TOWARDS A THEORY OF REPRESENTATIONS

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1. Introduction

Definitions of Type 2 computability, i.e. computability on sets with cardinality not greater than that of the continuum, have been given in several ways (see e.g. Egli/Constable [1], Scott [6], Rogers [5]). Most of these definitions are equivalent or at least dependent from each other but there is no generally accepted approach as in the case of computability on denumerable sets.

This paper presents the concept of representations as a foundation for a unified Type 2 computability theory. Its basic idea is that real world computers cannot operate on abstract elements of a set M but only on names. We have chosen the set \( \mathcal{IF} \) of sequences of natural numbers as a standard set of names and have defined computability on \( \mathcal{IF} \) explicitly (see Weihrauch [7]). Computability on other sets M then can be derived from computability on \( \mathcal{IF} \) by means of representations i.e. (partial) mappings from \( \mathcal{IF} \) onto M. The same computability theory could be obtained by using sets like \( \mathcal{P}_\omega \) as standard sets but considering the applications of our theory \( \mathcal{IF} \) seems to be the better one. For example infinite objects are often defined by sequences of finite objects (e.g. Cauchy sequences, chains etc.) and not by sets of finite objects. Furthermore the computation model for functions on \( \mathcal{IF} \) is easy to understand and allows studying computational complexity.

Computable functions turn out to be continuous in general and in most cases functions which are not computable are not even continuous. Hence topological considerations are fundamental for Type 2 theory and continuity w.r.t. representations will also be studied. Therefore two versions of Type 2 theory are developed simultaneously, a topological (t-) and a computable (c-) one.

We assume the reader is familiar with ordinary recursion theory and some basic properties of numberings (Mal'cev[4], Rogers [5], Ershov [2]).

By \( \mathbb{N} \) we denote the set of all natural numbers by \( \mathcal{W}(\mathbb{N}) \) the set of all finite words over \( \mathbb{N} \). \( \varepsilon \) is the empty word and \( \text{lg}(w) \) is the length of the word \( w \).

If \( w \in \mathcal{W}(\mathbb{N}) \) and \( w = x_0 x_1 \ldots x_n \) (where \( x_i \in \mathbb{N} \)) then we define \( w(i) = x_i \) for \( 0 \leq i \leq n \). By \( f: A \rightarrow B \) (with dotted arrow) we denote a partial function from A to B, where "partial" means \( \text{dom } f \subseteq A \). As usual we write \( <i_1, \ldots, i_n> \) instead of \( \pi(n)(i_1, \ldots, i_n) \) where \( \pi(n): \mathbb{N}^n \rightarrow \mathbb{N} \) is Cantor's bijection.

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\( \varphi \) denotes the standard numbering of the unary partial recursive functions.

Define \( \mathcal{I} := \{ p : \mathbb{N} \to \mathbb{N} \} \) and \( \mathbb{B} := W(\mathbb{N}) \cup \mathcal{I} \). For \( a, b, \in \mathbb{B} \) define
\[
 a \leq b : \iff a \text{ is a prefix of } b. \quad \text{For } p \in \mathcal{I} \text{ and } i \in \mathbb{N} \text{ let}
\]
\[
 p[i] := p(0) \ldots p(i-1) \in W(\mathbb{N}) \quad \text{and conversely for } v \in W(\mathbb{N}) \text{ let}
\]
\[
 [v] := \{ p \in \mathcal{I} | v \leq p \}. \quad \text{On } \mathbb{B} \text{ we consider the topology defined by the basis}
\]
\[
 \{ [v]|v \in W(\mathbb{N}) \} \quad \text{where } 0 := \{ b \in \mathbb{B} | v \leq b \}. \quad \text{The induced topology on } \mathcal{I} \text{ is the well known Baire's topology. On } \mathbb{N} \text{ we consider the discrete topology.}
\]

2. Type 2 Recursion theory

Unlike to ordinary (Type 1) recursion theory for Type 2 recursion theory there is no generally accepted formalism. We now outline a unified approach which is formally similar to the Type 1 formalism. More details can be found in Weihrauch's paper \([7]\). We start with the definition of a standard representation \( \Psi \) of \( [\mathcal{I} \to \mathbb{B}] \), the set of continuous functions from \( \mathcal{I} \) to \( \mathbb{B} \).

From \( \Psi \) we derive representations of certain continuous functions from \( \mathcal{I} \) to \( \mathcal{I} \) and from \( \mathcal{I} \) to \( \mathbb{N} \). The construction of \( \Psi \) rests on the following property. Let \( \gamma : W(\mathbb{N}) \to W(\mathbb{N}) \) be isotone (w.r.t. \( \leq \)). Then the function \( \bar{\gamma} : \mathcal{I} \to \mathbb{B} \), defined by
\[
 \bar{\gamma}(p) := \sup \{ \gamma(w) | w \leq p \},
\]
is continuous. And for any continuous function \( \Gamma : \mathcal{I} \to \mathbb{B} \), \( \Gamma = \bar{\gamma} \) for some isotone \( \gamma : W(\mathbb{N}) \to W(\mathbb{N}) \). The function \( \gamma \) specifies, how from prefixes of \( p \in \mathcal{I} \) sufficiently many prefixes of \( \Gamma(p) = \bar{\gamma}(p) \) can be determined. A function \( \Gamma : \mathcal{I} \to \mathbb{B} \) is called computable, iff \( \Gamma = \bar{\gamma} \) for some computable function \( \gamma \). The computable functions \( \Gamma : \mathcal{I} \to \mathbb{B} \) can easily be characterized by oracle Turing machines which on input \( p \in \mathcal{I} \) from time to time read a value \( p(i) \) and from time to time write one of the values \( q(0), q(1), \ldots \) (in this order) of the result \( q \in \mathbb{B} \). For transforming \( n \)-ary functions on \( \mathcal{I} \) to unary ones, the following tupling functions \( \Pi^{(n)} : \mathcal{I} \to \mathcal{I} \) are used:
\[
 \Pi^{(1)}(p) := p,
\]
\[
 \Pi^{(n+1)}(p_1, \ldots, p_{n+1})(x) = (\Pi^{(n)}(p_1, \ldots, p_n))(i) \text{ if } x = 2i, (\Pi^{(n+1)}(p_1, \ldots, p_{n+1})(i) \text{ if } x = 2i+1).
\]

Also \( \omega \)-ary tupling is possible: \( \Pi^{(\omega)}(p_0, p_1, \ldots) \in \Pi^{(n)}(p_0, p_1, \ldots) \). The functions \( \Pi^{(n)} \) and \( \Pi^{(\omega)} \) are homeomorphisms w.r.t. the product topologies. The projections of their inverses are computable.

The definition of \( \bar{\gamma} \) is effective in the following sense. There is a computable (by an oracle Turing machine) operator \( \Gamma_{\mathcal{I}} : \mathcal{I} \to \mathbb{B} \) with the following property. On input \( p, q \) it determines \( \bar{\gamma}(q) \) if \( \gamma := \nu q \nu q \nu N \nu N \) is isotone, \( \Gamma(q) \) for some continuous \( \Gamma : \mathcal{I} \to \mathbb{B} \) otherwise. Then by \( \Psi(p)(q) := \Gamma_{\mathcal{I}} <p, q> \) a representation \( \Psi : \mathcal{I} \to [\mathcal{I} \to \mathbb{B}] \) of the continuous functions from \( \mathcal{I} \) to \( \mathbb{B} \) is defined, which satisfies the "universal Turing machine theorem" and the "smn-theorem".
THEOREM

1. \( \psi_p(q) = \Gamma_u <p,q> \) for some computable \( \Gamma_u \in [\mathbb{N} \to \mathbb{N}] \).

2. \( \psi_p(q,r) = \tilde{\psi}_p <p,q>(r) \) for some computable \( \Sigma \in [\mathbb{N} \to \mathbb{N}] \) with range \( \Sigma \subseteq \mathbb{N} \).

Notice that \( \Gamma_u \) and \( \Sigma \) are not only continuous but even computable. Similar to Type 1 recursion theory the utm-theorem and the smn-theorem characterize the representation \( \psi \) uniquely up to (computable) equivalence (see Chapter 3). More interesting than \( \psi \) itself are two representations derived from \( \psi \).

DEFINITION

1. Define a set \([\mathbb{N} \to \mathbb{N}] \) of partial functions from \( \mathbb{N} \) to \( \mathbb{N} \) and a representation \( \chi: \mathbb{N} \to [\mathbb{N} \to \mathbb{N}] \) by: \( \chi_p(q) := \chi(p)(q) := \text{div if } \psi_p(q) = \epsilon \in \mathbb{N}, \) otherwise.

2. Define a set \([\mathbb{N} \to \mathbb{N}] \) of partial functions from \( \mathbb{N} \) to \( \mathbb{N} \) and a representation \( \tilde{\psi}: \mathbb{N} \to [\mathbb{N} \to \mathbb{N}] \) by: \( \tilde{\psi}_p(q) := \tilde{\psi}(p)(q) := \psi_p(q) \) if \( \psi_p(q) \in \mathbb{N} \), otherwise.

This definition extends well known concepts of computable operators and functionals to a uniform topological description, where the elements computable w.r.t. a given representation are those with computable names. The functions from \([\mathbb{N} \to \mathbb{N}] \) and from \([\mathbb{N} \to \mathbb{N}] \) have natural domains (c.f. domains of partial recursive functions). But the set of domains is sufficiently rich such that any continuous function is essentially considered.

Theorem

1. \([\mathbb{N} \to \mathbb{N}] \) is the set of all continuous functions \( \Sigma: \mathbb{N} \to \mathbb{N} \) such that \( \text{dom}(\Sigma) \) is open. For any continuous function \( \Gamma: \mathbb{N} \to \mathbb{N} \) there is some \( \Sigma \in [\mathbb{N} \to \mathbb{N}] \) which extends \( \Gamma \).

2. A valid statement is obtained by substituting "\( \mathbb{N} \)" by "\( \mathbb{N} \)" and "open" by "Gδ-subset" in (1).

Also the representations \( \chi \) and \( \tilde{\psi} \) satisfy the utm- and the smn-theorem. This leads to a rich theory for continuity and of computability which is formally similar to Type 1 recursion theory. From the above theorem we conclude that by \( \omega'(p) := \text{dom}(\chi_p) \) a representation \( \omega' \) of the open subsets of \( \mathbb{N} \) is defined, which corresponds to the numbering \( i \to \text{dom}(\chi_p(i)) \) of the r.e. subsets of \( \mathbb{N} \). We call a subset \( A \subseteq \mathbb{N} \) t-open (c-open) iff \( A = \omega'(p) \) for some (computable) \( p \). \( A \) is t-clopen (c-clopen), iff \( A \) and \( \mathbb{N} \setminus A \) are t-open (c-open). The t-open (c-open) sets are exactly the projections of the t-clopen (c-clopen) sets. The self applicability and the halting problem of \( \chi \) can be formulated. They are c-open, not t-clopen, c-complete and c-productive. Also effective inseparability can be defined. The sets \( \{p|\chi_p(p) = 0\} \) and \( \{p|\chi_p(p) = 1\} \) are c-effectively inseparable. This property can be used in the study of precomplete representations.
Many other properties can be proved easily but more questions are still unsolved in this theory of continuity and computability on $\text{IF}$.

3. Theory of representations

In order to define computability and constructivity on a set $M$ with cardinality not greater than that of the continuum, we represent $M$ by a surjective mapping $\delta: \text{IF} \rightarrow M$, called representation of $M$. Some examples for representations are the enumeration representation $\delta_M: \text{IF} \rightarrow P_\omega$ with $\delta_M(p) := \{i| i+1 \in \text{range } p\}$, the representation $\delta_{\text{cf}}$ of $P_\omega$ by characteristic functions with $\delta_{\text{cf}}(p) := \{i| p(i) = 0\}$, and the representations $\psi: \text{IF} \rightarrow [\text{IF} \rightarrow \text{IN}], \psi: \text{IF} \rightarrow [\text{IF} \rightarrow \text{IF}],$

$\chi: \text{IF} \rightarrow [\text{IF} \rightarrow \text{IN}], \omega': \text{IF} \rightarrow \{x \in \text{IF}|x \text{ is open}\}$ introduced in chapter 2.

Effectivity properties of theorems, functions, sets, predicates etc. can be expressed by effectivity of correspondences (i.e. multivalued functions) which are triples $f = (M,M',P)$ where $P \subseteq M \times M'$.

Definition

Let $\delta, \delta'$ be representations of $M$ resp. $M'$ and let $f = (M,M',P)$ be a correspondence. $f$ is called weakly $(\delta, \delta') - t - (c-) effective$ iff there is some (computable) $\Gamma \in [\text{IF} \rightarrow \text{IF}]$ such that

$$(\delta q, \delta' \Gamma q) \in P \quad \text{for all } q \in \delta^{-1} \text{dom}(f).$$

$f$ is called $(\delta, \delta') - t - (c-) effective$, iff in addition $\Gamma(q)$ is undefined for all $q \in \delta^{-1}(M \setminus \text{dom } f)$.

$(\delta, \nu)$-effectivity of a correspondence $f = (M,S,P)$ where $\nu$ is a numbering of $S$ is defined accordingly using $[\text{IF} \rightarrow \text{IN}]$ instead of $[\text{IF} \rightarrow \text{IF}]$. For convenience we shall say "continuous" instead of "$t$-effective" and "computable" instead of "$c$-effective".

Since a partial function is a single valued correspondence the above definition is applicable to functions. A subset $A \subseteq M$ can either be characterized as the domain of a partial function or by its characteristic function.

A set $A \subseteq M$ is called $\delta - (c-) open$ iff $d_A := (M, \text{IN}, A \times \text{IN})$ is $(\delta, \text{id}_\text{IN}) - t - (c-) effective$. $A$ is called $\delta - (c-) clopen$ iff $c_A := (M, \text{IN}, \{(x,0)| x \in A\} \cup \{(y,1)| y \in M \setminus A\})$ is $(\delta, \text{id}_\text{IN}) - t - (c-) effective$. Usually we say "provable" instead of "$c$-open" and "decidable" instead of "$c$-open".

The $\delta$-effectivity on $M$ strongly depends on the representation $\delta$. Consider the two questions whether complementation on $P_\omega$ is effective and whether countable union on $P_\omega$ is effective. There is no absolute answer but only one relative to the considered representation: Complementation is $(\delta_{\text{cf}}, \delta_{\text{cf}})$-computable but not even $(M, M)$-continuous, countable union is computable w.r.t. $M$ but not even weakly continuous w.r.t. $\delta_{\text{cf}}$ (use $\Pi^0_2$ for formalization). This difference can be explained using the intuitive concept of finitely (or continuously) accessible
(f.a.) information. Every true information \( n \in \mathcal{E}(p) \) is f.a. from \( p \), no true information \( n \notin \mathcal{E}(p) \) is f.a. from \( p \). But every true information \( n \in \delta_{cf}(p) \) or \( m \notin \delta_{cf}(p) \) is f.a. from \( p \).

Representations may be changed in a certain way without changing the induced effectiveness. For any two representations \( \delta, \delta' \) of \( M \) resp. \( M' \) define
\[
\delta \leq_{t} \delta' \iff M \subseteq M' \quad \text{and} \quad \text{id}_{M,M'} \text{ is } (\delta, \delta') \text{-effective},
\]
\[
\delta \equiv_{t} \delta' \iff \delta \leq_{t} \delta' \text{ and } \delta' \leq_{t} \delta.
\]

C-reducibility \( (\leq_{c}) \) and C-equivalence \( (\equiv_{c}) \) is defined accordingly. It is easy to show that \( \delta_{cf} \leq_{c} \mathcal{E} \) and that \( \mathcal{E} \) and \( \delta_{cf} \) are not t-equivalent.

Since effective functions are closed under composition two representations are t- (c-) equivalent if and only if the define the same continuity (computability) theory.

**Theorem**

Let \( \delta, \delta' \) be representations of \( M \). Then (1),(2) and (3) are equivalent.

1. \( \delta \leq_{t} \delta' \)
2. For any representation \( \delta_{1} : \mathcal{E} \rightarrow M_{1} \) and any correspondence \( f = (M, M_{1}, P) \):
   \( f \) (weakly) \( (\delta_{1}, \delta) \) t-effective \( \Rightarrow \) \( f \) (weakly) \( (\delta_{1}, \delta') \) t-effective
3. For any representation \( \delta_{2} : \mathcal{E} \rightarrow M_{2} \) and any correspondence \( g = (M_{2}, M, P) \):
   \( g \) (weakly) \( (\delta, \delta_{2}) \) t-effective \( \Rightarrow \) \( g \) (weakly) \( (\delta, \delta_{2}) \) t-effective.

Every representation \( \delta : \mathcal{E} \rightarrow M \) induces a topology \( \tau_{\delta} \) on \( M \) by \( x \in \tau_{\delta} : \iff \delta^{-1}x = A \cap \text{dom } \delta \) for some open subset \( A \subseteq \mathcal{E} \). \( \tau_{\delta} \) is called the final topology of \( \delta \) and it consists exactly of all the \( \delta \)-open subsets of \( M \). For example \( \tau_{\mathcal{E}} \), the final topology of the enumeration representation of \( \mathcal{P} \), is determined by the basis \( \{ O_{e} \mid e \subseteq \mathcal{E}, \text{ finite} \} \) where \( O_{e} := \{ x \in \mathcal{E} \mid e \subseteq x \} \). Clearly t-equivalent representations have the same final topologies but the converse does not hold in general (a counterexample is presented in [8]). If on \( M \) already a topology \( \tau \) is given then \( \tau = \tau_{\delta} \) should hold for any "reasonable" representation of \( M \). (In some special cases there might be reasons for choosing \( \tau_{\delta} + \tau \).) For separable \( T_{0} \)-spaces representations equivalent to a standard representation defined as follows seem to be the most natural ones.

**Definition**

Let \( (M, \tau) \) be a separable \( T_{0} \)-space and let \( U \) be a numbering of some basis of \( \tau \).
For \( x \in M \) let \( \epsilon_{U}(x) := \{ i \mid x \in U_{i} \} \).

A standard representation \( \delta_{u} \) of \( (M, \tau) \) is defined by
\[
\text{dom } \delta_{u} := \mathcal{E}_{u}(M) \quad \text{and} \quad \delta_{u}(p) := \epsilon_{u}^{-1}(M(p)) \text{ whenever } p \in \text{dom } \delta_{u}.
\]
A standard representation \( \delta_u \) of a separable \( T_0 \)-space has some remarkable properties:

1. \( \delta_u \) is continuous and open, especially \( \tau = \tau_{\delta_u} \).
2. For any space \((M', \tau')\) and any \( H : M \rightarrow M' \):
   \[ H \circ \delta_u \text{ is continuous } \iff H \text{ is continuous}, \]
3. \( \xi \leq \delta \) for any continuous \( \xi : \mathbb{F} \rightarrow M \).

An immediate consequence is that all the standard representations of the same space are \( t \)-equivalent and therefore the equivalence class \( \{ \delta | \delta \equiv_{t-u} \delta \} \) does not depend on the numbering \( U \). Since every representation equivalent to \( \delta_u \) induces the same continuity theory, we call a representation \( \delta \) of a separable \( T_0 \)-space \( t \)-effective (or admissible) iff \( \delta \equiv_{t-u} \delta_u \) for some standard representation \( \delta_u \). The representations \( \mathbb{M} \) and \( \delta_{cf} \) of \( \mathbb{P} \) are admissible. The decimal representation of the real numbers is not (see [8]).

For admissible representations of a space \((M, \tau)\) the final topology is identical with \( \tau \). Furthermore topological continuity and continuity w.r.t. these representations are closely related.

**Theorem**

Let \((M_i, \tau_i)\) be separable \( T_0 \)-spaces and let \( \delta_i : \mathbb{F} \rightarrow M_i \) be admissible representations \((i = 1, 2)\). Let \( F : M_1 \rightarrow M_2 \), then:

1. \( F (\tau_1, \tau_2) \)-continuous \( \iff \) \( F \) weakly \((\delta_1, \delta_2)\)-continuous,
2. \( F (\tau_1, \tau_2) \)-continuous \( \land \) \( \text{dom } F \in G_0(\tau_1) \) \( \implies \) \( F (\delta_1, \delta_2) \)-continuous.

For some representations the converse of (2) also holds (e.g. for the representation \( \rho \) of \( \mathbb{R} \) by normed Cauchy-sequences - see [7]).

There are many other aspects of representations which should be studied, for example recursion-theoretic properties, computable elements, the structure of equivalence degrees, closure properties etc. There are also natural representations the final topologies of which are not separable. See Kreitz & Weihrauch [3] for further discussion.
References


