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Involutions in the Group of Automorphisms of an Algebraically Closed Field

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1. INTRODUCTION

This paper is concerned with the group of automorphisms of algebraically closed fields. It is well known that the nontrivial subgroups of finite order in the group of automorphisms of an algebraically closed field A are subgroups of order 2, i.e., subgroups which are generated by an involution (cf. [2]). In [3], R. Baer has shown the existence of involutions in the group of automorphisms if A has characteristic zero. In the following article, A stands for an algebraically closed field of characteristic zero.

We are interested in the number of involutions in the automorphism group of A, and in particular, in the number of conjugacy classes of involutions. In [1, 3], it was shown that the number of conjugacy classes depends on the transcendency degree of A over the prime field $\mathbf{Q}$; specifically, the involutions of an algebraically closed field A are conjugate with each other if and only if A is algebraic over $\mathbf{Q}$.

2. NOTATION

$\mathbf{Q}$	the field of rational numbers
$\mathbf{F}$	the field of algebraic numbers
$\mathbf{R}$	the field of real numbers
$\mathbf{C}$	the field of complex numbers
$A \subset B$	A is a proper subfield of B
i	solution of $x^2+1=0$
$\lvert T\rvert$	cardinality of the set T
Aut A	automorphism group of the field A

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Fix σ	fixed field of the automorphism σ
$\sigma\vert_A$	restriction of the automorphism σ to A
$[A:B]_T$	transcendency degree of A over B

3. Conjugacy Classes of Involutions in the Automorphism Group of an Algebraically Closed Field with Positive Transcendency Degree over Q

Dieudonné has shown that if the transcendency degree of A over $\mathbf{Q}$ is finite and equal to n, then there are at least $n+1$ conjugacy classes of involutions in Aut A; further, if the transcendency degree is infinite, then there are an infinite number of conjugacy classes of involutions in Aut A (cf. [4]).

We prove the following stronger result:

Theorem 1. *For an algebraically closed field A the following statements are equivalent*:

(a) *A is of positive transcendency degree over* $\mathbf{Q}$.

(b) *There are infinitely many conjugacy classes of involutions in* Aut A.

Proof. From Baer's result, we only have to prove that there exist infinitely many conjugacy classes of involutions whenever A has positive transcendency degree over $\mathbf{Q}$.

The basic idea is to construct maximal subfields which remain fixed under an involution of A instead of the involutions themselves; i.e., we are looking for real closed fields R with $R(i)=A$.

Let T be a transcendency base for A over its prime subfield $\mathbf{Q}$. We consider an arbitrary ordering of $\mathbf{Q}(T)$, and define R to be the real closure of $\mathbf{Q}(T)$. Then R is real closed, and, because of its transcendency degree, we have $R(i)=A$. Thus we have only to define suitable orderings on $\mathbf{Q}(T)$ so that the corresponding real closures in pairs are nonisomorphic.

Accordingly to our assumption, there is an element $t\in T$. Hence we have the following disjunctive decomposition: $T=\{t\}\cup T'$, where $t\notin T'$. Let $x\in\mathbf{R}$ be transcendental over $\mathbf{Q}$; then $\mathbf{Q}(x)$ is an Archimedean subfield of $\mathbf{R}$. The isomorphism $\mathbf{Q}(t)\cong\mathbf{Q}(x)$, defined by $t\mapsto x$, induces an Archimedean ordering on $\mathbf{Q}(t)$.

We now order $\mathbf{Q}(T)=\mathbf{Q}(t)(T')$ lexicographically over $\mathbf{Q}(t)$, and denote the real closure of $\mathbf{Q}(T)$ by R_x.

Next we show that, for algebraically independent elements $x, y\in\mathbf{R}$, the corresponding real closures R_x and R_y are not isomorphic. Otherwise, the maximal Archimedean subfields M_x and M_y of R_x and R_y would be isomorphic: From the construction of R_x and R_y, it follows easily that

$$[M_x : \mathbf{Q}]_T = [M_y : \mathbf{Q}]_T = 1.$$

Since $M_x \cong M_y$, it follows that M_x and M_y are isomorphic to a subfield $B \subseteq \mathbf{R}$ for which $[B : \mathbf{Q}]_T = 1$. The isomorphisms are given by $\phi_x : M_x \to B$ and $\phi_y : M_y \to B$. Since $\mathbf{Q}(t) \cong \mathbf{Q}(x)$ and $\mathbf{Q}(t) \cong \phi_x(\mathbf{Q}(t))$ are both order isomorphisms, the subfields $\mathbf{Q}(x)$ and $\phi_x(\mathbf{Q}(t))$ of $\mathbf{R}$ are order isomorphic, and hence equal; i.e., $\mathbf{Q}(x) = \phi_x(\mathbf{Q}(t)) \subseteq B$. Analogously it follows that $\mathbf{Q}(y) = \phi_y(\mathbf{Q}(t)) \subseteq B$. This implies that $\mathbf{Q}(x, y) \subseteq B$ and hence $[B : \mathbf{Q}]_T \geqslant 2$, which contradicts our assumption. Therefore M_x and M_y are not isomorphic, and hence R_x and R_y are also not isomorphic.

Since $[\mathbf{R} : \mathbf{Q}]_T = 2^{\aleph_0}$ there are infinitely many nonisomorphic real closed fields R_x. Consequently, we have infinitely many conjugacy classes of involutions, which proves the theorem.

4. Examples of Involutions

In case of the complex numbers, we have a well known example of an involution, namely the involution with fixed field $\mathbf{R}$. In this section we give further examples of involutions of algebraically closed fields.

We first discuss whether every involution of a subfield of A extends to an involution of A. For example, we examine the involution $\phi : \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}(\sqrt{2})$ given by $\sqrt{2} \mapsto -\sqrt{2}$. We can extend ϕ to an automorphism of $\mathbf{C}$ (cf. [5]). However, every extension has infinite order: Any extension $\Phi \in \operatorname{Aut} \mathbf{C}$ of ϕ of finite order would be an involution, the corresponding fixed field R would be real closed, and $R(i) = \mathbf{C}$. In particular, $\Phi(i) = -i$ which implies $\Phi(\sqrt{2}\, i) = \Phi(\sqrt{2}) \cdot \Phi(i) = \sqrt{2}\, i$, i.e., $\sqrt{2}\, i \in R$ and hence $\mathbf{Q}(\sqrt{2}\, i) \subset R$. Since -1 is a sum of squares of elements of $\mathbf{Q}$ $(\sqrt{2}\, i)$, the subfield $\mathbf{Q}(\sqrt{2}\, i)$ is not formally real. But R is real closed and hence formally real. This is a contradiction.

A sufficient condition to obtain an extension of order 2 is that the initial subfield be algebraically closed:

Lemma 2. *Let A be an algebraically closed field. Then every involution of an algebraically closed subfield of A extends to an involution of A.*

Proof. Let K be an algebraically closed subfield of A and let τ be an involution of K with fixed field P. From Artin and Schreier's result, it follows that P is real closed and $P(i) = K$. For a transcendency base T of A over P we define R as the real closure of $P(T)$, and $R(i) = A$. The involution σ of A which is defined by $\sigma(a+bi)=a-bi$ with $a, b \in R$ is obviously an extension of τ to A.

In the case of $\mathbf{C}$ and the subfield $\mathbf{F}$ of algebraic numbers, we obtain the following result:

LEMMA 3. *Let τ be an involution of* $\mathbf{F}$. *Then in each conjugacy class $\bar{\sigma}$ of an involution σ of* $\operatorname{Aut} \mathbf{C}$ *there is an extension of τ to* $\mathbf{C}$.

Proof. Let σ be an involution of $\mathbf{C}$. Then $\sigma|_{\mathbf{F}}$ is an involution of $\mathbf{F}$, and we use P to denote its fixed field. Due to Baer all involutions of $\mathbf{F}$ are conjugate in $\operatorname{Aut} \mathbf{F}$. Hence the corresponding fixed fields are isomorphic. Consequently, P and the fixed field B of τ are isomorphic. Let $\phi: B \to P$ denote this isomorphism; then ϕ extends to an automorphism Φ of $\mathbf{C}$.

We show that $\Phi^{-1}\sigma\Phi$ is an extension of τ. In order to prove this, it is sufficient to show that $\operatorname{Fix} \Phi^{-1}\sigma\Phi|_{\mathbf{F}} = B$. Let $x \in \operatorname{Fix} \Phi^{-1}\sigma\Phi|_{\mathbf{F}}$, i.e., $\sigma\Phi(x) = \Phi(x)$. Since the restriction of Φ to F is an automorphism of $\mathbf{F}$, it follows that $\Phi(x)$ is an element of $\operatorname{Fix} \sigma|_{\mathbf{F}} = P$, and therefore $x \in \Phi^{-1}(P) = \phi^{-1}(P) = B$. Conversely, let $x \in B$. Then $\Phi(x) = \phi(x) \in P$, and hence $\Phi^{-1}\sigma(\Phi(x)) = \Phi^{-1}\Phi(x) = x$; i.e., $x \in \operatorname{Fix} \Phi^{-1}\sigma\Phi|_{\mathbf{F}}$.

The following lemma provides an alternative mean of obtaining involutions with special properties.

LEMMA 4. *Let σ be an involution of the algebraically closed field A. Then, for each pair of algebraically independent numbers $x, y \in A$, there exists an involution which is conjugate with σ, and permutes x and y.*

Proof. Let $x, y \in A$ be algebraically independent over $\mathbf{Q}$, hence $[A:\mathbf{Q}]_T \geqslant 2$. The desired involutions will be obtained as extensions of a bijective mapping between two transcendency bases of A over $\mathbf{Q}$.

Since A over R is algebraic, $R := \operatorname{Fix} \sigma$ contains a transcendency base T of A over $\mathbf{Q}$. Because $[A:\mathbf{Q}]_T \geqslant 2$, we can find two distinct elements $a_0, b_0 \in T$. We denote $z_0 = a_0 + i \cdot b_0$, and therefore $\sigma(z_0) = a_0 - i \cdot b_0$. The subset $\{z_0, \sigma(z_0)\} \subseteq A$ is also algebraically closed since

$$[\mathbf{Q}(z_0, \sigma(z_0)) : \mathbf{Q}]_T = [\mathbf{Q}(a_0, b_0) : \mathbf{Q}]_T = 2.$$

We complete $\{z_0, \sigma(z_0)\}$ to a transcendency base $D = \{z_j \mid j \in I\}$ of A over $\mathbf{Q}$ with $z_{j_0} = z_0$, $z_{j_1} = \sigma(z_0)$, and also $\{x, y\}$ to a transcendency base $S = \{x_j \mid j \in I\}$ of A over $\mathbf{Q}$ with $x_{j_0} = x$, $x_{j_1} = y$. The bijective mapping $D \to S$ given by $z_j \mapsto x_j$ defines an isomorphism $\mathbf{Q}(D) \to \mathbf{Q}(S)$ which extends to an automorphism $\phi \in \operatorname{Aut} A$. The conjugate involution $\tau := \phi\sigma\phi^{-1}$ permutes x and y:

$$\tau(x) = \phi\sigma\phi^{-1}(x_{j_0}) = \phi\sigma(z_{j_0}) = \phi(z_{j_1}) = x_{j_1} = y.$$

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