# Separating Words Problem on Groups

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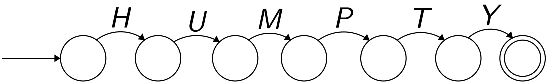

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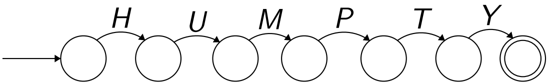


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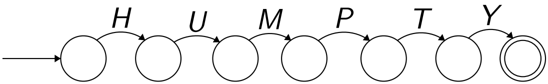


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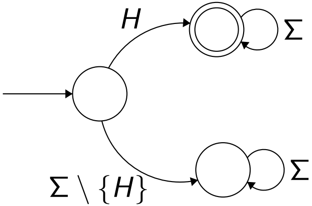

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- Separating Words Problem is NP-Complete [BKSS17]

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**Question:** What if the automaton is restricted?

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- Robson [Rob89] :For any two words of length $n$, we can construct a permuting automaton with $O(\sqrt{n})$ states that separates them.
- Each permuting automaton is associated with a subgroup of $S_n$.
- Motivated by this : we define the separating words problem over groups.

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the group $\mathbb{Z}_p$ with prime $p = O(\log n)$, with $\phi(1) = 1$, $\phi(0) = 0$ (or vice versa), will separate $w$ and $x$.

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- Yes! Group associated with Robson's Permuting Automaton.
- The group is a subgroup of $Sym\left(\sqrt{n}\right)$.

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A class of groups $\mathcal{G}$ is said to be *universal* if for any two words $w, x \in \Sigma^*$, there exists a group $G \in \mathcal{G}$ for which a separating substitution map exists such that the yields of the words under the map are distinct.

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Question : Which classes of groups are *universal*?

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  We show that SepGroupWords is NP-Complete

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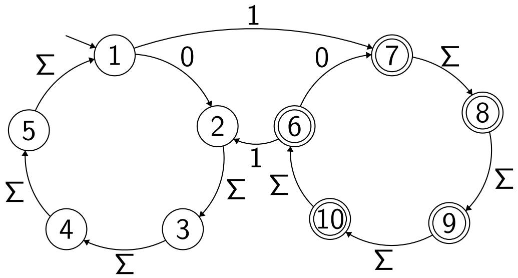


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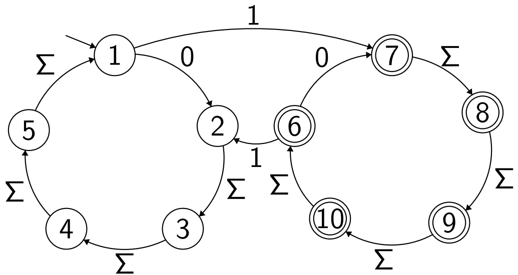


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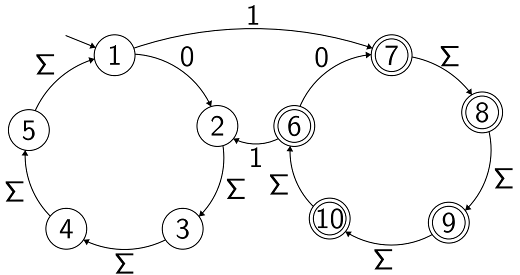


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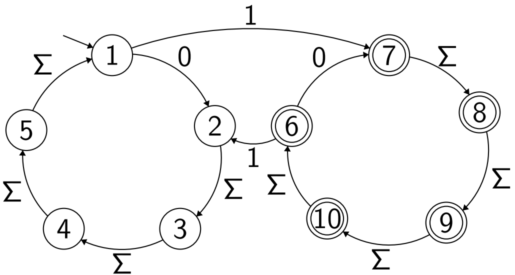


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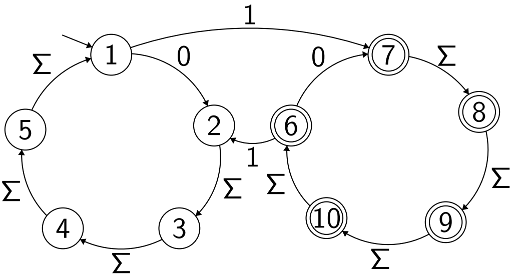


Robson's group $G_m = \langle g, h \rangle$ where,

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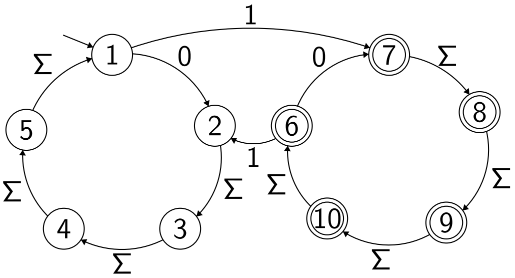


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We know $|G_m| = |H_m| * [G_m : H_m]$, $H_m$ is the commutator subgroup of $G_m$ and $[G_m : H_m]$ is the index (equal to the number of left cosets).

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2. $[G_m : H_m] = 2m$

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**Lemma**

*Consider a Robson's group $G_m$. Let $H_m$ be the commutator subgroup of $G_m$. Consider $S = \{(1, m+1)(m-i+1, 2m-i+1) | \forall i \in [m-1]\}$. Then $H_m = \langle S \rangle$*

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**Lemma**

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**Lemma (from group theory)**

*For any group $G$, $[G : H]$ where $H$ is a commutator subgroup of $G$ is number of group homomorphisms $\phi : G \longrightarrow \mathbb{C}^{\times}$.*

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Thus, we have the following constraints:

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- Number of distinct solutions to the equation is $2m$.
- There are $2m$ many one dimensional representations of $G_m$

# Estimating the Size of Robson's Group

Size of Robson's Group $G_m$ with commutator subgroup $H_m$ is

$$\begin{aligned}|G_m| &= |H_m| * [G_m : H_m] \\ &= 2^{m-1} * 2m \\ &= m2^m\end{aligned}$$

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**Theorem**

*For any $w, x \in \Sigma^*$, with $|w| = |x| = n$, there is a group of size $O\left(\sqrt{n}2^{\sqrt{n}}\right)$ that separates them.*

# Our Results

- **Size Bounds :** $\forall w, x \in \{0,1\}^n$, a group of size $O(\sqrt{n}2^{\sqrt{n}})$ separating $w$ and $x$. ✓
- **Universality :**
  - Class of solvable groups, nilpotent groups, in particular, p-groups, are universal.
  - Class of Abelian groups and dihedral groups are **not** universal.
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- **Computational Version :** SepGroupWords Problem - Given two words $w, x \in \Sigma^*$, a set of permutations $S$ that generates a group $G \leq S_n$ and a function $\phi : \Sigma \to S$, with the guarantee that yield($w$)$\neq$ yield($x$) and an integer $k$, check if there is an automaton of size $k$ which separates $w$ and $x$.

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**Lemma**

*Given a pair of distinct words $w, x \in \{0,1\}^n$ there exists $0 \leq i < m \leq [2n]$ and $m = 2^k$ for some $k \in \mathbb{N}$ such that this 2-group separates $w, x$.*

- $p$-groups are universal.
- Nilpotent groups are universal.

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  Consider the following cases:

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  Case 2: Suppose at least one of the elements gets mapped to some $s_j$. Then $(s_j)^{2k} = (s_j s_j)^k = (r_0)^k = r_0$. Then $yield(w) = h^{2k} r_0 = r_0 h^{2k} = yield(x)$, $h \in G \setminus \{s_j\}$.

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# Sufficiency Conditions for Non-Universality

**Lemma**

*Let $\mathcal{G}$ be a family of groups such that $\forall G \in \mathcal{G}, \forall g \in G$ order of $g \leq k$ for some finite $k \in \mathbb{N}$. Then $\mathcal{G}$ is not universal.*

**Lemma**

*If $\exists c\ \forall m$ such that $H_m \leq G_m$ is a maximal Abelian subgroup of $G_m$ and $\mathrm{lcm}(G_m \setminus H_m) \leq c$ then $\{G_m\}_{m \geq 0}$ is not universal*

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# Separating Group Words Problem

**Theorem**

$\textsc{SepGroupWords}$ *is* NP*-complete.*

- By reducing Separating Words Problem to $\textsc{SepGroupWords}$.
- $\textsc{SepWordProblem}(w, x, k) \iff \textsc{SepGroupWords}(w, x, S, \phi, k)$
  - $\exists$ Robson's permuting automaton $O(\sqrt{n})$.
  - $S$ is the set generators of Robson's group.
  - Given $w, x \in \{0, 1\}^n$, computing $S$ can be done in poly($n$) time.

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- Bound on the size of $p$-groups used in Group Separation.

# Thank You!

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