoint of P
  - Any other Fitting fixpoint extends the Fitting semantics
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- P has three total Fitting fixpoints:
  $\langle\{a, c\}, \{b, d, e, f\}\rangle$
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  P has three supported models, two of them are stable models

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Let P be a normal logic program,
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- Let $\mathbf{\Phi}_P\langle T, F\rangle = \langle T', F'\rangle$
- If X is a stable model of P such that $T \subseteq X$ and $X \cap F = \emptyset$, then $T' \subseteq X$ and $X \cap F' = \emptyset$

  That is, $\mathbf{\Phi}_P$ is stable model preserving

  Hence, $\mathbf{\Phi}_P$ can be used for approximating stable models and so for propagation in ASP-solvers

- However, $\mathbf{\Phi}_P$ is still insufficient, because total fixpoints correspond to supported models, not necessarily stable models

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$$P = \left\{ \begin{array}{lcl} a & \leftarrow & b \\ b & \leftarrow & a \end{array} \right\}$$

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- That is, Fitting semantics cannot assign *false* to a and b, although they can never become *true* !

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Outline

4 Partial Interpretations

5 Fitting Operator

6 Unfounded Sets

7 Well-Founded Operator

Unfounded sets

Let P be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation

- A set $U \subseteq atom(P)$ is an unfounded set of P wrt $\langle T, F\rangle$, if for each rule $r \in P$ such that $head(r) \in U$, we have that

  1. $body(r)^+ \cap F \neq \emptyset$ or $body(r)^- \cap T \neq \emptyset$ or
  2. $body(r)^+ \cap U \neq \emptyset$

- Intuitively, $\langle T, F\rangle$ is what we already know about P
- Rules satisfying Condition 1 are not usable for further derivations
- Condition 2 is the unfounded set condition treating cyclic derivations: All rules still being usable to derive an atom in U require an(other) atom in U to be true

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Example

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- $\emptyset$ is an unfounded set (by definition)
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- Analogously for $\{b\}$
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Greatest unfounded sets

Let P be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation

- Observation The union of two unfounded sets is an unfounded set
- The greatest unfounded set of P wrt $\langle T, F\rangle$ is the union of all unfounded sets of P wrt $\langle T, F\rangle$

  It is denoted by $\mathbf{U}_P\langle T, F\rangle$
- Alternatively, we may define

$$\mathbf{U}_P\langle T, F\rangle = atom(P) \setminus Cn(\{r \in P \mid body(r)^+ \cap F = \emptyset\}^T)$$

- Note $Cn(\{r \in P \mid body(r)^+ \cap F = \emptyset\}^T)$ contains all non-circularly derivable atoms from P wrt $\langle T, F\rangle$

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Greatest unfounded sets

Let P be a normal logic program,
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- Observation The union of two unfounded sets is an unfounded set
- The greatest unfounded set of P wrt $\langle T, F\rangle$ is the union of all unfounded sets of P wrt $\langle T, F\rangle$

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Outline

4 Partial Interpretations

5 Fitting Operator

6 Unfounded Sets

7 Well-Founded Operator

Well-founded operator

Let P be a normal logic program,
and let $\langle T, F\rangle$ be a partial interpretation

- Observation Condition 1 (in the definition of an unfounded set) corresponds to $\mathbf{F}_P\langle T, F\rangle$ of Fitting's $\mathbf{\Phi}_P\langle T, F\rangle$
- Idea Extend (negative part of) Fitting's operator $\mathbf{\Phi}_P$

  That is,
  - keep definition of $\mathbf{T}_P\langle T, F\rangle$ from $\mathbf{\Phi}_P\langle T, F\rangle$ and
  - replace $\mathbf{F}_P\langle T, F\rangle$ from $\mathbf{\Phi}_P\langle T, F\rangle$ by $\mathbf{U}_P\langle T, F\rangle$
- In words, an atom must be *false*
  if it belongs to the greatest unfounded set
- Definition $\mathbf{\Omega}_P\langle T, F\rangle = \langle \mathbf{T}_P\langle T, F\rangle, \mathbf{U}_P\langle T, F\rangle\rangle$

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Example

$$P = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, \sim d & e \leftarrow b, \sim f \\ b \leftarrow \sim a & d \leftarrow \sim c, \sim e & e \leftarrow e \end{array} \right\}$$

- Let's iterate $\mathbf{\Omega}_{P_1}$ on $\langle \{c\}, \emptyset \rangle$:

$$\begin{array}{rcl} \mathbf{\Omega}_P \langle \{c\}, \emptyset \rangle & = & \langle \{a\}, \{d, f\} \rangle \\ \mathbf{\Omega}_P \langle \{a\}, \{d, f\} \rangle & = & \langle \{a, c\}, \{b, e, f\} \rangle \\ \mathbf{\Omega}_P \langle \{a, c\}, \{b, e, f\} \rangle & = & \langle \{a\}, \{b, d, e, f\} \rangle \\ \mathbf{\Omega}_P \langle \{a\}, \{b, d, e, f\} \rangle & = & \langle \{a, c\}, \{b, e, f\} \rangle \\ & \vdots & \end{array}$$

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Well-founded semantics

- Define the iterative variant of $\mathbf{\Omega}_P$ analogously to $\mathbf{\Phi}_P$:

$$\mathbf{\Omega}_P^0\langle T, F\rangle = \langle T, F\rangle \qquad \mathbf{\Omega}_P^{i+1}\langle T, F\rangle = \mathbf{\Omega}_P\mathbf{\Omega}_P^i\langle T, F\rangle$$

- Define the well-founded semantics of a normal logic program P as the partial interpretation:

$$\bigsqcup_{i\geq 0}\mathbf{\Omega}_P^i\langle\emptyset,\emptyset\rangle$$

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Properties

Let P be a normal logic program

- $\mathbf{\Omega}_P\langle\emptyset,\emptyset\rangle$ is monotonic
  That is, $\mathbf{\Omega}_P^i\langle\emptyset,\emptyset\rangle \sqsubseteq \mathbf{\Omega}_P^{i+1}\langle\emptyset,\emptyset\rangle$
- The well-founded semantics of P is
  - not conflicting,
  - and generally not total
- We have $\bigsqcup_{i\geq 0}\mathbf{\Phi}_P^i\langle\emptyset,\emptyset\rangle \sqsubseteq \bigsqcup_{i\geq 0}\mathbf{\Omega}_P^i\langle\emptyset,\emptyset\rangle$

Well-founded fixpoints

Let P be a normal logic program,
and let $\langle T, F \rangle$ be a partial interpretation

- Define $\langle T, F \rangle$ as a well-founded fixpoint of P if $\mathbf{\Omega}_P\langle T, F \rangle = \langle T, F \rangle$
  - The well-founded semantics is the $\sqsubseteq$-least well-founded fixpoint of P
  - Any other well-founded fixpoint extends the well-founded semantics
  - Total well-founded fixpoints correspond to stable models

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$$P = \left\{ \begin{array}{lll} a \leftarrow & c \leftarrow a, \sim d & e \leftarrow b, \sim f \\ b \leftarrow \sim a & d \leftarrow \sim c, \sim e & e \leftarrow e \end{array} \right\}$$

- P has two total well-founded fixpoints:
  - $\langle\{a, c\}, \{b, d, e, f\}\rangle$
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Let P be a normal logic program,
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- Let $\mathbf{\Omega}_P\langle T, F\rangle = \langle T', F'\rangle$
- If X is a stable model of P such that $T \subseteq X$ and $X \cap F = \emptyset$, then $T' \subseteq X$ and $X \cap F' = \emptyset$

  That is, $\mathbf{\Omega}_P$ is stable model preserving

  Hence, $\mathbf{\Omega}_P$ can be used for approximating stable models and so for propagation in ASP-solvers
- In contrast to $\mathbf{\Phi}_P$, operator $\mathbf{\Omega}_P$ is sufficient for propagation because total fixpoints correspond to stable models
- Note In addition to $\mathbf{\Omega}_P$, most ASP-solvers apply backward propagation, originating from program completion (although this is unnecessary from a formal point of view)

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Proof-theoretic Characterization: Overview

8 Tableau Calculi

9 Tableau Calculi for ASP

10 Tableau Calculi characterizing ASP solvers

11 Proof complexity

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- Goal Analyze computations in ASP solvers
- Wanted A declarative and fine-grained instrument for characterizing operations as well as strategies of ASP solvers
- Idea View stable model computations as derivations in an inference system

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Tableau calculi

- Traditionally, tableau calculi are used for
  - automated theorem proving and
  - proof theoretical analysis

  in classical as well as non-classical logics
- General idea Given an input, prove some property by decomposition
  Decomposition is done by applying deduction rules
- For details, see *Handbook of Tableau Methods*, Kluwer, 1999

General definitions

- A tableau is a (mostly binary) tree
- A branch in a tableau is a path from the root to a leaf
- A branch containing $\gamma_1, \dots, \gamma_m$ can be extended by applying tableau rules of form

$$\frac{\gamma_1, \dots, \gamma_m}{\begin{array}{c}\alpha_1\\ \vdots\\ \alpha_n\end{array}} \qquad\qquad \frac{\gamma_1, \dots, \gamma_m}{\beta_1 \mid \dots \mid \beta_n}$$

Rules of the former format append entries $\alpha_1, \dots, \alpha_n$ to the branch

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Example

- A simple tableau calculus for proving unsatisfiability of propositional formulas, composed from $\neg$, $\wedge$, and $\vee$, consists of rules

$$\frac{\neg\neg\alpha}{\alpha} \qquad \frac{\alpha_1 \wedge \alpha_2}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta_1 \vee \beta_2}{\beta_1 \mid \beta_2}$$

- All rules are semantically valid, when interpreting entries in a branch conjunctively and distinct (sub-)branches as connected disjunctively
- A propositional formula φ is unsatisfiable iff there is a tableau with φ as the root node such that
  1. all other entries can be produced by tableau rules and
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(1)	$a \land ((\neg b \land (\neg a \lor b)) \lor \neg\neg\neg a)$	$[\varphi]$
(2)	a	[1]
(3)	$(\neg b \land (\neg a \lor b)) \lor \neg\neg\neg a$	[1]

(4)	$\neg b \land (\neg a \lor b)$	[3]	**(9)**	$\neg\neg\neg a$	[3]
(5)	$\neg b$	[4]	**(10)**	$\neg a$	[9]
(6)	$\neg a \lor b$	[4]			

(7)	$\neg a$	[6]	**(8)**	b	[6]

All three branches of the tableau are contradictory (cf 2, 5, 7, 8, 10)

Hence, $a \land ((\neg b \land (\neg a \lor b)) \lor \neg\neg\neg a)$ is unsatisfiable

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Hence, $a \wedge ((\neg b \wedge (\neg a \vee b)) \vee \neg\neg\neg a)$ is unsatisfiable

Example

(1)	$a \wedge ((\neg b \wedge (\neg a \vee b)) \vee \neg\neg\neg a)$	[φ]
(2)	a	[1]
(3)	$(\neg b \wedge (\neg a \vee b)) \vee \neg\neg\neg a$	[1]

(4)	$\neg b \wedge (\neg a \vee b)$	[3]		(9)	$\neg\neg\neg a$	[3]
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Right branch:

(9)	$\neg\neg\neg a$	[3]
(10)	$\neg a$	[9]

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Outline

8 Tableau Calculi

9 Tableau Calculi for ASP

10 Tableau Calculi characterizing ASP solvers

11 Proof complexity

Tableaux and ASP

- A tableau rule captures an elementary inference scheme in an ASP solver
- A branch in a tableau corresponds to a successful or unsuccessful computation of a stable model
- An entire tableau represents a traversal of the search space

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ASP-specific definitions

- A (signed) tableau for a logic program P is a binary tree such that
  - the root node of the tree consists of the rules in P;
  - the other nodes in the tree are entries of the form $\boldsymbol{T}v$ or $\boldsymbol{F}v$, called signed literals, where v is a variable,
  - generated by extending a tableau using deduction rules (given below)
- An entry $\boldsymbol{T}v$ ($\boldsymbol{F}v$) reflects that variable v is *true* (*false*) in a corresponding variable assignment

  A set of signed literals constitutes a partial assignment
- For a normal logic program P,
  - atoms of P in $atom(P)$ and
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Tableau rules for ASP at a glance

$$\text{(FTB)}\quad \frac{p \leftarrow l_1,\dots,l_n \qquad \boldsymbol{t}l_1,\dots,\boldsymbol{t}l_n}{\boldsymbol{T}\{l_1,\dots,l_n\}}$$

$$\text{(BFB)}\quad \frac{\boldsymbol{F}\{l_1,\dots,l_i,\dots,l_n\} \qquad \boldsymbol{t}l_1,\dots,\boldsymbol{t}l_{i-1},\boldsymbol{t}l_{i+1},\dots,\boldsymbol{t}l_n}{\boldsymbol{f}l_i}$$

$$\text{(FTA)}\quad \frac{p \leftarrow l_1,\dots,l_n \qquad \boldsymbol{T}\{l_1,\dots,l_n\}}{\boldsymbol{T}p}$$

$$\text{(BFA)}\quad \frac{p \leftarrow l_1,\dots,l_n \qquad \boldsymbol{F}p}{\boldsymbol{F}\{l_1,\dots,l_n\}}$$

$$\text{(FFB)}\quad \frac{p \leftarrow l_1,\dots,l_i,\dots,l_n \qquad \boldsymbol{f}l_i}{\boldsymbol{F}\{l_1,\dots,l_i,\dots,l_n\}}$$

$$\text{(BTB)}\quad \frac{\boldsymbol{T}\{l_1,\dots,l_i,\dots,l_n\}}{\boldsymbol{t}l_i}$$

$$\text{(FFA)}\quad \frac{\boldsymbol{F}B_1,\dots,\boldsymbol{F}B_m}{\boldsymbol{F}p}\quad (\S)$$

$$\text{(BTA)}\quad \frac{\boldsymbol{T}p \qquad \boldsymbol{F}B_1,\dots,\boldsymbol{F}B_{i-1},\boldsymbol{F}B_{i+1},\dots,\boldsymbol{F}B_m}{\boldsymbol{T}B_i}\quad (\S)$$

$$\text{(WFN)}\quad \frac{\boldsymbol{F}B_1,\dots,\boldsymbol{F}B_m}{\boldsymbol{F}p}\quad (\dagger)$$

$$\text{(WFJ)}\quad \frac{\boldsymbol{T}p \qquad \boldsymbol{F}B_1,\dots,\boldsymbol{F}B_{i-1},\boldsymbol{F}B_{i+1},\dots,\boldsymbol{F}B_m}{\boldsymbol{T}B_i}\quad (\dagger)$$

$$\text{(FL)}\quad \frac{\boldsymbol{F}B_1,\dots,\boldsymbol{F}B_m}{\boldsymbol{F}p}\quad (\ddagger)$$

$$\text{(BL)}\quad \frac{\boldsymbol{T}p \qquad \boldsymbol{F}B_1,\dots,\boldsymbol{F}B_{i-1},\boldsymbol{F}B_{i+1},\dots,\boldsymbol{F}B_m}{\boldsymbol{T}B_i}\quad (\ddagger)$$

$$\text{(Cut}[X]\text{)}\quad \frac{}{\boldsymbol{T}v \mid \boldsymbol{F}v}\quad (\sharp[X])$$

More concepts

- A tableau calculus is a set of tableau rules
- A branch in a tableau is conflicting,
  if it contains both $\boldsymbol{T}v$ and $\boldsymbol{F}v$ for some variable v
- A branch in a tableau is total for a program P,
  if it contains either $\boldsymbol{T}v$ or $\boldsymbol{F}v$ for each $v \in atom(P) \cup body(P)$
- A branch in a tableau of some calculus $\mathcal{T}$ is closed,
  if no rule in $\mathcal{T}$ other than *Cut* can produce any new entries
- A branch in a tableau is complete,
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Example

- Consider the program

$$P = \left\{ \begin{array}{l} a \leftarrow \\ c \leftarrow {\sim}b, {\sim}d \\ d \leftarrow a, {\sim}c \end{array} \right\}$$

having stable models $\{a, c\}$ and $\{a, d\}$

(Previewed) Example

$$
\begin{array}{c}
a \leftarrow \\
c \leftarrow \sim b, \sim d \\
d \leftarrow a, \sim c
\end{array}
$$

$$
\begin{array}{rccccc}
(\mathrm{FTB}) & & & \boldsymbol{T}\emptyset & & \\
(\mathrm{FTA}) & & & \boldsymbol{T}a & & \\
(\mathrm{FFA}) & & & \boldsymbol{F}b & & \\
(\mathrm{Cut}[atom(P)]) & & \boldsymbol{T}c & & & \boldsymbol{F}c \\
& (\mathrm{BTA}) & \boldsymbol{T}\{\sim b, \sim d\} & & (\mathrm{BFA}) & \boldsymbol{F}\{\sim b, \sim d\} \\
& (\mathrm{BTB}) & \boldsymbol{F}d & & (\mathrm{BFB}) & \boldsymbol{T}d \\
& (\mathrm{FFB}) & \boldsymbol{F}\{a, \sim c\} & & (\mathrm{FTB}) & \boldsymbol{T}\{a, \sim c\}
\end{array}
$$

(Previewed) Example

$$a \leftarrow$$
$$c \leftarrow \sim b, \sim d$$
$$d \leftarrow a, \sim c$$

(FTB)		$\boldsymbol{T}\emptyset$		
(FTA)		$\boldsymbol{T}a$		
(FFA)		$\boldsymbol{F}b$		
(Cut[$atom(P)$])		$\boldsymbol{T}c$		$\boldsymbol{F}c$
	(BTA)	$\boldsymbol{T}\{\sim b, \sim d\}$	(BFA)	$\boldsymbol{F}\{\sim b, \sim d\}$
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(Previewed) Example

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\begin{array}{rcrc}
 & & \multicolumn{2}{l}{a \leftarrow} \\
 & & \multicolumn{2}{l}{c \leftarrow \sim b, \sim d} \\
 & & \multicolumn{2}{l}{d \leftarrow a, \sim c} \\
(FTB) & & \multicolumn{2}{l}{\boldsymbol{T}\emptyset} \\
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Auxiliary definitions

- For a literal l, define conjugation functions $\boldsymbol{t}$ and $\boldsymbol{f}$ as follows

$$\boldsymbol{t}l = \begin{cases} \boldsymbol{T}l & \text{if } l \text{ is an atom} \\ \boldsymbol{F}a & \text{if } l = {\sim}a \text{ for an atom } a \end{cases}$$

$$\boldsymbol{f}l = \begin{cases} \boldsymbol{F}l & \text{if } l \text{ is an atom} \\ \boldsymbol{T}a & \text{if } l = {\sim}a \text{ for an atom } a \end{cases}$$

- Examples $\boldsymbol{t}a = \boldsymbol{T}a$, $\boldsymbol{f}a = \boldsymbol{F}a$, $\boldsymbol{t}{\sim}a = \boldsymbol{F}a$, and $\boldsymbol{f}{\sim}a = \boldsymbol{T}a$

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Auxiliary definitions

- Some tableau rules require conditions for their application
- Such conditions are specified as provisos

$$\frac{\mathit{prerequisites}}{\mathit{consequence}}\ (\mathit{proviso})$$

proviso: some condition(s)

- Note All tableau rules given in the sequel are stable model preserving

Forward True Body (FTB)

- Prerequisites All of a body's literals are *true*
- Consequence The body is *true*
- Tableau Rule FTB

$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_n \\ \boldsymbol{t}l_1, \dots, \boldsymbol{t}l_n \end{array}}{\boldsymbol{T}\{l_1, \dots, l_n\}}$$

- Example

$$\frac{\begin{array}{c} a \leftarrow b, \sim c \\ \boldsymbol{T}b \\ \boldsymbol{F}c \end{array}}{\boldsymbol{T}\{b, \sim c\}}$$

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Backward False Body (BFB)

- Prerequisites A body is *false*, and all its literals except for one are *true*
- Consequence The residual body literal is *false*
- Tableau Rule BFB

$$\frac{\begin{array}{c}\boldsymbol{F}\{l_1,\dots,l_i,\dots,l_n\}\\ \boldsymbol{t}l_1,\dots,\boldsymbol{t}l_{i-1},\boldsymbol{t}l_{i+1},\dots,\boldsymbol{t}l_n\end{array}}{\boldsymbol{f}l_i}$$

- Examples

$$\frac{\begin{array}{c}\boldsymbol{F}\{b,\sim c\}\\ \boldsymbol{T}b\end{array}}{\boldsymbol{T}c} \qquad \frac{\begin{array}{c}\boldsymbol{F}\{b,\sim c\}\\ \boldsymbol{F}c\end{array}}{\boldsymbol{F}b}$$

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- Examples

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Forward False Body (FFB)

- Prerequisites Some literal of a body is *false*
- Consequence The body is *false*
- Tableau Rule FFB

$$\frac{p \leftarrow l_1, \ldots, l_i, \ldots, l_n \qquad \boldsymbol{f}\, l_i}{\boldsymbol{F}\{l_1, \ldots, l_i, \ldots, l_n\}}$$

- Examples

$$\frac{a \leftarrow b, \sim c \qquad \boldsymbol{F}\, b}{\boldsymbol{F}\{b, \sim c\}} \qquad \frac{a \leftarrow b, \sim c \qquad \boldsymbol{T}\, c}{\boldsymbol{F}\{b, \sim c\}}$$

Forward False Body (FFB)

- Prerequisites Some literal of a body is *false*
- Consequence The body is *false*
- Tableau Rule FFB

$$\frac{p \leftarrow l_1, \dots, l_i, \dots, l_n \quad \boldsymbol{f} l_i}{\boldsymbol{F}\{l_1, \dots, l_i, \dots, l_n\}}$$

- Examples

$$\frac{a \leftarrow b, \sim c \quad \boldsymbol{F} b}{\boldsymbol{F}\{b, \sim c\}} \qquad \frac{a \leftarrow b, \sim c \quad \boldsymbol{T} c}{\boldsymbol{F}\{b, \sim c\}}$$

Backward True Body (BTB)

- Prerequisites A body is *true*
- Consequence The body's literals are *true*
- Tableau Rule BTB

$$\frac{\boldsymbol{T}\{l_1, \ldots, l_i, \ldots, l_n\}}{\boldsymbol{t}\, l_i}$$

- Examples

$$\frac{\boldsymbol{T}\{b, \sim c\}}{\boldsymbol{T}\, b} \qquad \frac{\boldsymbol{T}\{b, \sim c\}}{\boldsymbol{F}\, c}$$

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Tableau rules for bodies

Consider rule body $B = \{l_1, \dots, l_n\}$

- Rules FTB and BFB amount to implication

$$l_1 \wedge \dots \wedge l_n \rightarrow B$$

- Rules FFB and BTB amount to implication

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- Together they yield

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Forward True Atom (FTA)

- Prerequisites Some of an atom's bodies is *true*
- Consequence The atom is *true*
- Tableau Rule FTA

$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_n \\ \boldsymbol{T}\{l_1, \dots, l_n\} \end{array}}{\boldsymbol{T}p}$$

- Examples

$$\frac{\begin{array}{c} a \leftarrow b, \sim c \\ \boldsymbol{T}\{b, \sim c\} \end{array}}{\boldsymbol{T}a} \qquad \frac{\begin{array}{c} a \leftarrow d, \sim e \\ \boldsymbol{T}\{d, \sim e\} \end{array}}{\boldsymbol{T}a}$$

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Backward False Atom (BFA)

- Prerequisites An atom is *false*
- Consequence The bodies of all rules with the atom as head are *false*
- Tableau Rule BFA

$$\frac{\begin{array}{c} p \leftarrow l_1, \dots, l_n \\ \boldsymbol{F}p \end{array}}{\boldsymbol{F}\{l_1, \dots, l_n\}}$$

- Examples

$$\frac{\begin{array}{c} a \leftarrow b, \sim c \\ \boldsymbol{F}a \end{array}}{\boldsymbol{F}\{b, \sim c\}} \qquad \frac{\begin{array}{c} a \leftarrow d, \sim e \\ \boldsymbol{F}a \end{array}}{\boldsymbol{F}\{d, \sim e\}}$$

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Forward False Atom (FFA)

- Prerequisites For some atom, the bodies of all rules with the atom as head are *false*
- Consequence The atom is *false*
- Tableau Rule FFA

$$\frac{\boldsymbol{F}B_1, \dots, \boldsymbol{F}B_m}{\boldsymbol{F}p} \quad (body_P(p) = \{B_1, \dots, B_m\})$$

- Example

$$\frac{\begin{array}{c}\boldsymbol{F}\{b, \sim c\} \\ \boldsymbol{F}\{d, \sim e\}\end{array}}{\boldsymbol{F}a} \quad (body_P(a) = \{\{b, \sim c\}, \{d, \sim e\}\})$$

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Backward True Atom (BTA)

- Prerequisites An atom is *true*, and the bodies of all rules with the atom as head except for one are *false*
- Consequence The residual body is *true*
- Tableau Rule BTA

$$\frac{\boldsymbol{T}p \quad \boldsymbol{F}B_1,\dots,\boldsymbol{F}B_{i-1},\boldsymbol{F}B_{i+1},\dots,\boldsymbol{F}B_m}{\boldsymbol{T}B_i}\ (\mathit{body}_P(p)=\{B_1,\dots,B_m\})$$

- Examples

$$\frac{\boldsymbol{T}a \quad \boldsymbol{F}\{b,\sim c\}}{\boldsymbol{T}\{d,\sim e\}}\ (*) \qquad \frac{\boldsymbol{T}a \quad \boldsymbol{F}\{d,\sim e\}}{\boldsymbol{T}\{b,\sim c\}}\ (*)$$

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Relationship with program completion

Let P be a normal logic program

- The eight tableau rules introduced so far essentially provide (straightforward) inferences from $CF(P)$

Preliminaries for unfounded sets

Let P be a normal logic program

- For $P' \subseteq P$, define the greatest unfounded set of P wrt P' as

$$\mathbf{U}_P(P') = atom(P) \setminus Cn((P')^{\emptyset})$$

- For a loop $L \in loop(P)$, define the external bodies of L as

$$EB_P(L) = \{body(r) \mid r \in P, head(r) \in L, body(r)^+ \cap L = \emptyset\}$$

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Well-Founded Negation (WFN)

- Prerequisites An atom is in the greatest unfounded set wrt rules whose bodies are *false*
- Consequence The atom is *false*
- Tableau Rule WFN

$$\frac{\boldsymbol{F}B_1,\dots,\boldsymbol{F}B_m}{\boldsymbol{F}p} \quad (p \in \mathbf{U}_P(\{r \in P \mid body(r) \notin \{B_1,\dots,B_m\}\}))$$

- Examples

$$\frac{\begin{array}{c} a \leftarrow \sim b \\ \boldsymbol{F}\{\sim b\} \end{array}}{\boldsymbol{F}a} \ (*) \qquad \frac{\begin{array}{c} a \leftarrow a \\ a \leftarrow \sim b \\ \boldsymbol{F}\{\sim b\} \end{array}}{\boldsymbol{F}a} \ (*)$$

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Well-Founded Justification (WFJ)

- Prerequisites A *true* atom is in the greatest unfounded set wrt rules whose bodies are *false*, if a particular body is made *false*
- Consequence The respective body is *true*
- Tableau Rule WFJ

$$\frac{\begin{array}{c}\boldsymbol{T}p\\ \boldsymbol{F}B_1,\dots,\boldsymbol{F}B_{i-1},\boldsymbol{F}B_{i+1},\dots,\boldsymbol{F}B_m\end{array}}{\boldsymbol{T}B_i}\quad(p\in\mathbf{U}_P(\{r\in P\mid body(r)\notin\{B_1,\dots,B_m\}\}))$$

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Well-founded tableau rules

- Tableau rules WFN and WFJ ensure non-circular support for *true* atoms
- Note
  1. WFN subsumes falsifying atoms via FFA,
  2. WFJ can be viewed as "backward propagation" for unfounded sets,
  3. WFJ subsumes backward propagation of *true* atoms via BTA

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Relationship with well-founded operator

Let P be a normal logic program, $\langle T, F \rangle$ a partial interpretation, and $P' = \{r \in P \mid body(r)^+ \cap F = \emptyset \text{ and } body(r)^- \cap T = \emptyset\}$.

- The following conditions are equivalent
  1. $p \in \mathbf{U}_P\langle T, F \rangle$
  2. $p \in \mathbf{U}_P(P')$
- Hence, the well-founded operator $\mathbf{\Omega}$ and WFN coincide
- Note In contrast to $\mathbf{\Omega}$, WFN does not necessarily require a rule body to contain a *false* literal for the rule being inapplicable

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Forward Loop (FL)

- Prerequisites The external bodies of a loop are *false*
- Consequence The atoms in the loop are *false*
- Tableau Rule FL

$$\frac{\boldsymbol{F}B_1, \dots, \boldsymbol{F}B_m}{\boldsymbol{F}p} \; (p \in L, L \in loop(P), EB_P(L) = \{B_1, \dots, B_m\})$$

- Example

$$a \leftarrow a$$
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Backward Loop (BL)

- Prerequisites An atom of a loop is *true*, and all external bodies except for one are *false*
- Consequence The residual external body is *true*
- Tableau Rule BL

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Tableau rules for loops

- Tableau rules FL and BL ensure non-circular support for *true* atoms
- For a loop L such that $EB_P(L) = \{B_1, \dots, B_m\}$, they amount to implications of form

$$\bigvee_{p \in L} p \rightarrow B_1 \vee \dots \vee B_m$$

- Comparison to well-founded tableau rules yields
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Relationship with loop formulas

- Tableau rules FL and BL essentially provide (straightforward) inferences from loop formulas
  - Impractical to precompute exponentially many loop formulas
- In practice, ASP solvers such as *smodels* and *clasp*
  - exploit strongly connected components of positive atom dependency graphs
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  - do not directly implement BL (and neither WFJ)
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- Up to now, all tableau rules are deterministic

  That is, rules extend a single branch but cannot create sub-branches
- In general, closing a branch leads to a partial assignment
- Case analysis is done by $Cut[\mathcal{C}]$ where $\mathcal{C} \subseteq atom(P) \cup body(P)$

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Case analysis by *Cut*

- Prerequisites None
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Well-known tableau calculi

- Fitting's operator $\mathbf{\Phi}$ applies forward propagation without sophisticated unfounded set checks

$$\mathcal{T}_{\mathbf{\Phi}} = \{FTB, FTA, FFB, FFA\}$$

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Outline

8 Tableau Calculi

9 Tableau Calculi for ASP

10 Tableau Calculi characterizing ASP solvers

11 Proof complexity

Tableau calculi characterizing ASP solvers

- ASP solvers combine propagation with case analysis
- We obtain the following tableau calculi characterizing

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\begin{aligned}
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Outline

8 Tableau Calculi

9 Tableau Calculi for ASP

10 Tableau Calculi characterizing ASP solvers

11 Proof complexity

Potassco

Proof complexity

- Proof complexity is used for describing the relative efficiency of different proof systems
  - It compares proof systems based on minimal refutations
  - It is independent of heuristics
- A proof system $\mathcal{T}$ polynomially simulates a proof system $\mathcal{T}'$, if every refutation of $\mathcal{T}'$ can be polynomially mapped to a refutation of $\mathcal{T}$
  - Otherwise, $\mathcal{T}$ does not polynomially simulate $\mathcal{T}'$
- For showing that proof system $\mathcal{T}$ does not polynomially simulate $\mathcal{T}'$, we have to provide an infinite witnessing family of programs such that minimal refutations of $\mathcal{T}$ asymptotically are exponentially larger than minimal refutations of $\mathcal{T}'$
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- Let $\{P_a^n\}$, $\{P_b^n\}$, and $\{P_c^n\}$ be infinite families of programs where

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- As witnessed by $\{P_a^n \cup P_c^n\}$ and $\{P_b^n \cup P_c^n\}$, respectively, $\mathcal{T}_{smodels}$ and $\mathcal{T}_{noMoRe}$ do not polynomially simulate one another
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  - both $\mathcal{T}_{smodels}$ and $\mathcal{T}_{noMoRe}$ are polynomially simulated by $\mathcal{T}_{nomore^{++}}$ and
  - $\mathcal{T}_{nomore^{++}}$ is polynomially simulated by neither $\mathcal{T}_{smodels}$ nor $\mathcal{T}_{noMoRe}$

- More generally, the proof system obtained with $Cut[atom(P) \cup body(P)]$ is exponentially stronger than the ones with either $Cut[atom(P)]$ or $Cut[body(P)]$
- Case analyses (at least) on atoms and bodies are mandatory in powerful ASP solvers

$\mathcal{T}_{smodels}$: Example tableau

(r_1)	$a \leftarrow \sim b$		(r_2)	$b \leftarrow d, \sim a$		(r_3)	$c \leftarrow b, d$
(r_4)	$c \leftarrow g$		(r_5)	$d \leftarrow c$		(r_6)	$d \leftarrow g$
(r_7)	$e \leftarrow f, \sim c$		(r_8)	$f \leftarrow \sim g$		(r_9)	$g \leftarrow \sim a, \sim f$

(1)	$\boldsymbol{T}a$	[*Cut*]
(2)	$\boldsymbol{T}\{\sim b\}$	[*BTA*: r_1, 1]
(3)	$\boldsymbol{F}b$	[*BTB*: 2]
(4)	$\boldsymbol{F}\{d, \sim a\}$	[*BFA*: r_2, 3]
(5)	$\boldsymbol{F}\{\sim a, \sim f\}$	[*FFB*: r_9, 1]
(6)	$\boldsymbol{F}g$	[*FFA*: r_9, 5]
(7)	$\boldsymbol{T}\{\sim g\}$	[*FTB*: r_8, 6]
(8)	$\boldsymbol{T}f$	[*FTA*: r_8, 7]
(9)	$\boldsymbol{F}\{b, d\}$	[*FFB*: r_3, 3]
(10)	$\boldsymbol{F}\{g\}$	[*FFB*: r_4, r_6, 6]
(11)	$\boldsymbol{F}c$	[*FFA*: r_3, r_4, 9, 10]
(12)	$\boldsymbol{F}\{c\}$	[*FFB*: r_5, 11]
(13)	$\boldsymbol{F}d$	[*FFA*: r_5, r_6, 10, 12]
(14)	$\boldsymbol{T}\{f, \sim c\}$	[*FTB*: r_7, 8, 11]
(15)	$\boldsymbol{T}e$	[*FTA*: r_7, 14]

(16)	$\boldsymbol{F}a$	[*Cut*]
(17)	$\boldsymbol{F}\{\sim b\}$	[*BFA*: r_1, 16]
(18)	$\boldsymbol{T}b$	[*BFB*: 17]
(19)	$\boldsymbol{T}\{d, \sim a\}$	[*BTA*: r_2, 18]
(20)	$\boldsymbol{T}d$	[*BTB*: 19]
(21)	$\boldsymbol{T}\{b, d\}$	[*FTB*: r_3, 18, 20]
(22)	$\boldsymbol{T}c$	[*FTA*: r_3, 21]
(23)	$\boldsymbol{F}\{f, \sim c\}$	[*FFB*: r_7, 22]
(24)	$\boldsymbol{F}e$	[*FFA*: r_7, 23]
(25)	$\boldsymbol{T}\{c\}$	[*FTB*: r_5, 22]

(26)	$\boldsymbol{T}f$	[*Cut*]
(27)	$\boldsymbol{F}\{\sim a, \sim f\}$	[*FFB*: r_9, 26]
(28)	$\boldsymbol{F}c$	[*WFN*: 27]

(29)	$\boldsymbol{F}f$	[*Cut*]
(30)	$\boldsymbol{T}\{\sim a, \sim f\}$	[*FTB*: r_9, 16, 29]
(31)	$\boldsymbol{T}g$	[*FTA*: r_9, 30]
(32)	$\boldsymbol{T}\{g\}$	[*FTB*: r_4, r_6, 31]
(33)	$\boldsymbol{F}\{\sim g\}$	[*FFB*: r_8, 31]

$\mathcal{T}_{noMoRe}$: Example tableau

(r_1)	$a \leftarrow \sim b$		(r_2)	$b \leftarrow d, \sim a$		(r_3)	$c \leftarrow b, d$	
(r_4)	$c \leftarrow g$		(r_5)	$d \leftarrow c$		(r_6)	$d \leftarrow g$	
(r_7)	$e \leftarrow f, \sim c$		(r_8)	$f \leftarrow \sim g$		(r_9)	$g \leftarrow \sim a, \sim f$	

(1)	$\boldsymbol{T}\{\sim b\}$	$[Cut]$
(2)	$\boldsymbol{T}a$	$[FTA\text{: } r_1, 1]$
(3)	$\boldsymbol{F}b$	$[BTB\text{: } 1]$
(4)	$\boldsymbol{F}\{d, \sim a\}$	$[BFA\text{: } r_2, 3]$
(5)	$\boldsymbol{F}\{\sim a, \sim f\}$	$[FFB\text{: } r_9, 2]$
(6)	$\boldsymbol{F}g$	$[FFA\text{: } r_9, 5]$
(7)	$\boldsymbol{T}\{\sim g\}$	$[FTB\text{: } r_8, 6]$
(8)	$\boldsymbol{T}f$	$[FTA\text{: } r_8, 7]$
(9)	$\boldsymbol{F}\{b, d\}$	$[FFB\text{: } r_3, 3]$
(10)	$\boldsymbol{F}\{g\}$	$[FFB\text{: } r_4, r_6, 6]$
(11)	$\boldsymbol{F}c$	$[FFA\text{: } r_3, r_4, 9, 10]$
(12)	$\boldsymbol{F}\{c\}$	$[FFB\text{: } r_5, 11]$
(13)	$\boldsymbol{F}d$	$[FFA\text{: } r_5, r_6, 10, 12]$
(14)	$\boldsymbol{T}\{f, \sim c\}$	$[FTB\text{: } r_7, 8, 11]$
(15)	$\boldsymbol{T}e$	$[FTA\text{: } r_7, 14]$

(16)	$\boldsymbol{F}\{\sim b\}$	$[Cut]$
(17)	$\boldsymbol{F}a$	$[FFA\text{: } r_1, 16]$
(18)	$\boldsymbol{T}b$	$[BFB\text{: } 16]$
(19)	$\boldsymbol{T}\{d, \sim a\}$	$[BTA\text{: } r_2, 18]$
(20)	$\boldsymbol{T}d$	$[BTB\text{: } 19]$
(21)	$\boldsymbol{T}\{b, d\}$	$[FTB\text{: } r_3, 18, 20]$
(22)	$\boldsymbol{T}c$	$[FTA\text{: } r_3, 21]$
(23)	$\boldsymbol{F}\{f, \sim c\}$	$[FFB\text{: } r_7, 22]$
(24)	$\boldsymbol{F}e$	$[FFA\text{: } r_7, 23]$
(25)	$\boldsymbol{T}\{c\}$	$[FTB\text{: } r_5, 22]$

(26)	$\boldsymbol{T}\{\sim g\}$	$[Cut]$
(27)	$\boldsymbol{F}g$	$[BTB\text{: } 26]$
(28)	$\boldsymbol{F}\{g\}$	$[FFB\text{: } r_4, r_6, 27]$
(29)	$\boldsymbol{F}c$	$[WFN\text{: } 28]$

(30)	$\boldsymbol{F}\{\sim g\}$	$[Cut]$
(31)	$\boldsymbol{T}g$	$[BFB\text{: } 30]$
(32)	$\boldsymbol{T}\{g\}$	$[FTB\text{: } r_4, r_6, 31]$
(33)	$\boldsymbol{F}f$	$[FFA\text{: } r_8, 30]$
(34)	$\boldsymbol{T}\{\sim a, \sim f\}$	$[FTB\text{: } r_9, 17, 33]$

$\mathcal{T}_{nomore^{++}}$: Example tableau

$$
\begin{array}{llllll}
(r_1) & a \leftarrow \sim b & (r_2) & b \leftarrow d, \sim a & (r_3) & c \leftarrow b, d \\
(r_4) & c \leftarrow g & (r_5) & d \leftarrow c & (r_6) & d \leftarrow g \\
(r_7) & e \leftarrow f, \sim c & (r_8) & f \leftarrow \sim g & (r_9) & g \leftarrow \sim a, \sim f
\end{array}
$$

(1)	$\boldsymbol{T}a$	[*Cut*]
(2)	$\boldsymbol{T}\{\sim b\}$	[*BTA*: r_1, 1]
(3)	$\boldsymbol{F}b$	[*BTB*: 2]
(4)	$\boldsymbol{F}\{d, \sim a\}$	[*BFA*: r_2, 3]
(5)	$\boldsymbol{F}\{\sim a, \sim f\}$	[*FFB*: r_9, 1]
(6)	$\boldsymbol{F}g$	[*FFA*: r_9, 5]
(7)	$\boldsymbol{T}\{\sim g\}$	[*FTB*: r_8, 6]
(8)	$\boldsymbol{T}f$	[*FTA*: r_8, 7]
(9)	$\boldsymbol{F}\{b, d\}$	[*FFB*: r_3, 3]
(10)	$\boldsymbol{F}\{g\}$	[*FFB*: r_4, r_6, 6]
(11)	$\boldsymbol{F}c$	[*FFA*: r_3, r_4, 9, 10]
(12)	$\boldsymbol{F}\{c\}$	[*FFB*: r_5, 11]
(13)	$\boldsymbol{F}d$	[*FFA*: r_5, r_6, 10, 12]
(14)	$\boldsymbol{T}\{f, \sim c\}$	[*FTB*: r_7, 8, 11]
(15)	$\boldsymbol{T}e$	[*FTA*: r_7, 14]

(16)	$\boldsymbol{F}a$	[*Cut*]
(17)	$\boldsymbol{F}\{\sim b\}$	[*BFA*: r_1, 16]
(18)	$\boldsymbol{T}b$	[*BFB*: 17]
(19)	$\boldsymbol{T}\{d, \sim a\}$	[*BTA*: r_2, 18]
(20)	$\boldsymbol{T}d$	[*BTB*: 19]
(21)	$\boldsymbol{T}\{b, d\}$	[*FTB*: r_3, 18, 20]
(22)	$\boldsymbol{T}c$	[*FTA*: r_3, 21]
(23)	$\boldsymbol{F}\{f, \sim c\}$	[*FFB*: r_7, 22]
(24)	$\boldsymbol{F}e$	[*FFA*: r_7, 23]
(25)	$\boldsymbol{T}\{c\}$	[*FTB*: r_5, 22]

(26)	$\boldsymbol{T}\{\sim g\}$	[*Cut*]
(27)	$\boldsymbol{F}g$	[*BTB*: 26]
(28)	$\boldsymbol{F}\{g\}$	[*FFB*: r_4, r_6, 27]
(29)	$\boldsymbol{F}c$	[*WFN*: 28]

(30)	$\boldsymbol{F}\{\sim g\}$	[*Cut*]
(31)	$\boldsymbol{T}g$	[*BFB*: 30]
(32)	$\boldsymbol{T}\{g\}$	[*FTB*: r_4, r_6, 31]
(33)	$\boldsymbol{F}f$	[*FFA*: r_8, 30]
(34)	$\boldsymbol{T}\{\sim a, \sim f\}$	[*FTB*: r_9, 16, 33]

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