The Word Problem for Finitary Automaton Groups

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Fact (Anisimov 1971)

G is finite $\iff$ $\mathrm{WP}_Q(G)$ is regular

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- Possible descriptions: Cayley tables, Cayley graphs, matrices, permutations, ...

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Fact

The word problem for groups given as Cayley tables

Input: *a Cayley table* $G \times G \to G, (g, h) \mapsto gh$ *of a finite group* G *and group elements* $g_1, \dots, g_n \in G$

Question: *is* $g_1 \cdot \ldots \cdot g_n = \mathbb{1}$*?*

is in LogSpace.

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Theorem (Lipton, Zalcstein 1977/Simon 1979)

The word problem of a finitely generated linear group

Constant: $G \leq \mathrm{GL}(d, \mathbb{F})$

Input: matrices $M_1, \dots, M_n \in G$

Question: is $M_1 \cdot \ldots \cdot M_n$ the identity matrix?

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Theorem (Cook, McKenzie 1987)

The problem

Input: *permutations* $\pi_1, \ldots, \pi_\ell$ *in cycle notation*

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In fact: this holds for any non-solvable finite group!
This yields: The uniform word problem for any group presentation (allowing A_5) is NC^1-hard!

Presenting Groups Using Automata

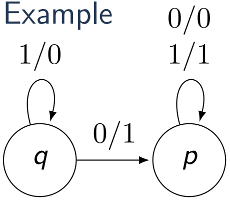
Automata

- In this setting, a $\mathscr{G}$-automaton is a
  - finite-state,
  - letter-to-letter

  transducer
  - without final or initial states

  which is
  - complete,
  - deterministic and
  - invertible.

Example



State Actions

- Idea: every state q induces a bijection $\Sigma^* \to \Sigma^*$ mapping input to output words

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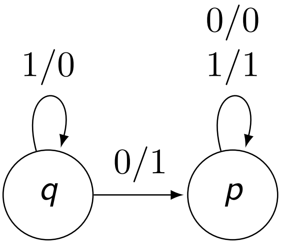

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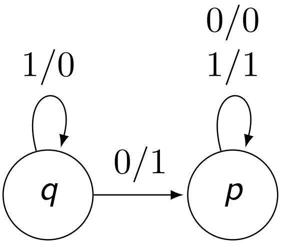

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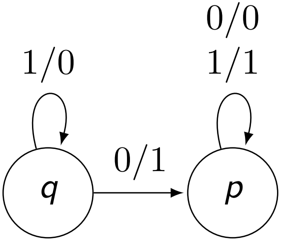

- p induces the identity function
- $$q \xrightarrow[1]{0} p \xrightarrow[0]{0} p \xrightarrow[0]{0} p \qquad q \circ 000 = 100$$

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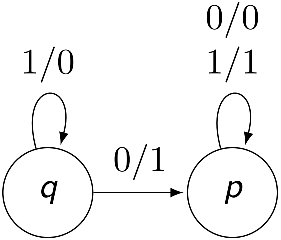

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  $$q \circ 000 = 100$$
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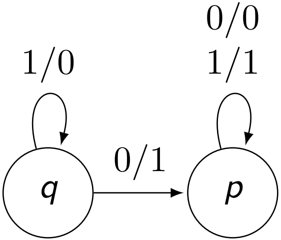

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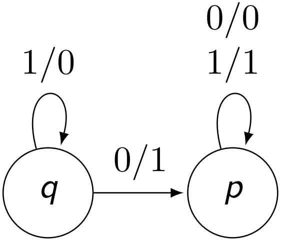

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  $\rightsquigarrow$ q increments (reverse) binary representation (least significant bit first)

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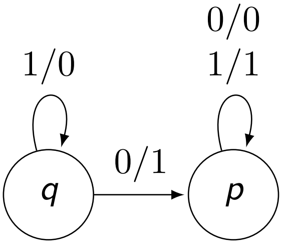

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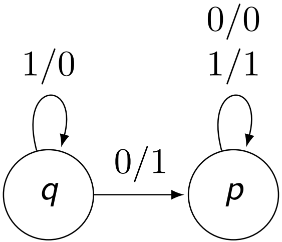

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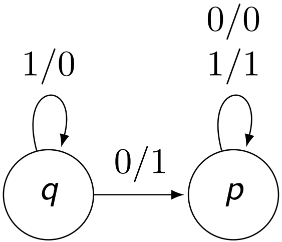

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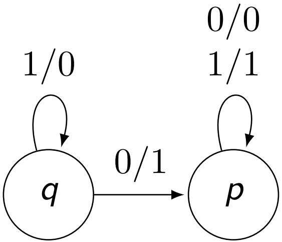

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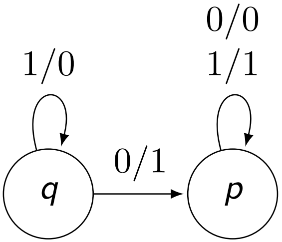

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$$\mathscr{G}(\mathcal{T}) = F(q) \simeq \mathbb{Z}$$

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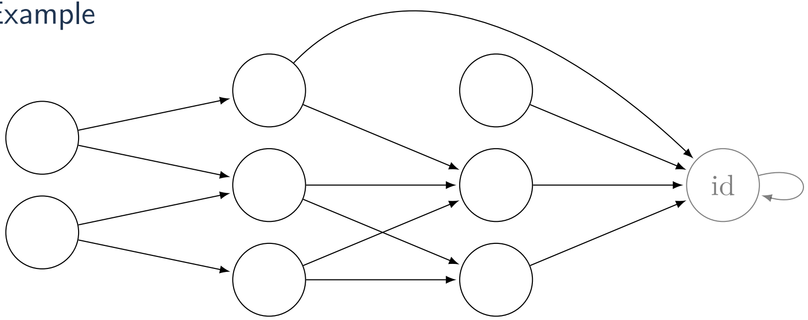

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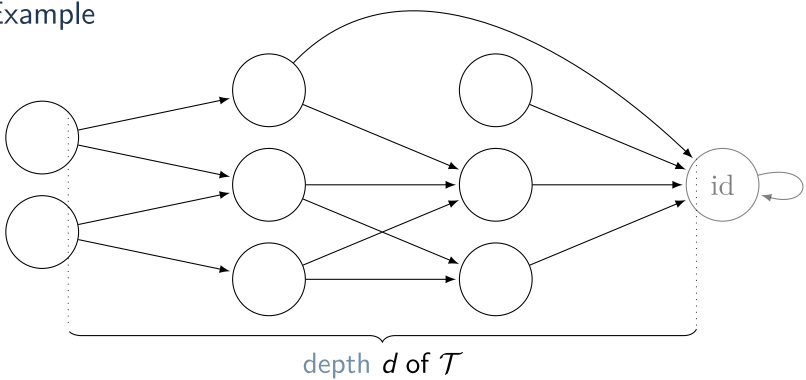

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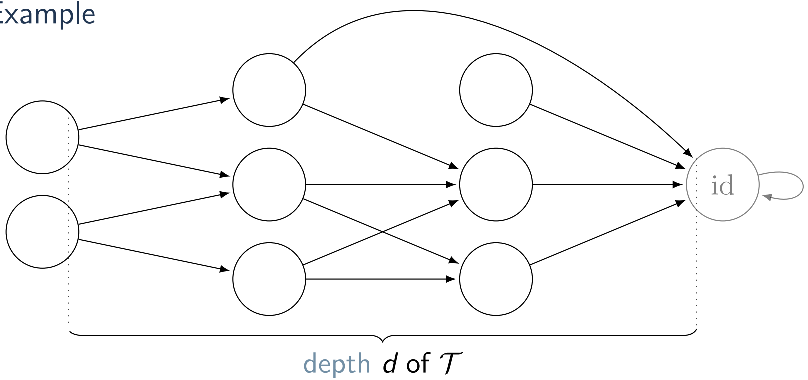

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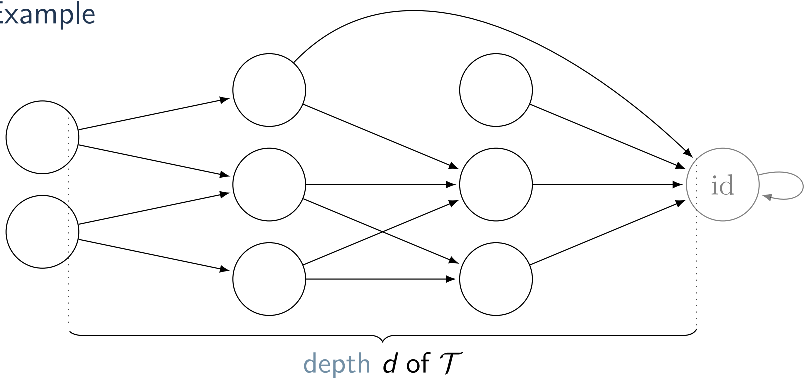

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⇝ all finitary automaton groups are finite

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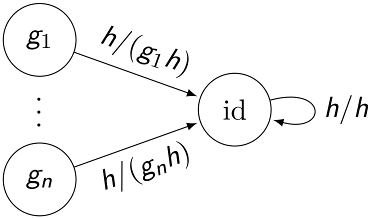

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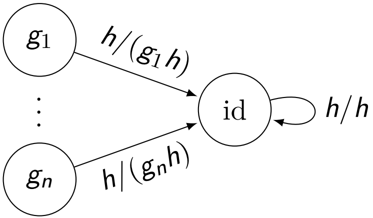

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Fact

G *is finite* $\iff$ G *is a finitary automaton group*

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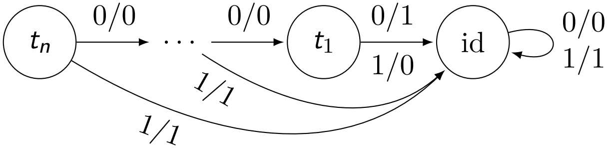

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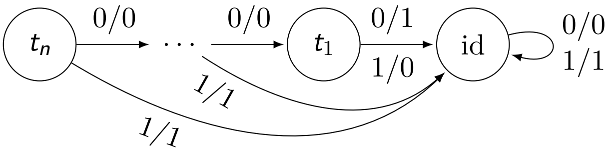

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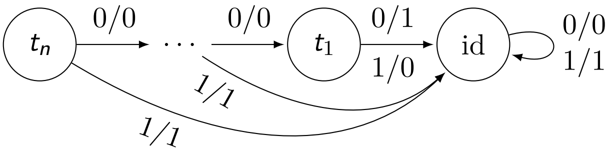

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$$|\mathcal{T}| = n + 1$$
$$\mathscr{G}(\mathcal{T}) = \operatorname{Aut} B_n$$
$$\implies |\mathscr{G}(\mathcal{T})| = 2^{2^n - 1}$$

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Theorem (Kotowsky, W.)

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…it is PSpace-complete for general automaton groups
W., Weiß (2020)

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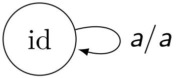

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- same for β

Proof (continued)

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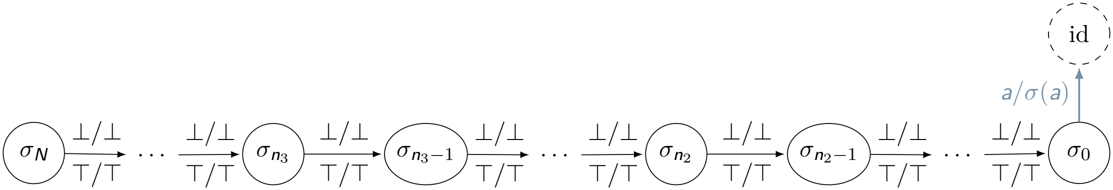

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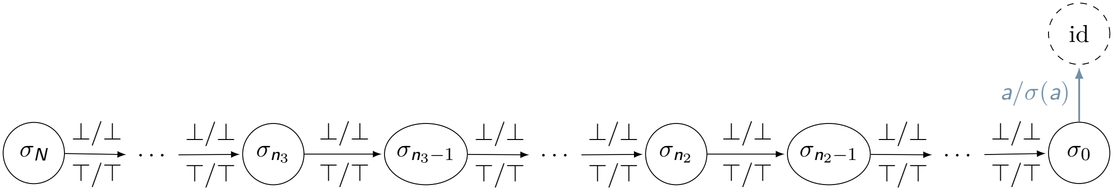

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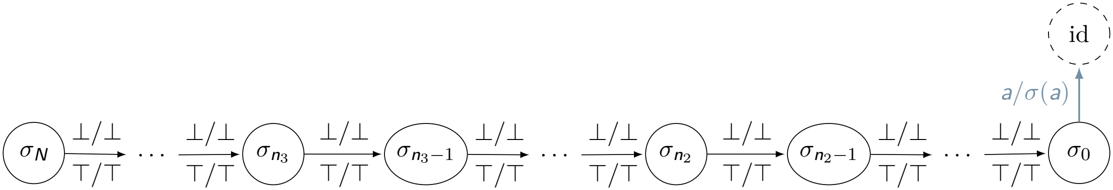

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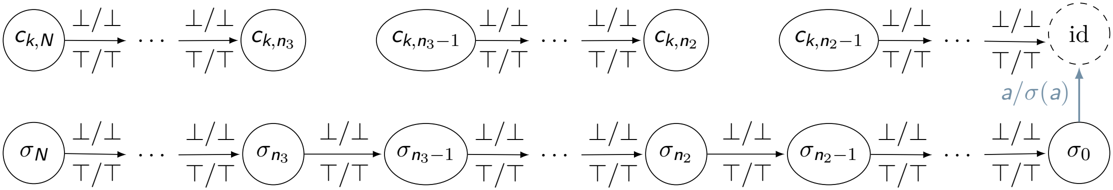

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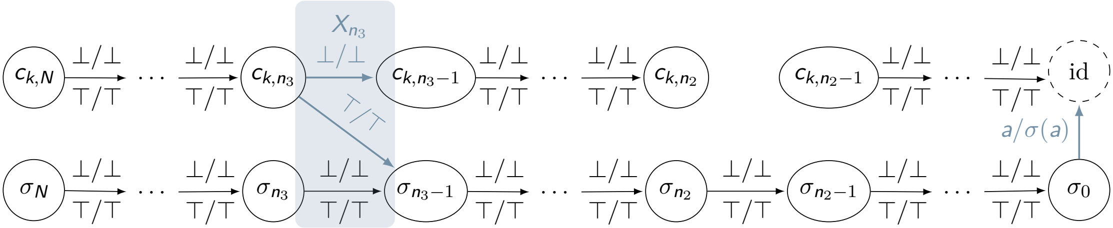

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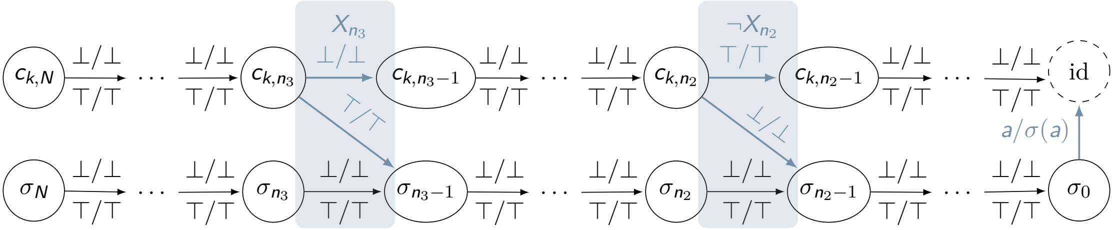

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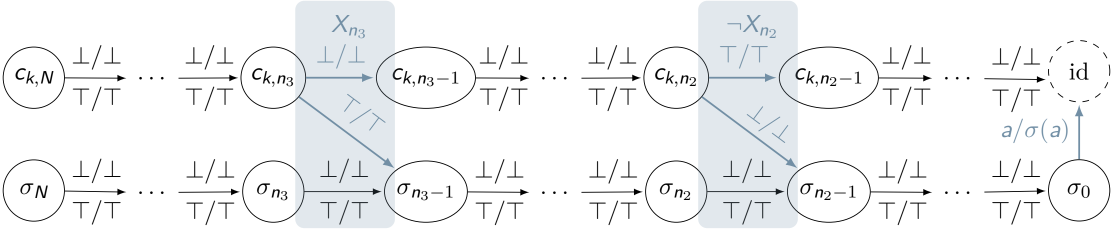

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- Invariant for $w \in \Sigma^N$: $c_k = c_{k,N} \overset{w}{\underset{w}{\longrightarrow}} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \mathrm{id} & \text{otherwise} \end{cases}$

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Let $w \in \Sigma^N$.

$$\begin{tikzcd} & w \arrow[d, -] & \\ c_1 \arrow[r, -] & {} \arrow[r] \arrow[d] & \sigma_0 \text{ or } \mathrm{id} \\ & w & \\ \vdots & \vdots & \vdots \\ & w \arrow[d, -] & \\ c_k \arrow[r, -] & {} \arrow[r] \arrow[d] & \sigma_0 \text{ or } \mathrm{id} \\ & w & \\ \vdots & \vdots & \vdots \\ & w \arrow[d, -] & \\ c_K \arrow[r, -] & {} \arrow[r] \arrow[d] & \sigma_0 \text{ or } \mathrm{id} \\ & w & \end{tikzcd}$$

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$$\left.\begin{array}{ccc} & w & \\ c_1 & \longrightarrow & \sigma_0 \text{ or } \mathrm{id} \\ & w & \\ \vdots & \vdots & \vdots \\ & w & \\ c_k & \longrightarrow & \sigma_0 \text{ or } \mathrm{id} \\ & w & \\ \vdots & \vdots & \vdots \\ & w & \\ c_K & \longrightarrow & \sigma_0 \text{ or } \mathrm{id} \\ & w & \end{array}\right\} = \begin{array}{c} a_1 \\ \sigma_0 \longrightarrow \mathrm{id} \\ \neq a_1 \end{array}$$

$B_N[$ $\quad B_0$

Proof (continued further)

Invariant for $w \in \Sigma^N$:

$$c_k \overset{w}{\underset{w}{+}} \begin{cases} \sigma_0 & \text{if } w = \langle \mathcal{A} \rangle, \mathcal{A} \models C_k \\ \mathrm{id} & \text{otherwise} \end{cases}$$

Goal:

φ is satisfiable $\iff$ $\boldsymbol{q} \neq \mathbb{1}$ in $\mathscr{G}(\mathcal{T})$

Set $\boldsymbol{q} = B_N[c_K, \dots, c_1]$

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The Uniform Compressed Word Problem for Finitary Automaton Groups

The uniform compressed word problem for finitary automaton groups

Input: a finitary $\mathscr{G}$-automaton $\mathcal{T} = (Q, \Sigma, \delta)$
a straight-line program encoding $\boldsymbol{q} \in (Q^{\pm 1})^*$

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- However: One may also finitely approximate various other groups with PSPACE-complete compressed word problem Bartholdi, Figelius, Lohrey, Weiß (2020)

Thank you!