The Word Problem for Finitary Automaton Groups

Maximilian Kotowsky[1] **Jan Philipp Wächter**[2,3]

[1]Universität Stuttgart, Institut für Formale Methoden der Informaik

[2]Politecnico di Milano, Dipartimento di Matematica

[3]funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 492814705

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The Word Problem of a Group

- Consider a group G generated by a finite set Q
  i. e. every element $g \in G$ can be written as $g = q_1^{\delta_1} \dots q_\ell^{\delta_\ell}$ with $q_i \in Q$, $\delta_i \in \{-1, 1\}$
- The word problem of G is the decision problem

  Constant: the group G generated by Q
  Input: a word $\boldsymbol{q} \in (Q^{\pm 1})^*$
  Question: is $\boldsymbol{q} = \mathbb{1}$ in G?

- ...as a formal language: $\mathrm{WP}_Q(G) = \{\boldsymbol{q} \in (Q^{\pm 1})^* \mid \boldsymbol{q} = \mathbb{1} \text{ in } G\}$

Fact (Anisimov 1971)

G *is finite* $\iff$ $\mathrm{WP}_Q(G)$ *is regular*

The Uniform Word Problem for Groups

But: We can also consider the group as part of the input!

Definition

The uniform word problem for groups is the decision problem

Input: a group G generated by Q and
a word $\boldsymbol{q} \in (Q^{\pm 1})^*$

Question: is $\boldsymbol{q} = \mathbb{1}$ in G?

- Problem: How can we give a group as an input to an algorithm?
- Typically: using a finite presentation $G = \langle Q \mid r_1 = \mathbb{1}, \ldots, r_k = \mathbb{1} \rangle$ "finitely presented"
- Today: We only consider finite groups! Possible: $Q = G$
- Possible descriptions: Cayley tables, Cayley graphs, matrices, permutations, ...

Some Known Results: Upper Bounds

Fact

The word problem for groups given as Cayley tables

Input: *a Cayley table* $G \times G \to G, (g, h) \mapsto gh$ *of a finite group* G *and group elements* $g_1, \ldots, g_n \in G$

Question: *is* $g_1 \cdot \ldots \cdot g_n = \mathbb{1}$*?*

is in LogSpace.

Theorem (Lipton, Zalcstein 1977/Simon 1979)

The word problem of a finitely generated linear group

Constant: $G \leq \mathrm{GL}(d, \mathbb{F})$

Input: *matrices* $M_1, \ldots, M_n \in G$

Question: *is* $M_1 \cdot \ldots \cdot M_n$ *the identity matrix?*

is in LogSpace.

Some Known Results: Lower Bounds

Theorem (Cook, McKenzie 1987)

The problem

Input: *permutations* $\pi_1, \ldots, \pi_\ell$ *in cycle notation*
Output: *the product* $\pi_1 \ldots \pi_\ell$ *in cycle notation*

is complete for functional LogSpace.

Theorem (Barrington 1986)

The word problem $\mathrm{WP}(A_5)$ *of the group of even permutations over* $\{a_1, \ldots, a_5\}$ *is* NC^1*-complete.* ← Boolean circuits, bounded fan-in, $\mathcal{O}(\log n)$ depth; $\mathrm{NC}^1 \subseteq$ LogSpace

In fact: this holds for any non-solvable finite group!

This yields: The uniform word problem for any group presentation (allowing A_5) is NC^1-hard!

Presenting Groups Using Automata

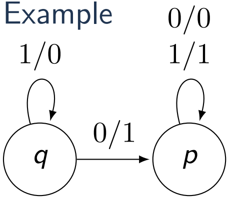
Automata

- In this setting, a $\mathscr{G}$-automaton is a
  - finite-state,
  - letter-to-letter

  transducer
  - without final or initial states

  which is
  - complete,
  - deterministic and
  - invertible.

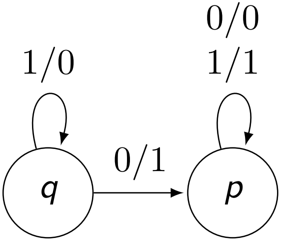
Example



State Actions

- Idea: every state q induces a bijection $\Sigma^* \to \Sigma^*$ mapping input to output words

Example



- p induces the identity function
- $$q \xrightarrow{0/1} p \xrightarrow{0/0} p \xrightarrow{0/0} p \qquad q \circ 000 = 100$$
  $$q \xrightarrow{1/0} q \xrightarrow{0/1} p \xrightarrow{0/0} p \qquad q \circ 100 = 010$$
  $$q \circ 010 = 110$$

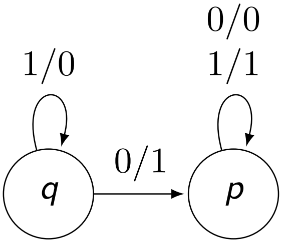
  $\rightsquigarrow$ q increments (reverse) binary representation (least significant bit first)

Automaton Groups

Such a group is an automaton group.

- A $\mathscr{G}$-automaton $\mathcal{T}$ with state set Q generates a group $\mathscr{G}(\mathcal{T})$:
  it is the closure under composition of the bijections induced by the states and their inverses.

Example



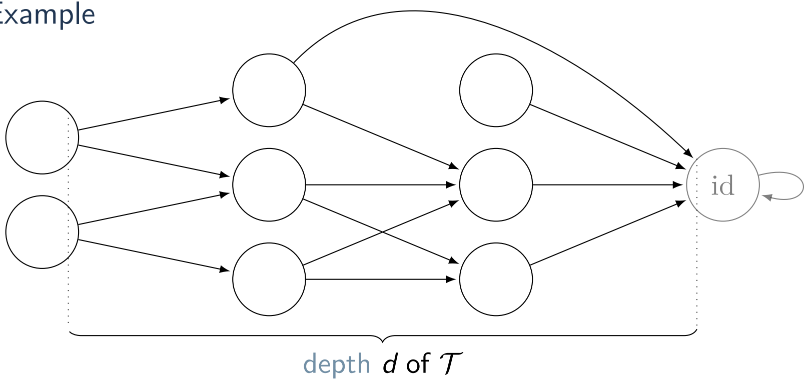
- p: identity
- q: increment, q^{-1}: decrement
- $qp = pq = q$ in $\mathscr{G}(\mathcal{T})$
- $qq \circ 000 = q \circ 100 = 010$
- q^n: "add n", q^{-n}: "subtract n"

$$\mathscr{G}(\mathcal{T}) = F(q) \simeq \mathbb{Z}$$

Finitary Automaton Groups as Finite Groups

A finitary automaton has no cycles except for self-loops at the identity state
⇝ it is a labeled directed acyclic graph

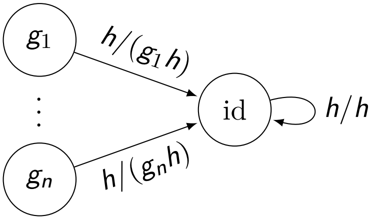
Example



Effectively all functions are $\Sigma^d \to \Sigma^d$
⇝ all finitary automaton groups are finite

Finite Groups as Finitary Automaton Groups

An arbitrary finite group $G = \{\mathrm{id}, g_1, \ldots, g_n\}$ is generated by the finitary $\mathscr{G}$-automaton



$h \in G$

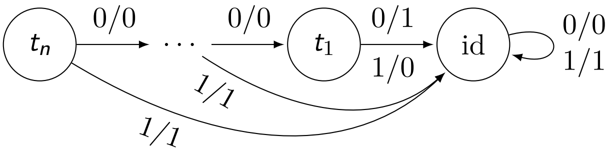
Fact

G is finite $\iff$ G is a finitary automaton group

Why Automata?

Because: the presentation using automata is powerful

- General case: Many groups with interesting properties are automaton groups
  For Example: Grigorchuk's group, which is not finitely presented.
  $\rightsquigarrow$ finite automata can encode groups without traditional finite presentations
- For finite groups: We can achieve a doubly exponential compression
  For Example:



Automorphism group of the regular binary tree of depth n

$|\mathcal{T}| = n + 1$

$\mathscr{G}(\mathcal{T}) = \operatorname{Aut} B_n$

$\implies |\mathscr{G}(\mathcal{T})| = 2^{2^n - 1}$

The Uniform Word Problem for Finitary Automaton Groups

Theorem (Kotowsky, W.)

The uniform word problem for finitary automaton groups

Input: a *finitary* $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
$\boldsymbol{q} \in (Q^{\pm 1})^*$

Question: is $\boldsymbol{q} \circ u = u$ for all $u \in \Sigma^*$ (i. e. $\boldsymbol{q} = \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$)?

is coNP*-complete.*

…it is PSpace-complete for general automaton groups
W., Weiß (2020)

Proof (complement is in NP).

- For the depth $d < |Q|$, we have:

$$\boldsymbol{q} \xrightarrow[v]{u} \mathrm{id}^{|\boldsymbol{q}|} \quad \text{for all } u \in \Sigma^{\geq d}.$$

- $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T}) \implies \exists u \in \Sigma^d : \boldsymbol{q} \circ u \neq u$
- Algorithm: "guess & check"
  - Guess witness u with $|u| < |Q|$ (in time $|Q|$).
  - Check $\boldsymbol{q} \circ u \neq u$ (in time $\approx |Q| \cdot |\boldsymbol{q}|$).

Barrington's Idea (1986)

A_5: Group of even permutations over $\{a_1, \ldots, a_5\}$

It's a logical conjunction!

Fact

$\sigma^\alpha = \alpha^{-1}\sigma\alpha$

There are $\sigma, \alpha, \beta \in A_5$ *with* $\sigma \neq \mathrm{id}$ *and* $\sigma = [\sigma^\beta, \sigma^\alpha]$.

$[h, g] = h^{-1}g^{-1}hg$

Definition (Balanced Commutator)

$$B[\boldsymbol{q}_1] = \boldsymbol{q}_1$$

$$B[\boldsymbol{q}_t, \ldots, \boldsymbol{q}_1] = \left[B[\boldsymbol{q}_t, \ldots, \boldsymbol{q}_{\lfloor \frac{t}{2} \rfloor + 1}]^\beta,\ B[\boldsymbol{q}_{\lfloor \frac{t}{2} \rfloor}, \ldots, \boldsymbol{q}_1]^\alpha\right]$$

Proposition

$g_1, \ldots, g_t \in \{\sigma, \mathrm{id}\}$

$$B[g_t, \ldots, g_1] = \begin{cases} \sigma & \forall i : g_i = \sigma \\ \mathrm{id} & \text{otherwise} \end{cases}$$

Proposition

$B[\boldsymbol{q}_t, \ldots, \boldsymbol{q}_1]$ can be computed in LogSpace.

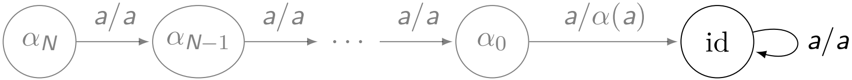
Proof (complement is NP-hard)

- We reduce $3\mathrm{SAT}$

  Input: boolean formula $\varphi = \bigwedge_{k=1}^{K} C_k$ with
  $C_k = (\neg)X_{n_{k,3}} \vee (\neg)X_{n_{k,2}} \vee (\neg)X_{n_{k,1}}$ over variables $\mathbb{X} = \{X_1, \dots, X_N\}$

  Question: $\exists \mathcal{A} : \mathbb{X} \to \mathbb{B} : \mathcal{A} \models \varphi$?

  to the complement of the word problem.
- We need a map $\varphi \mapsto (\mathcal{T}, \boldsymbol{q})$ in logarithmic space s. t. φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$
- alphabet: $\Sigma = \{a_1, \dots, a_5\} \ni \bot, \top$ $\quad \langle \mathcal{A} \rangle \in \{\bot, \top\}^N$: encoding of $\mathcal{A}$
- technical states:

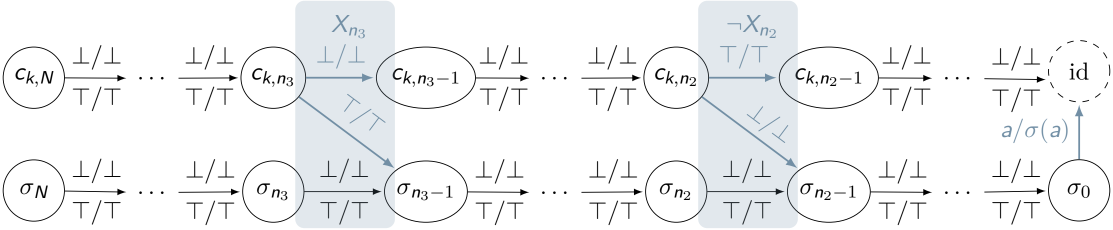


for all $a \in \Sigma$

- same for β

Proof (continued)

- Important part: Example: $C_k = X_{n_3} \vee \neg X_{n_2} \vee X_{n_1}$ (w. l. o. g.: $n_3 < n_2 < n_1$)



missing transition go to id with b/b

- Invariant for $w \in \Sigma^N$: $c_k = c_{k,N} \overset{w}{\underset{w}{\dashv\!\!\!\vdash}} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \text{id} & \text{otherwise} \end{cases}$

Proof (continued further)

Invariant for $w \in \Sigma^N$:

$$c_k \overset{w}{\underset{w}{\longrightarrow}} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \mathrm{id} & \text{otherwise} \end{cases}$$

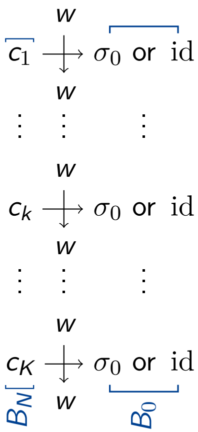
Goal:

φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Set $\boldsymbol{q} = B_N[c_K, \dots, c_1]$

Convention: B_n uses α_n and β_n instead of α and β

Let $w \in \Sigma^N$.

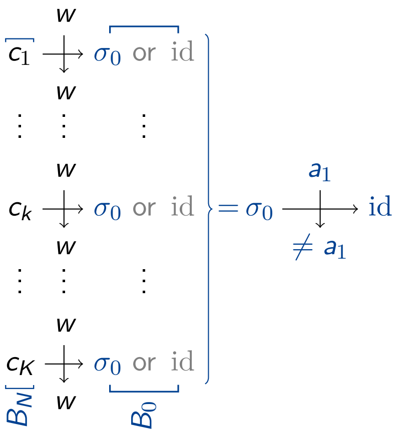


Proof (continued further)

Invariant for $w \in \Sigma^N$:

$$c_k \overset{w}{\underset{w}{\longrightarrow}} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \mathrm{id} & \text{otherwise} \end{cases}$$

Let $w = \langle \mathcal{A} \rangle$ for $\mathcal{A} \models \varphi$.



Goal:

φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Set $\boldsymbol{q} = B_N[c_K, \ldots, c_1] \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Convention: B_n uses α_n and β_n instead of α and β

Proof (continued further)

Invariant for $w \in \Sigma^N$:

$$c_k \xrightarrow[w]{w} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \text{id} & \text{otherwise} \end{cases}$$

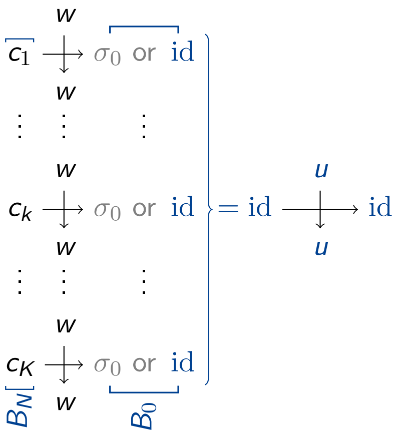
Goal:

φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Set $\boldsymbol{q} = B_N[c_K, \ldots, c_1]$

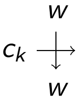
Convention: B_n uses α_n and β_n instead of α and β

Let $w = \langle \mathcal{A} \rangle$ for $\mathcal{A} \not\models \varphi$.



Proof (continued further)

Invariant for $w \in \Sigma^N$:



$$\begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \mathrm{id} & \text{otherwise} \end{cases}$$

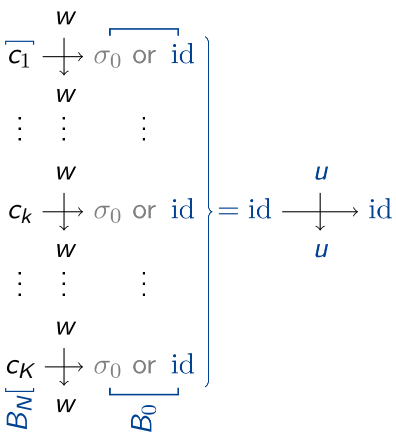
Goal:

φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Set $\boldsymbol{q} = B_N[c_K, \dots, c_1]$

Convention: B_n uses α_n and β_n instead of α and β

Let $w \notin \{\bot, \top\}^N$.



The Uniform Compressed Word Problem for Finitary Automaton Groups

Theorem (Kotowsky, W.)

The uniform compressed word problem for finitary automaton groups

Input: a finitary $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
a straight-line program encoding $\boldsymbol{q} \in (Q^{\pm 1})^*$

Question: is $\boldsymbol{q} = \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$?

is PSPACE-complete.

a context-free grammar generating a single word

…it is ExpSpace-complete for general automaton groups
W., Weiß (2020)

- We prove this using a similar reduction form QBF.
- However: One may also finitely approximate various other groups with PSPACE-complete compressed word problem Bartholdi, Figelius, Lohrey, Weiß (2020)

Thank you!