Considerations on Default Logics

by

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Abstract

The ability to reason seems to be one of the distinguishing marks of intelligent behavior. However, the available underlying information is quite often incomplete. As a result, conclusions frequently have to be drawn in the absence of information. A formal approach to reasoning in the absence of information, or to reason by default, is given by *default logic* — a logical system developed by Raymond Reiter in [1980].

The approach taken by default logic is extensively studied and further developed in this thesis. After surveying various approaches to default reasoning, we thoroughly investigate Reiter's original approach in order to identify its properties and limitations.

The major contributions of this thesis are threefold. First, an alternative approach to default logic is developed in order to address the limitations of the original approach and subsequent variants. The resulting system is called *constrained default logic*. The approach has clear semantical foundations and remedies the problems encountered in the original approach in an arguably simpler way than other proposals. Second, we provide differing semantical characterizations for several default logics. In particular, we develop a uniform semantical framework for default logics in terms of Kripke structures. This approach provides a simple but meaningful instrument for comparing existing default logics in a unified setting. Third, we examine in detail the relationships among the variations of default logic. As a result, we obtain several criteria for the coincidence of the examined approaches. Moreover, we provide a general approach for incorporating nonmonotonic lemmas into default logics.

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The major part of the results presented in this thesis has been published as refereed papers in the proceedings of international conferences, eg. [Schaub, 1991a; Schaub, 1991b; Schaub, 1992a]. Some parts of Chapter 4 and 5, although original work of the author, appear in a joint work with James Delgrande and Ken Jackson [Delgrande *et al.*, 1992]. They provided many valuable suggestions on content and presentation of the respective parts of this thesis. I gratefully acknowledge their help. In fact, James Delgrande and Ken Jackson have independently developed in [Delgrande and Jackson, 1991] a slightly different variant of constrained default logic. Their individual contributions are explicitly indicated in the text. Chapter 6 in its entirety is joint work with Philippe Besnard [Besnard and Schaub, 1992]. I gratefully acknowledge his contribution.

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Chapter 1

Introduction

"It is reasonable to expect that the relationship between computation and mathematical logic will be as fruitful in the next century as that between physics and analysis in the past."

John McCarthy, 1963

The ability to reason is a fundamental property of any intelligent being. As a consequence, any "intelligent" computer system has to have some capacity for reasoning. Accordingly, the study of reasoning has become one of the major research topics in Intellectics.[1]

Traditionally, the task of reasoning has been accomplished by standard logical systems such as propositional and first-order predicate logic.[2] For instance, predicate logic has been shown to be sufficient to characterize axiomatic methods in mathematics. Traditional standard logics are concerned with reasoning from given premises. Hence, the addition of new premises increases the set of derivable conclusions. Therefore, standard logics are said to be *monotonic*. In particular, the premises are regarded as an entire description of a certain domain.

In real life, however, we are faced with incomplete information. In particular, human commonsense reasoning is strongly based upon the ability to draw conclusions in the absence of information. We often expect things to happen in a usual way. So, if we are asked whether something is going to happen, we normally apply general rules for prediction. Usually, these rules allow for exceptions. Hence, if we get to know more specific information, a former conclusion has sometimes to be withdrawn. For instance, it happens that we expect something to hold in a certain situation. However, we might have to retract this expectation if we find out that things happened under abnormal circumstances.

Let us illustrate this by means of the following example, which will be used throughout the introductory sections. Assume that we know a little girl called *Larissa*. Also, as everybody knows, children like ice-cream. It is a hot day and we are wondering whether she would like some ice-cream. We could represent this as follows.

"Larissa is a child" (1.1)

[1] Ie. the fields of artificial intelligence and cognitive science [Bibel, 1992].
[2] For the sake of clarity, we refer to propositional and first-order logic as *standard* logics.

"Children like ice-cream" (1.2)

Of course, it is reasonable to conclude from (1.1) and (1.2) that

"Larissa likes ice-cream". (1.3)

First of all, notice that the "rule" (1.2) is not identical to the sentence *"All children like ice-cream"* since (1.2) allows for exceptions. Indeed, there are many unforeseeable exceptions, eg. *Larissa* may have toothache, she may be sleeping, she might even be allergic to milk, etc. For instance, if somebody tells us that *Larissa's* teeth are aching these days, we tend to withdraw our former conclusion (1.3) since toothache usually destroys any predilection for ice-cream.

In fact, we have concluded (1.3) in the absence of information. In real life, however, this is the rule, not an exception. There are always gaps in our knowledge. However, we are not paralyzed by missing information and still arrive at plausible conclusions. As a result, conclusions like (1.3) are always subject to revision. In the light of evidence for the contrary, they may have to be withdrawn. Formally, the tentative nature of human commonsense reasoning is referred to as the property of *nonmonotonicity.* That is, the addition of premises may decrease the set of conclusions. Therefore, many people refer to this kind of reasoning as *nonmonotonic reasoning.*[3] In general, there are several slightly different types of nonmonotonic reasoning.[4] Here, we are dealing with patterns of the form

" ... in the absence of information to the contrary, assume ... ".

This kind of reasoning is called *default reasoning.* In this case, conclusions are somehow sanctioned *by default* and may be rejected, given subsequent information. For instance, we have adopted the conclusion (1.3) in the absence of information to the contrary, say that *Larissa* is allergic to milk. In a way, we have jumped to the conclusion that *"Larissa likes ice-cream"* without explicitly showing that she is *not* allergic to milk, that she is *not* sleeping, etc. In a sense, we have assumed by default that things happened under normal circumstances. Therefore, default reasoning can also be seen as a means to represent statements of the form *"normally, a property holds"* or *"typically, this is the case."*

In all, there seems to be two reasons for the failure of standard predicate logics for formalizing nonmonotonic reasoning. First, standard logics are monotonic. Hence, they cannot account for the tentative nature of nonmonotonic reasoning. Second, even if we could enumerate all exceptions of a default rule, we would have to refute all of the exceptions before we could apply the rule. Of course, this is impossible in practice.

We have seen the need for nonmonotonic reasoning. However, we do not want to give up the advantages of traditional logics. As a result, several approaches have been proposed during the last decade in order to extend standard logics (cf. Chapter 2). Accordingly, all of these approaches share standard predicate logic as their underlying framework.

Finally, a short remark on "intelligent" computer systems. The foremost aim of any formalism in the field of *knowledge representation and reasoning* is to give a formal description of the world or a particular domain. As regards complex systems, however, this is beyond manageability as long as only explicit knowledge is representable. In order to obtain plausible conclusions, there have to be means to represent implicit and, notably, absent knowledge. Otherwise, there is no chance of incorporating any notion of artificial intelligence into computer systems of any kind.

[3] This term was originally proposed by Minsky in [1975].

[4] For instance, Brewka distinguishes in [1991c] between *default reasoning*, *autoepistemic reasoning* (cf. Section 2.2.2), *representation conventions*, and *reasoning in the presence of inconsistent information.* However, these types mainly constitute different views on nonmonotonic reasoning and, therefore, are not substantially different.

1.1 Contributions

This thesis presents several innovations in the field of default logic. The major contributions of this thesis are as follows:

- An alternative approach to default logic is developed in order to address the limitations of the original approach and subsequent variants. The resulting system is called constrained default logic. The approach has clear semantical foundations and remedies the problems encountered in the original approach in an arguably simpler way than other proposals.

- We provide two different semantical approaches to default logics. First, a semantical characterization for constrained default logic is given which is referred to as the focused models semantics. This semantics also provides a first semantical characterization for a variant of default logic introduced in [Brewka, 1991b]. As a result, the focused models semantics supplies us with several semantical insights into the two aforementioned default logics and their properties.

  Second, a uniform semantical framework for default logics in terms of Kripke structures is developed. This semantical approach provides a simple but meaningful instrument for comparing existing default logics in a unified setting. No other semantics for any default logic offers this generality. But apart from its unique generality, the approach also remedies several difficulties encountered in previous proposals aiming at individual default logics.

- The relationships among the variations of default logic, namely default logic [Reiter, 1980], justified default logic [Łukaszewicz, 1988], cumulative default logic [Brewka, 1991b], and constrained default logic, are examined in detail. As a consequence, several results establishing criteria for the coincidence of the examined approaches are obtained.

Furthermore, a general approach to incorporate lemmas into default logics is developed in order to overcome the failure of default logic to handle lemmas stemming from default conclusions. In particular, the underlying formal property of cumulativity[5] [Gabbay, 1985; Makinson, 1989] in default logics is thoroughly investigated.

A detailed and more technical overview of this thesis and its main results is given in the next section.

1.2 Overview

This section provides a detailed overview of the remaining sections along with their respective innovations.

In Chapter 2, we survey the most common logical approaches to default reasoning in an informal way. We start with default logic in Section 2.1 and continue in Section 2.2 with a survey of the modal approaches to default reasoning. In Section 2.3, we describe circumscription as the third dominating approach to default reasoning. Finally, we sketch the similarities and differences of the various approaches.

Chapter 3 is devoted to the formal development of Reiter's original default logic. First, we formally account for the notion of an extension in Reiter's approach (in Section 3.1). In addition, we repeat some of the definitions already given in Section 2.1 in order to make the treatment self-contained. In Section 3.2, we give the basic properties of default logic. In Section 3.3, we discuss

[5]Intuitively, cumulativity stipulates that the addition of a theorem to the set of premises does not change the theory under consideration (cf. Section 3.5).

some limitations of the approach and describe subsystems which avoid these problems. One such subsystem is detailed in Section 3.4. Section 3.5 deals with the formal property of cumulativity and its most important practical impact: the capability of handling (nonmonotonic) lemmas. After a thorough discussion of the failure of cumulativity in default logic in Section 3.5.1, we introduce in Section 3.5.2 the proof-oriented notion of lemma default rules as a general approach for using and generating nonmonotonic lemmas[6] in default logics. Finally, we give in Section 3.6 a semantical characterization of default logic and reflect semantically some of the problems discussed in Section 3.3 and 3.5.

In Chapter 4, we develop a new variant of default logic which avoids several shortcomings of the original approach. In particular, we introduce the notion of a constrained extension and, therefore, call the resulting system constrained default logic. We start in Section 4.1 with a brief discussion of some limitations of Reiter's default logic in order to motivate the approach taken by constrained default logic. Section 4.2 is devoted to the formal development of constrained default logic. The approach is further elaborated in Section 4.3. There, we examine the properties of constrained default logic and show how it copes with the problems encountered in Reiter's default logic. Section 4.4 investigates the relationship between constrained default logic and its classical counterpart. Section 4.5 presents a model-theoretic semantics for constrained default logic — the so-called focused models semantics — which provides useful semantical insights into the enhancements of the underlying approach. In Section 4.6, we show that prerequisite-free default theories preserve cumulativity whenever we are reasoning skeptically.[7] This important subsystem of constrained default logic is further investigated in Section 4.7. In Section 4.8, we slightly extend constrained default logic in order to integrate a predetermined set of constraints. This system is called pre-constrained default logic and it serves as a basis for further extensions of the approach taken by constrained default logic. Section 4.9 presents another extension of constrained default logic which allows for incorporating priorities among default rules. This system is called prioritized constrained default logic. In Section 4.10, we introduce lemma default rules for constrained default logic and discuss in detail how lemma default rules deal with nonmonotonic lemmas (as the most important practical impact of cumulativity).

Chapter 5 contains an extensive study of the relationships between the various derivatives of default logic; thereby, benefiting from constrained default logic, as an instrument for comparing the examined approaches. In Section 5.2, we discuss Łukaszewicz' variant of default logic and detail its relationship to constrained default logic. Section 5.3 discusses Brewka's cumulative default logic. Aside from characterizing the relationship between cumulative and constrained default logic, we provide a first semantical characterization of Brewka's variant by means of the focused models semantics. Also, we compare Brewka's approach to restore cumulativity to default logic with that taken by lemma default rules (cf. Section 3.5.2 and 4.10). Section 5.4 deals with Poole's approach to default reasoning, which turns out to be a proper subsystem of constrained default logic. Finally, we survey the examined approaches by comparing their respective properties.

In Chapter 6, we introduce a uniform semantical framework for various default logics in terms of Kripke structures. This approach provides a simple but meaningful instrument for comparing existing default logics in a unified setting. The possible worlds semantics is introduced by means of constrained default logic. Also, it easily deals with Brewka's cumulative default logic. The semantics is then extended to Reiter's original default logic as well as Łukaszewicz' variant. Notably, the approach remedies several difficulties encountered in former proposals aiming at individual default logics. For instance, the possible worlds semantics provides the first

[6]The term 'nonmonotonic lemmas' is used in order to indicate lemmas stemming from default conclusions.
[7]See page 20.

semantical characterization of Łukaszewicz' default logic which is purely model-theoretic. Since the semantical framework is presented from the perspective of "commitment to assumptions", we also obtain a very natural modal interpretation of the notion of commitment.

Chapter 7 summarizes the contributions of this thesis and discusses some important problems and topics for further research. For continuity, all proofs of the stated theorems are given in the appendices B, C, D, and E.

1.3 Preliminaries

We assume the reader to be familiar with the basic concepts of first-order logic (cf. [Enderton, 1972; Bibel, 1987]) as well as some acquaintance with modal logics (cf. [Bowen, 1979; Chellas, 1980]). We shall be dealing with a standard first-order language $\mathcal{L}$. That is, $\mathcal{L}$ is the set of all first-order formulas which can be formed using an alphabet consisting of countably many variables $x, y, z, \ldots$; countably many function symbols $a, b, c, \ldots, f, g, h, \ldots$; countably many predicate symbols $\mathsf{P}, \mathsf{Q}, \ldots$; the usual punctuation signs, the symbols $\top$ (for verum) and $\perp$ (for falsum), and the standard logical connectives $\neg$ (for negation), $\wedge$ (for conjunction), $\vee$ (for disjunction), $\rightarrow$ (for implication), $\leftrightarrow$ (for equivalence), and quantifiers $\forall$ (for universal quantification) and $\exists$ (for existential quantification). Letters $A, B, C, \ldots$ denote propositional variables or simply atoms; Greek letters $\alpha, \beta, \gamma, \eta, \upsilon, \ldots$ are variables for arbitrary formulas; letters S, T, U, $V, \ldots$ denote sets of formulas.

We denote first-order derivability by $\vdash$ and the corresponding consequence operator by Th, that is $Th(S) = \{\alpha \mid S \vdash \alpha\}$. We denote first-order interpretations by π and first-order entailment by $\models$. The class of all models of a set of formulas S is written as $\mathrm{MOD}(S)$. That is, $\mathrm{MOD}(S) = \{\pi \mid \pi \models S\}$. Further definitions and conventions will be introduced when they occur for the first time.

Chapter 2

Logical approaches to default reasoning

This chapter surveys the most widely used logical approaches to default reasoning in an informal way. We start with Reiter's default logic in Section 2.1 and continue in Section 2.2 with an overview of the modal approaches to default reasoning. Section 2.3 deals with circumscription as the third dominating approach to default reasoning. Finally, we sketch the similarities and differences of the various approaches.

2.1 Default logic

Default logic was introduced by Reiter in [1980] as a formal account of reasoning in the absence of complete information. It has since proved to be one of the most widely-used formalizations of default reasoning. On the one hand, default logic has been employed in various areas in order to formalize different applications, eg. [Etherington and Reiter, 1983; Froidevaux, 1990; Mercer, 1988; Perrault, 1987; Reiter, 1987b]. On the other hand, it turns out that default logic subsumes other approaches, eg. circumscription [Lifschitz, 1990]. Also, it is sometimes more expressive than competing approaches. For instance, it has been shown to be superior to autoepistemic logic in formalizing logic programs with negation [Truszczyński, 1991].

In addition, default logic incorporates default reasoning into the framework of standard logic in a very natural way. Default knowledge is added to standard first-order logic by means of *default rules* as nonstandard rules of inference. These rules differ from standard inference rules in sanctioning inferences that rely upon given as well as absent knowledge. Such inferences therefore could not be made in a standard framework. Hence, default rules can be seen as rules of conjecture whose role it is to augment an underlying incomplete first-order theory.

Formally, a default rule is any expression of the form[1]

$$\frac{\alpha(\vec{x}) \,:\, \beta_1(\vec{x}), \ldots, \beta_m(\vec{x})}{\gamma(\vec{x})},$$

where $\alpha(\vec{x}), \beta_1(\vec{x}), \ldots, \beta_m(\vec{x})$ and $\gamma(\vec{x})$ are first-order formulas whose free variables are among $\vec{x} = (x_1, \ldots, x_n)$. $\alpha(\vec{x})$ is called the prerequisite, $\gamma(\vec{x})$ the consequent and the $\beta_i(\vec{x})$ are called the justifications of the default rule. If none of $\alpha(\vec{x}), \beta_i(\vec{x})$ and $\gamma(\vec{x})$ contain free variables, the default rule is said to be *closed*. Usually, open default rules are regarded as schemata and represent

[1] Observe that in the case of $m = 0$ default rules behave like standard inference rules.

all instantiations of the considered default rule.[2] Informally, a default rule is applicable if its prerequisite holds and its justifications are consistent, ie. the negations of the justifications do not hold.[3]

As an example, take the default *"typically, children like ice-cream"*. This piece of commonsense knowledge can be expressed by the default rule

$$\frac{\mathsf{child}(x)\ :\ \mathsf{likes\text{-}ice\text{-}cream}(x)}{\mathsf{likes\text{-}ice\text{-}cream}(x)} \tag{2.1}$$

which is to be interpreted as *"If x is a child and it is consistent to assume that x likes ice-cream, then infer that x likes ice-cream"*. Observe that usually default rules differ from standard inference rules in consisting of domain-specific formulas. Thus, they are not general inference schemata such as modus ponens.

A *default theory*, (D, W), consists of a set of closed first-order formulas W and a set of default rules D. The set of *facts* W is supposed to be a logically valid, but generally incomplete, description of the world. The default rules D, however, represent hypothetical or non-strict rules and, therefore, sanction plausible but not necessarily true conclusions. A default theory is said to be closed if all of its default rules are closed. For example, the fact that *"Larissa is a child"*,

$$\mathsf{child}(Larissa), \tag{2.2}$$

and the default rule given in (2.1) constitute the following default theory.

$$(D, W) = \left(\left\{\frac{\mathsf{child}(x)\ :\ \mathsf{likes\text{-}ice\text{-}cream}(x)}{\mathsf{likes\text{-}ice\text{-}cream}(x)}\right\}, \{\mathsf{child}(Larissa)\}\right) \tag{2.3}$$

Now, in the presence of the fact $\mathsf{child}(Larissa)$, the appropriately instantiated prerequisite of the default rule is trivially derivable. Also, nothing prevents us from consistently assuming $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$, so that we can infer $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$, ie. *"Larissa likes ice-cream"*.

A set of conclusions sanctioned by a given default theory is called an *extension*. Informally, an extension of the initial set of facts W is defined as the set of all formulas derivable from W using standard inference rules and all specified default rules. Reiter states in [1980, p 88] that the "*... intuitive idea which must be captured is that of a set of defaults D inducing an extension of some underlying incomplete set of first-order wffs W*". Therefore, he demands three properties to hold for an extension E:

1. E should contain our initial set of facts W, ie. $W \subseteq E$.
2. E should be deductively closed, ie. $Th(E) = E$.
3. E should contain each consequent of any applicable default rules, ie. for any default rule $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D$, if $\alpha \in E$ and $\neg\beta_1, \ldots, \neg\beta_n \notin E$ then $\gamma \in E$.

Section 3.1 describes how an extension is characterized formally. However, in the case of the default theory given in (2.3) the extension amounts to the deductive closure of the formulas $\mathsf{child}(Larissa)$ and $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$.

Since default conclusions are drawn in the absence of information, they are subject to revision. In the presence of more specific information that denies a prior consistency assumption, the

[2]In [Reiter, 1980], skolemization is used to generate all ground instances of an open default rule. In contrast, [Lifschitz, 1990] treats free variables in open default rules as genuine objects.

[3]See page 20 for an example violating this consistency condition.

conclusions have to be withdrawn. Notably, this is exactly the point where nonmonotonicity comes in. Let us illustrate this by means of the above example. Imagine, we get to know that *"Larissa's teeth are aching"*. Furthermore, we are convinced that *"having toothache destroys any predilection to ice-cream"*. We formalize this by the following axioms and add these to the facts of the default theory given in (2.3).

$$\mathsf{has\text{-}toothache}(Larissa), \tag{2.4}$$

$$\forall x.\mathsf{has\text{-}toothache}(x) \rightarrow \neg\mathsf{likes\text{-}ice\text{-}cream}(x) \tag{2.5}$$

First, recall that we have concluded from the default theory (2.3) that *"Larissa likes ice-cream"* because we were unable to derive the opposite. Now, however we derive by instantiation of (2.5) and modus ponens that *"Larissa does not like ice-cream"*. Hence, the default rule (2.1) is not applicable since the negation of its justification holds, namely $\neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$. As a result, the augmented default theory yields a different set of conclusions, among others containing $\neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$.

In general, default theories allow for more than one extension. To see this, let us weaken the above implication (2.5) and, reformulate it by means of the following default rule which expresses that *"normally, having toothache destroys any predilection to ice-cream"*.

$$\frac{\mathsf{has\text{-}toothache}(x)\ :\ \neg\mathsf{likes\text{-}ice\text{-}cream}(x)}{\neg\mathsf{likes\text{-}ice\text{-}cream}(x)} \tag{2.6}$$

Then, we obtain a default theory consisting of the axioms (2.2) and (2.4) and the default rules (2.1) and (2.6).

$$\left(\left\{\begin{array}{c}\frac{\mathsf{child}(x)\,:\,\mathsf{likes\text{-}ice\text{-}cream}(x)}{\mathsf{likes\text{-}ice\text{-}cream}(x)},\\ \frac{\mathsf{has\text{-}toothache}(x)\,:\,\neg\mathsf{likes\text{-}ice\text{-}cream}(x)}{\neg\mathsf{likes\text{-}ice\text{-}cream}(x)}\end{array}\right\},\left\{\begin{array}{c}\mathsf{child}(Larissa),\\ \mathsf{has\text{-}toothache}(Larissa)\end{array}\right\}\right) \tag{2.7}$$

Potentially, both default rules are applicable since their prerequisites hold in the case where we instantiate the variable x with *Larissa*. However, both default rules cannot contribute to the same extension since each consequent contradicts the other default rule's justification. Therefore, we obtain two extensions, one containing $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$ and another one including $\neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$.

Reiter's [1980] original idea was that "... *the purpose of default reasoning is to determine one consistent set of beliefs about the world, ie. one extension, and to reason within this extension until such time as the evidence at hand forces a revision of those beliefs, in which case a switch to a new extension may be called*". Such reasoning (ie. accepting each extension as a possible set of beliefs) is referred to as being *credulous*. Complementary to this, accepting only the intersection of all extensions as the set of consequences, is *skeptical reasoning*.

Almost all default theories discussed in the literature [Reiter, 1987a] fall into the class of *singular* default theories, ie. default theories whose default rules have only one justification.[4] Two classes of singular default theories are distinguished: *normal* and *semi-normal* default theories.

Normal default theories consist of *normal* default rules of the form[5]

$$\frac{\alpha\ :\ \gamma}{\gamma}.$$

[4]As an exception, logic programs with negation can only be captured by default theories consisting of general default rules with multiple justifications [Gelfond and Lifschitz, 1990].

[5]Henceforth, we do not indicate free variables.

That is, normal default rules are default rules whose single justification is equivalent to the consequent. As a consequence, normal default rules guarantee that once they have been applied, their justification remains consistent. Because of this, normal default theories possess numerous desirable properties which do not hold in general. We will discuss these properties in detail in Section 3.3. In addition, normal default theories capture many common patterns of reasoning (eg. the closed world assumption [Reiter, 1977]). Originally, Reiter stated in [1980] that "*... I know of no naturally occurring default which cannot be represented in this form*".

However, semi-normal default theories were introduced by Reiter and Criscuolo in [1981] in order to avoid several shortcomings induced by interacting normal default rules. Similar to normal default theories, semi-normal default theories consist of *semi-normal* default rules which are of the form

$$\frac{\alpha : \beta \wedge \gamma}{\gamma}.$$

That is, in general semi-normal default rules are default rules whose single justification implies the consequent. Although semi-normal default rules allow for more expressiveness (eg. when regarding priorities between defaults), semi-normal default theories lack many of the desired properties possessed by normal default theories (cf. Section 3.3).

So far, we have defined default logic purely by proof-theoretic means. We will semantically account for default logic in Section 3.6 and 6.3. However, the proof-theoretic and, therefore, pure syntactic treatment of default logic is very appealing as we have seen my means of our example.

2.2 Modal approaches to default reasoning

In the previous section, we saw how inference rules, namely default rules, are used in order to augment standard first-order logic. In contrast, the approaches described in this section extend the first-order language in order to represent the notions of consistency and/or belief. Therefore, they employ modal propositions but not necessarily modal logics for deriving these propositions.

The axiomatic treatment of defaults, as used in the modal approaches discussed in the present section, has the advantage of allowing for combining and nesting defaults in an arbitrary way. This may lead to a greater flexibility in modelling hypothetical knowledge, since this representation allows for formalizing domain knowledge as well as statements about it.

2.2.1 Nonmonotonic logics

In [1980], McDermott and Doyle proposed an approach called *nonmonotonic logic*. In this approach, a standard first-order language is augmented with a unary modal operator M, where a formula $\mathsf{M}\alpha$ is to be read as "α is consistent". As an example, the schema[6]

$$\mathsf{child}(x) \wedge \mathsf{M}(\mathsf{likes\text{-}ice\text{-}cream}(x)) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(x) \tag{2.8}$$

can be used to represent our default statement that *"typically, children like ice-cream"*. By adding the fact that *"Larissa is a child"*, $\mathsf{child}(Larissa)$, we obtain the analogous world description as given by means of the default theory (2.3). So far, however, there is no way to conclude that *"Larissa likes ice-cream"* since standard logics do not provide a way to derive the modal proposition $\mathsf{M}(\mathsf{likes\text{-}ice\text{-}cream}(Larissa))$.

[6] In this section, we simply deal with schemata instead of universal formulas, since quantification over modalities is a non-trivial matter (cf. [Konolige, 1989; Levesque, 1990]).

Therefore, McDermott and Doyle [1980] allow us to "derive" formulas like

$$\mathsf{M}(\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa}))$$

whenever $\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa})$ is consistent. More formally, they define an *extension*, E, obtained from certain facts, W, as

$$\mathit{Th}(W \cup \{\mathsf{M}\alpha \mid \neg\alpha \notin E\}).$$

In our example, the set of facts W contains the fact $\mathsf{child}(\mathit{Larissa})$ and the default statement given in (2.8). Since we can consistently assume $\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa})$, we obtain the modal proposition $\mathsf{M}(\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa}))$ which allows for deriving $\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa})$, ie. that *"Larissa likes ice-cream"*

Although nonmonotonic logic manages our *Larissa* example, it is too weak in general. This weakness stems mainly from the poor inferential relation between formulas with and without modality. For instance, McDermott and Doyle's system tolerates axioms that reflect an incoherent notion of consistency. That is, a set of premises containing $\mathsf{M}A$ and $\neg A$ may have a consistent extension. This amounts to asserting that A is consistent while $\neg A$ holds. This clearly violates the intuition described above.

This weakness has led to several attempts to strengthen the logic. One of them was made by McDermott himself in [1982], where he tried to close the "inferential gap" by using modal logic. Another, quite more successful, attempt was made by Moore in [1985]. This approach is described in the next section.

2.2.2 Autoepistemic logic

In order to overcome the problems encountered in McDermott and Doyle's approach [1980; 1982], Moore provides in [1985] a reconstruction of their logics which is based on the notion of *belief* rather then *consistency.* As a result, Moore proposes *autoepistemic logic* as a means for modelling an "*ideally rational agent's reasoning about his own beliefs*" [1985, p. 75]. Accordingly, he also augments a standard first-order language[7] with a unary modal operator L, where a formula $\mathsf{L}\alpha$ is to be read as "α is believed". In analogy to standard modal logic, we can roughly interpret L to be the dual operator to McDermott and Doyle's modal operator M.

For instance, our default, stating that *"children normally like ice-cream"*, can be represented by the schema (cf. (2.8))

$$\mathsf{child}(x) \wedge \neg\mathsf{L}(\neg\mathsf{likes\text{-}ice\text{-}cream}(x)) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(x) \tag{2.9}$$

However, the modal literal $\neg\mathsf{L}(\neg\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa}))$ is now interpreted as *"it is not believed that Larissa does not like ice-cream"*. As in the previous section, there is initially no way to derive this modal literal.

In contrast to McDermott and Doyle's approach, autoepistemic logic allows us to "derive" positive and negative modal literals. Given some set of facts, W, an *autoepistemic extension*, E, is given by

$$\mathit{Th}(W \cup \{\mathsf{L}\alpha \mid \alpha \in E\} \cup \{\neg\mathsf{L}\alpha \mid \alpha \notin E\}).$$

Compared to the definition in the previous section, autoepistemic extensions are additionally based upon the set $\{\mathsf{L}\alpha \mid \alpha \in E\}$. As a result, there is no consistent set of conclusions obtainable from a set of premises containing $\neg\mathsf{L}A$ and A, which caused the counterintuitive result in McDermott and Doyle's approach mentioned above.

[7] Originally, Moore dealt only with propositional logic. In [Levesque, 1990], this approach is generalized to the first-order case.

In our example, we conclude from $\mathsf{child}(Larissa)$ and (2.9) that *"Larissa likes ice-cream"*, $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$. This is because we are *"disbelieving that Larissa does not like ice-cream"* which is expressed by the modal literal $\neg\mathsf{L}(\neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa))$.

As observed by Konolige in [1987], autoepistemic logic allows for "ungrounded" beliefs, ie. conclusions drawn by means of circular chains of reasoning. For example, the set

$$\{\mathsf{L}A \rightarrow A\} \tag{2.10}$$

has two alternative autoepistemic extensions. The first one contains $\neg\mathsf{L}A$. This seems to be intuitively appropriate since an agent who believes $\mathsf{L}A \rightarrow A$ has no reason to believe A. However, the second autoepistemic extension of (2.10) contains A and $\mathsf{L}A$, which seems to be counter-intuitive. In this case, an agent may first assume A, and hence also $\mathsf{L}A$, and then justify his former assumption by $\mathsf{L}A \rightarrow A$. Obviously, this is a circular chain of reasoning that results in self-grounded beliefs. Konolige therefore writes in [1988, p. 352]: "*This certainly seems to be an anomalous situation, since the agent can, simply by choosing to assume a belief or not, be justified in either believing or not believing a fact about the world.*" The notion of *groundedness* in autoepistemic logic has been extensively studied in [Konolige, 1988].

Finally, let us turn to the relationship between autoepistemic and default logic. This relationship has first been investigated in [Konolige, 1988] and then pursued further in [Marek and Truszczyński, 1989; Truszczyński, 1991]. According to [Konolige, 1988], a default rule of the form $\frac{\alpha:\beta}{\gamma}$ corresponds[8] to a modal formula of the form

$$\mathsf{L}\alpha \wedge \neg\mathsf{L}\neg\beta \rightarrow \gamma.$$

For instance, as regards our example, the modal formula (2.9) corresponds to the default rule (2.1) and vice versa.

However, there remain some slight differences between the two approaches. As we have seen above, autoepistemic logic allows for conclusions which are not "grounded" in the underlying premises. This is not the case in default logic as is illustrated next. According to the above transformation the default theory

$$\left(\left\{\frac{A\ :}{A}\right\}, \emptyset\right) \tag{2.11}$$

corresponds to the modal formula (2.10). This default theory, however, has only one extension which simply contains the set of all tautologies. Obviously, this default extension corresponds to the autoepistemic extension containing $\neg\mathsf{L}A$. Hence, default logic does not permit "ungrounded" conclusions.

Another difference between autoepistemic and default logic lies in their disagreement in the representation of defaults. Consider the default theory (due to [Marek and Truszczyński, 1989])

$$\left(\left\{\frac{A\ :}{A}, \frac{:\ \neg A}{A}\right\}, \emptyset\right) \tag{2.12}$$

and its autoepistemic counterpart

$$\{\mathsf{L}A \rightarrow A, \neg\mathsf{L}A \rightarrow A\}. \tag{2.13}$$

The default theory (2.12) has no extension. In contrary, we obtain an autoepistemic extension from (2.13) which contains A. Marek and Truszczyński regard this in [1989] as the most significant difference between autoepistemic and default logic. They describe this very aptly as follows: "*Defaults* [default rules] *work as inference rules and their prerequisites and justifications do not interplay as they are capable to do in the autoepistemic framework*" [1989, p. 281].

[8]The exact correspondence can only be proved between so-called *super-strongly grounded* autoepistemic theories and general default theories (cf. [Łukaszewicz, 1990, Sec. 5.9]).

2.3 Circumscription

John McCarthy's *circumscription* [1980; 1986] is based on the idea that "... *the objects that can be shown to have a certain property P by reasoning from certain facts W are all the objects that satisfy P*" [1980, p. 171]. Accordingly, we would like *Larissa* to be the only child involved in our above example. That is, we would like to infer that there are no other children apart from *Larissa* in the described scenario.

McCarthy formalizes this idea in [1986][9] by means of a second-order formula. That is, the *circumscription* of a predicate P in a formula W by varying the predicate Q is defined as[10]

$$W \wedge \forall \Phi, \Psi (W\{\mathsf{P}/\Phi, \mathsf{Q}/\Psi\} \wedge (\forall x.\Phi(x) \rightarrow \mathsf{P}(x)) \rightarrow (\forall x.\mathsf{P}(x) \rightarrow \Phi(x))) \tag{2.14}$$

where $W\{\mathsf{P}/\Phi, \mathsf{Q}/\Psi\}$ denotes the result of substituting all occurrences of P and Q by Φ and Ψ. The second conjunct is called the *circumscription axiom.*

Let us illustrate this by means of our *Larissa* example. In order to formalize that *"Larissa is a child"* and that, *"normally, children like ice-cream"*, we introduce an additional predicate abnormal (cf. [McCarthy, 1986]) along with its obvious meaning. This results in the following formula.

$$\mathsf{child}(\mathit{Larissa}) \wedge \forall x.\mathsf{child}(x) \wedge \neg\mathsf{abnormal}(x) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(x)$$

Intuitively, the purpose of the literal $\neg\mathsf{abnormal}(x)$ in the above formula, say W, is to indicate that only "abnormal" children violate the rule. In standard first-order logic, however, there is no way to derive $\neg\mathsf{abnormal}(\mathit{Larissa})$ in order to conclude that *"Larissa likes ice-cream"*. Exactly this task is accomplished by the circumscription of the predicate abnormal. In particular, the idea underlying circumscription carries over to the predicate abnormal in a natural way. That is, in our example, simply no individuals should be "abnormal" since there are no objects which can be shown to have the property abnormal by reasoning from W.

In addition, we have to "vary" the predicate likes-ice-cream in order to allow for the variation of the individuals satisfying likes-ice-cream.[11] Then, the circumscription of abnormal in W by varying likes-ice-cream yields the formula

$$\begin{aligned} W \wedge \forall \Phi, \Psi (& W\{\mathsf{abnormal}/\Phi, \mathsf{likes\text{-}ice\text{-}cream}/\Psi\} \\ & \wedge (\forall x.\Phi(x) \rightarrow \mathsf{abnormal}(x)) \rightarrow (\forall x.\mathsf{abnormal}(x) \rightarrow \Phi(x))) \end{aligned}$$

By substituting

$$\Phi \quad \text{by} \quad \lambda x.\bot \qquad \text{and} \qquad \Psi \quad \text{by} \quad \lambda x.\top,$$

and then expanding $W\{\mathsf{abnormal}/\bot, \mathsf{likes\text{-}ice\text{-}cream}/\top\}$, we obtain the formula

$$W \wedge \forall x.\neg\mathsf{abnormal}(x).$$

Notably, the circumscription axiom is reduced to the formula $\forall x.\neg\mathsf{abnormal}(x)$ which exactly meets our intuition that no individuals should be "abnormal". As a consequence, we can derive $\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa})$ from the circumscribed theory.

In contrast to the rather complex circumscription axiom, the semantics of circumscription is very clear, since it relies on the notion of *minimal entailment.* In standard first-order logic, a formula α is entailed by a formula W if it holds in all models of W. As regards minimal entailment,

[9] Since the approach taken in [McCarthy, 1986] generalizes the one originally proposed in [McCarthy, 1980], we shall deal with the more recent approach.

[10] This definition extends to tuples of predicates in a naturally way.

[11] See [McCarthy, 1986] on varying predicates.

we are faced with the notion of *minimal models*, ie. a formula α is minimally entailed by W if it holds in all minimal models of W. In terms of circumscription, this amounts to restricting entailment to those models of W in which the fewest objects fulfill the circumscribed predicate P. More formally, two models are comparable if they agree except for their interpretation of P and Q.[12] Then, a model is minimal if its extension of P is minimal compared with all other comparable models. As shown in [Lifschitz, 1985], circumscription is correct and complete wrt the minimal model semantics.

For instance, the above circumscription of abnormal in W by varying likes-ice-cream semantically amounts to entailment regarding only those models of W with a minimal extension of abnormal while disregarding the respective extension of likes-ice-cream. Therefore, circumscription eliminates all non-minimal models of W. As a result, we simply consider those models of W which have an empty extension of abnormal.

Finally, let us deal with the relationship between circumscription and default logic. Circumscribing a predicate corresponds superficially to the use of prerequisite-free normal default theories (ie. normal default theories consisting of default rules with no prerequisite). For instance, we can simulate the above circumscription by the following default theory.

$$\left(\left\{\frac{:\neg\mathsf{abnormal}(x)}{\neg\mathsf{abnormal}(x)}\right\}, W\right) \tag{2.15}$$

As in circumscription, we can derive $\neg\mathsf{abnormal}(\mathit{Larissa})$ which allows for the derivation of $\mathsf{likes\text{-}ice\text{-}cream}(\mathit{Larissa})$ (cf. Section 2.1).

However, it is impossible to derive in default logic that all objects are "normal", provided that open default rules are treated as schemata. That is, instead of deriving $\forall x.\neg\mathsf{abnormal}(x)$ from the default theory (2.15) we can only derive $\neg\mathsf{abnormal}(t)$ for each ground term t. Nonetheless, Etherington shows in [1987b] that under the domain-closure assumption[13] and complete information about equality, skeptical reasoning from prerequisite-free default theories like the one above corresponds to the circumscription of abnormal while all other predicates are allowed to vary. If either of the above requirements fails, default logic allows for conclusions that cannot be obtained by circumscription. For instance, circumscription does not allow us to circumscribe equality.

In [1990], Lifschitz treats open variables in default rules as genuine object variables. That is, open variables are replaced by "names" for domain elements (cf. [Lifschitz, 1990]). As a consequence, his approach allows us to derive $\forall x.\neg\mathsf{abnormal}(x)$ from the default theory (2.15). In particular, Lifschitz shows that circumscription is subsumed by default logic according to his approach. Notably, the restrictions imposed in [Etherington, 1987b] are not necessary any longer.

2.4 Conclusion

All of the approaches introduced in the previous sections share standard predicate logic as an underlying framework. All of them provide means to allow for additional conclusions that are beyond standard derivability or entailment. Accordingly, we can formulate the common idea as follows. Given a set of facts W about the world, or a certain domain, and a formula α (which is usually not entailed by W), a nonmonotonic formalism (roughly) aims at providing a set of formulas A, say assumptions, such that $W \cup A$ entails α and $W \cup A$ is consistent.

[12] As above, P denotes the circumscribed predicate and Q the varying one.

[13] The domain-closure assumption says that the only individuals to be considered are those given by the Herbrand universe.

	First-order logic	Circumscription	Autoepistemic logic	Default logic
Language	First-order	Second-order	First-order modal	First-order
Monotonicity	yes	no	no	no
Cumulativity	yes	no[15]	no	no[16]
Nonmonotonic feature	—	circumscription axiom	modal propositions, $L\alpha, \neg L\alpha$	default rules, $\frac{\alpha:\beta}{\gamma}$
Multiple extensions	no	no	yes	yes
Contraposition	yes	yes	no	no[17]
Reasoning by cases	yes	yes	yes	no[17]

Table 2.1: Approaches to default reasoning.

We have observed how closely the different formalisms are related to each other, although they were originally aiming at differing facets of commonsense reasoning. Autoepistemic logic deals with beliefs in order to model the *reasoning of an ideally rational agent.* Circumscription deals with *closed world reasoning* in the sense that it commits to given knowledge and denies all absent information. Finally, default logic aims at the heart of default reasoning and addresses the notions of *normality* and *typicality* by dealing with the notion of *consistency.* In particular, "*it* [default logic] *adequately handles the idea of an exception*" [Sombé, 1990, p. 452] which is indispensable for default reasoning. On the other hand, we have seen in the previous sections how all of these approaches can be used for default reasoning. In particular, we have illustrated how circumscription and autoepistemic logic are related to default logic.

However, there remain some differences among the major formalisms[14] which are due to the different ways of representing and reasoning about absent information. Therefore, we close this chapter with a brief survey on the respective peculiarities. This survey is given in Table 2.1. Many of the criteria given in this table have already been discussed in the previous sections. There, we have seen that all nonmonotonic approaches differ in their underlying logical language. Circumscription uses a second-order language, autoepistemic logic employs a first-order modal language, and default logic sticks to a first-order language. The use of different languages stems mainly from the different ways default assumptions are incorporated into the respective approach. Circumscription uses a second-order axiom, autoepistemic logic employs additional modal literals, whereas default logic simply adds rules of inference and, therefore, avoids extending the first-order language. Also, it is worth mentioning that circumscription does not admit multiple extensions, which is the case for the two other approaches. In fact, circumscription does only allow for skeptical reasoning, whereas autoepistemic and default logic allow for credulous as well as skeptical reasoning.

The remaining criteria given in Table 2.1 like cumulativity, contraposition, or reasoning by cases are only listed for completeness since they will be discussed in the following chapters. A detailed discussion of the differences and similarities of the various approaches to nonmonotonic reasoning can be found in [Reiter, 1987a; Besnard, 1989; Sombé, 1990; Brewka, 1991c; Bibel *et al.*, 1993].

[14] We have omitted McDermott and Doyle's nonmonotonic logic, since it is somehow subsumed by autoepistemic logic.

[15] Circumscription is cumulative if the existence of minimal models is guaranteed [Makinson, 1989].

[16] Cf. Section 3.5.

[17] Holds for prerequisite-default theories (cf. Section 3.4).

In addition, the aforementioned literature gives an excellent survey on other approaches to nonmonotonic reasoning, which are unfortunately beyond the scope of this thesis. Among them, approaches investigating the formal properties of nonmonotonic consequence relations [Gabbay, 1985; Kraus *et al.*, 1990], approaches relying on conditional logic [Delgrande, 1987; Delgrande, 1988], preferential-models approaches [Bossu and Siegel, 1985; Shoham, 1988], or even approaches dealing with probabilistic reasoning [Pearl, 1988].

Chapter 3

Classical default logic

This chapter is devoted to the formal development of *classical default logic*.[1] In Section 3.1, we formally account for the notion of a classical extension. In order to make the treatment self-contained, we also repeat some of the definitions already given in Section 2.1. Section 3.2 gives the basic properties of classical default logic and its extensions. In Section 3.3, we discuss some limitations of general default theories and describe subclasses which avoid these problems. The class of prerequisite-free default theories is separately discussed in Section 3.4 since they allow us to retain the properties of standard implications (unless explicitly blocked). Section 3.5 deals with the formal property of cumulativity, which is not satisfied by general default theories. After a thorough discussion of the failure of cumulativity, we account for the most important practical impact of cumulativity, which is the capability of handling (nonmonotonic) lemmas. Therefore, we introduce the proof-theoretic notion of lemma default rules as a general approach to admit the use and generation of nonmonotonic lemmas in default logics. Finally, we describe in Section 3.6 a semantical characterization of classical default logic and reflect semantically some of the problems discussed in Section 3.3 and 3.5.

3.1 Formal development of classical default logic

Classical default logic was defined by Reiter in [1980] as a formal account of reasoning in the absence of complete information. It is based on first-order logic, whose sentences are hereafter simply referred to as formulas (instead of closed formulas). As introduced in Section 2.1, a default theory (D, W) consists of a set of formulas W and a set of default rules D. A (singular) default rule is any expression of the form

$$\frac{\alpha : \beta}{\gamma},$$

where α, β and γ are formulas. α is called the prerequisite, β the justification, and γ the consequent of the default rule. In the sequel, we shall consider only closed singular default rules, ie. closed default rules with one justification.[2] For convenience, we denote the prerequisite of a default rule δ by $\mathit{Prereq}(\delta)$, its justification by $\mathit{Justif}(\delta)$ and its consequent by $\mathit{Conseq}(\delta)$.[3] We recall that a default theory is said to be normal whenever the justification and the consequent of each default rule are logically equivalent, eg. $\frac{\alpha : \beta}{\beta}$, and it is called semi-normal whenever the

[1] For the sake of clarity, we will refer to Reiter's original default logic as *classical* default logic.

[2] In particular, we will see in Chapter 4 and Section 5.3 that in constrained and cumulative default logic multiple justifications correspond to their conjunction.

[3] These projections extend to sets of default rules in the obvious way.

justification of each default rule implies the corresponding consequent, eg. $\frac{\alpha:\beta\wedge\gamma}{\gamma}$. A default rule $\frac{\alpha:\beta}{\gamma}$ is applicable, if its prerequisite α is known and its justification β is *consistent*, ie. the negation of the justification, $\neg\beta$, does not hold.

An extension E of the initial set of facts W is defined as all formulas derivable from W using standard inference rules and all specified default rules. According to Reiter [1980], E should be the smallest set of formulas containing the initial set of facts W, being deductively closed and including each consequent of each applicable default rule. An extension is formally characterized in the following definition.

Definition 3.1.1 *Let (D, W) be a default theory. For any set of formulas S, let $\Gamma(S)$ be the smallest set of formulas S' such that*

1. $W \subseteq S'$,
2. $Th(S') = S'$,
3. *For any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $\neg\beta \notin S$ *then* $\gamma \in S'$.

A set of formulas E is a classical extension of (D, W) iff $\Gamma(E) = E$.

The interplay of classical extensions and default rules is illustrated in Figure 3.1. There, we see that the consequent γ of a default rule $\frac{\alpha:\beta}{\gamma}$ is added to a classical extension E iff its prerequisite α is in E and if its justification β is consistent with E.

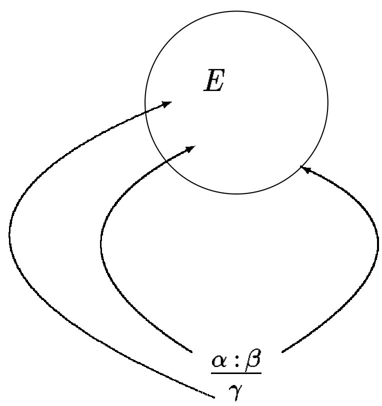

Figure 3.1: The interplay of classical extensions and default rules.

An interesting point in the last definition is how classical default logic deals with absent information. Default conclusions are drawn by lack of belief in the negation of the corresponding justifications. Therefore, we agree with Etherington [1988] who says that "... *the term 'justification' is seen to be somewhat misleading, since justifications need not to be known, merely consistent.*" Of course, justifications cannot be known in advance since otherwise default logic could not handle incomplete information. However, we will argue in Chapter 4 that the technical notion of a justification should be strengthened towards unverifiable reasons for belief instead of simple consistency warranties based on disbelief.

Finally, let us give an example dealing with several classical extensions.

Example 3.1.1 *The default theory*

$$\left(\left\{\frac{:\neg B}{A}, \frac{:\neg A}{B}\right\}, \{A \to C, B \to C\}\right)$$

has two classical extensions: $Th(\{A, C\})$ *and* $Th(\{B, C\})$.

In the last example, we obtain two classical extensions. As described in Section 2.1, there are two ways of theory formation: the credulous one which considers each extension as an acceptable set of beliefs, and the skeptical[4] way of theory formation that regards only the intersection of both extensions, namely $Th(\{A \vee B, C\})$, as the set of consequences. In the sequel, we will adopt neither the credulous nor the skeptical view. We will simply consider all extensions and we will explicitly refer to one or the other as needed.

3.2 Properties of classical default logic

In this section, we present the major properties of classical extensions. The definition of a classical extension given in the last section is rather complex. The next theorem [Reiter, 1980, Theorem 2.1] provides a more intuitive characterization of classical extensions.

Theorem 3.2.1 *Let* (D, W) *be a default theory and let* E *be a set of formulas. Define*

$$E_0 = W$$

and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

E *is a classical extension of* (D, W) *iff* $E = \bigcup_{i=0}^{\infty} E_i$.

The above characterization is not strictly iterative due to the occurrence of E in the definition of E_{i+1}. Even though it still lacks constructivity, it provides a more intuitive and very useful characterization of classical extensions, because it allows for inductive and, therefore, easier proofs.

As immediate corollaries of Theorem 3.2.1, [Reiter, 1980] obtains the following ones.

Corollary 3.2.2 *Let* (D, W) *be a default theory and* E *a classical extension of* (D, W). *Then, we have*

- E *is inconsistent iff* W *is inconsistent,*
- *if* E *is inconsistent then* E *is the only classical extension of* (D, W).

The next theorem accounts for the *maximality* of classical extensions.

Theorem 3.2.3 (Maximality) *Let* (D, W) *be a default theory and let* E *and* E' *be classical extensions of* (D, W). *Then,* $E \subseteq E'$ *implies* $E = E'$.

Another very useful characterization of classical extensions can be given by means of the *generating default rules.*

[4] See page 20.

Definition 3.2.1 *Let (D, W) be a default theory and S a set of formulas. The set of generating default rules for S wrt D is defined as*

$$GD_D^S = \left\{ \tfrac{\alpha : \beta}{\gamma} \in D \;\middle|\; \alpha \in S, \ \neg\beta \notin S \right\}.$$

Then, as shown in [Reiter, 1980], "... *the next theorem justifies the terminology of 'generating default'*".

Theorem 3.2.4 *Let E be a classical extension of a default theory (D, W). We have*

$$E \;=\; Th(W \cup Conseq(GD_D^E)).$$

That is, any extension can be characterized by means of the initial set of facts and the consequents of the set of generating default rules.

Now, we can formulate, by means of Definition 3.2.1, the following corollary to Theorem 3.2.1, which was first formulated in [Schwind, 1990]. This corollary expresses the property of groundedness.

Theorem 3.2.5 (Groundedness) *Let E be a classical extension of (D, W). Then, there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of GD_D^E such that for $i \in I$*

$$W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i).$$

We will find this property again in each variant of classical default logic discussed in Chapter 4, and Sections 5.2 and 5.3. On the whole, the property of groundedness can be seen as an important characteristic of default logics. In particular, autoepistemic logic does not satisfy the property of groundedness (cf. Section 2.2.2). A thorough discussion of the notion of groundedness in classical default logic and autoepistemic logic can be found in [Konolige, 1988].

The next section provides us with several more properties of classical extensions which hold for some restricted classes of default theories.

3.3 Problems with classical default logic

Classical default logic deals with many commonsense examples very well. However, there are several limitations encountered in the case of general default theories. In this section, we describe some major shortcomings of classical default logic and illustrate them by means of some canonical examples. The property of cumulativity is postponed and dealt with in Section 3.5.

In particular, we will see that many shortcomings of classical default logic vanish whenever we confine ourselves to normal default theories. This is basically due to the fact that the justifications and consequents of each normal default rule coincide. Thus, after successfully checking the consistency of a justification, we also believe its validity by adding it to our set of beliefs.

Recently, a slightly more general subclass of default theories has been isolated by Dix in [1992]. A default theory is *C-normal*[5] if all of its default rules are of the form

$$\frac{\alpha \,:\, \beta \wedge \hat{C}}{\beta},$$

where $\hat{C}$ is the conjunction of all formulas contained in a finite set of formulas C. Thus, C-normal default rules are semi-normal default rules whose non-normal part of the justification is fixed.

[5] Originally, [Dix, 1992] refers to C-normal default theories as default theories of *Poole-type* (cf. Section 5.4).

In other words, C-normal default theories are normal default theories along with an additional global consistency condition. As a consequence, C-normal default theories share many desired properties with normal default theories.

However, (C-)normal default theories lack expressibility. For instance, Reiter and Criscuolo [1981] describe this deficiency as follows: "*Although most commonly occurring default rules are normal when viewed in isolation, they can interact with each other in ways that lead to the derivation of anomalous default assumptions. In order to deal with such anomalies it is necessary to re-represent these rules, in some way by introducing non-normal defaults.*" As a response, Reiter and Criscuolo argued in favor of semi-normal default theories. That the class of semi-normal default theories is sufficiently expressive (opposed to normal default theories) is a result by Marek and Truszczyński [1993]. They show that any default theory can be represented as a semi-normal default theory. This is accomplished by extending the underlying language and by introducing new default rules. Thus, the new semi-normal default theory may be much larger than the original one (cf. [Marek and Truszczyński, 1993]). Unfortunately, semi-normal default theories have none of the nice properties of normal default theories. Rather they are as complicated as general default theories, as we will see below.

3.3.1 Coherence or the existence of extensions

Extensions play a central role in default logic. However, there are default theories that lack classical extensions. Etherington [1986] refers to them as *incoherent* default theories. The simplest incoherent default theory [Reiter, 1980, Example 2.6] is the following.

Example 3.3.1 *The default theory*

$$\left(\left\{\frac{:\neg A}{A}\right\}, \emptyset\right)$$

has no classical extension.

To see that the last default theory is incoherent, assume first A is not contained in a classical extension. Then, the default rule $\frac{:\neg A}{A}$ is applicable and consequently A would be included contradictory to our prior assumption. Analogously, assume A is included in a classical extension. Then, the default rule $\frac{:\neg A}{A}$ would not be applicable. Consequently, A cannot be derived again in contradiction to our prior assumption.

The simplest class of default theories which guarantees the existence of classical extensions consists of default theories without any default rules. Clearly, any default theory of the form $(\emptyset, W)$ has a unique extension $Th(W)$.

An important class which guarantees the existence of classical extensions is that of normal default theories. [Reiter, 1980, Theorem 3.1] states the following.

Theorem 3.3.1 (Existence of extensions) *Every normal default theory has a classical extension.*

The same result holds for C-normal default theories as shown in [Dix, 1992].

Etherington [1986] shows that semi-normal default theories which are ordered in a certain way have at least one classical extension. However, this is not the case for general semi-normal default theories. Etherington [1986] gives the following example.

Example 3.3.2 *The default theory*

$$\left(\left\{\frac{:C\wedge\neg D}{C}, \frac{:D\wedge\neg E}{D}, \frac{:E\wedge\neg C}{E}\right\}, \emptyset\right)$$

has no classical extension.

Etherington [1987a] himself describes splendidly the cause of incoherence in the last example: "*... applying any one default leaves one other applicable. Applying any two, however, results in the denial of the non-normal part of the justifications of at least one of them. Any set small enough to be an extension is too small; any set large enough is too large. This behavior is characteristic of theories with no extension; the requirement that extensions be closed under the default rules forces the application of defaults whose consequents lead to the denial of justifications of other applied defaults.*"

Similar to [Etherington, 1986], Zhang and Marek investigate in [1990] conditions for the existence of extensions and introduce the notion of "stratified sets of default rules".

3.3.2 Semi-monotonicity

A very important property which holds for normal default theories in classical default logic is *semi-monotonicity*.[6] Semi-monotonicity stands for monotonicity wrt the default rules and stipulates that adding a set of default rules to a default theory can only preserve or enlarge existing extensions. In the case of normal default theories, this formally amounts to the following theorem [Reiter, 1980, Theorem 3.2].

Theorem 3.3.2 (Semi-monotonicity) *Let (D, W) be a normal default theory and D' a set of default rules such that $D \subseteq D'$. If E is a classical extension of (D, W), then there is a classical extension E' of (D', W) such that $E \subseteq E'$.*

The importance of semi-monotonicity stems from the fact that it allows for local proof procedures which may discard some of the default rules. That is, in order to prove a proposition from a given normal default theory (D, W) it is enough to consider a proper finite subset of D. As a consequence, only the relevant default rules have to be taken into account for proving a given query. The same "locality" property holds for theories in standard first-order logic (due to compactness and monotonicity). In default logics, however, always all axioms and theorems of a default theory have to be considered (for checking consistency). Otherwise, default logic would be monotonic. Thus, we fully agree with [Besnard, 1989] who states that "*Nonmonotonicity requires such a global property...*".

We illustrate the failure of semi-monotonicity for general default theories in the next example.

Example 3.3.3 *The default theory*

$$\left(\left\{\frac{:B}{C}\right\}, \emptyset\right) \tag{3.1}$$

has one classical extension: Th($\{C\}$).

Adding the default rule $\frac{:D}{\neg B}$ yields the default theory

$$\left(\left\{\frac{:B}{C}, \frac{:D}{\neg B}\right\}, \emptyset\right) \tag{3.2}$$

whose classical extension is Th($\{\neg B\}$).

From the default theory (3.1) we conclude C under the assumption that B is consistent. However, this assumption is violated in the presence of the default rule $\frac{:D}{\neg B}$ which "overrides" the default rule in (3.1). Hence, the default theory (3.2) yields only one classical extension which contains $\neg B$ but neither C nor $\neg C$. On the one hand, this might be desired in order to establish

[6] Again, the same result holds for C-normal default theories (cf. [Dix, 1992]).

a priority between the two default rules in (3.2). However, this is a very implicit way of encoding priorities between default rules.[7] We will describe in Section 4.9 a way to handle priorities at the meta-level.

On the other hand, there is a priori no priority between the two consistency assumptions, namely the justifications B and D, in (3.2). In order to illustrate this, let us replace D in Example 3.3.3 by $\neg B$.

Example 3.3.4 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{\neg B}\right\}, \emptyset\right)$$

has one classical extension: $Th(\{\neg B\})$.

As in Example 3.3.3, we obtain only one classical extension containing $\neg B$. But now this seems counterintuitive: if we know nothing else originally, then there seems to be no good reason that the first default rule is not applicable. As Delgrande puts it in [1992], the argument might run:

> *"Initially I know nothing at all; hence B is consistent, and I can conclude that C. However, $\neg B$ is inconsistent with my original assumption, and so I cannot apply the second default rule."*

Similar reasoning beginning with the second default rule yields a second extension.

The above argument also shows that every semi-monotonic default logic guarantees the existence of extensions. In this case, we may start with no default rules and then successively apply one default rule after another with no risk of destroying any previous partial extension.

Technically, the reason for the lack of semi-monotonicity is that the *continued consistency* of the justifications of applied default rules is in general not preserved. As we have seen in both previous examples, it happens that the justifications of previously applied default rules may be denied by default rules subsequently applied. Thus, we can also refer to semi-monotonicity as "weak commitment to assumptions", since every semi-monotonic formalism commits to the justifications of previously applied default rules. In Section 3.3.4, we will discuss an even stronger notion of "commitment to assumptions" which additionally requires the joint consistency of the justifications of applying default rules.

3.3.3 Orthogonality

Default theories may have alternative classical extensions which are consistent, ie. not *orthogonal* to each other. The problem with orthogonality is principally of a cognitive nature. Namely, why should we adopt alternative beliefs if they are compatible with each other? In other words, what is the source of different sets of beliefs if not their incompatibility? For instance, the property of orthogonality holds in autoepistemic logic (cf. Section 2.2.2), where agents can only adopt incompatible sets of beliefs.

Let us illustrate the failure of orthogonality for general default theories by means of the following example.

Example 3.3.5 *The default theory*

$$\left(\left\{\frac{:\neg B}{C}, \frac{:\neg C}{B}\right\}, \emptyset\right)$$

has two classical extensions: $Th(\{C\})$ and $Th(\{B\})$.

[7] See [Brewka, 1991b] for a different opinion.

Analogously to Example 3.1.1, we see that once we have applied one of the two default rules the other one becomes blocked. Thus, we obtain two alternative sets of beliefs which are not contradictory to each other. As can be observed, multiple extensions as such stem from inconsistencies between themselves and justifications of inapplicable default rules. However, these inconsistencies and the corresponding justifications remain invisible. In this respect, classical extensions simply lack transparency.

Since normal default theories reveal the justifications of the applying default rules their classical extensions enjoy orthogonality. Or as [Reiter, 1980] puts it, in the case of normal default theories "*... one can attempt to simultaneously hold beliefs in two distinct extensions only at the risk of inconsistency*".

Theorem 3.3.3 (Orthogonality) *Let (D, W) be a normal default theory. If E and E' are distinct classical extensions of (D, W), then $E \cup E'$ is inconsistent.*

Finally, the same result holds for C-normal default theories, as shown in [Dix, 1992].

3.3.4 Commitment to assumptions

Another problem has been addressed by Poole who shows in [1989] that classical default logic does not commit to assumptions. That is, a classical extension may be justified by contradictory consistency assumptions. Thus, the term *"commitment to assumptions"* stands for the joint consistency of an extension with all of its underlying consistency assumptions, ie. the justifications of all applying default rules. In particular, such inconsistencies among the justifications may lead to counterintuitive results as is illustrated next.

Example 3.3.6 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right)$$

has one classical extension: $Th(\{C, D\})$.

We observe that both default rules apply, although they have contradictory justifications. Informally, there is no commitment to the assumption B nor $\neg B$. However, this seems to be rather counterintuitive since the default conclusion C relies on B, while the conclusion D relies on $\neg B$ being consistent. Therefore, the conclusion $C \wedge D$ is in some sense justified by simultaneously assuming the consistency of B and its negation.

Poole himself describes the problem in [1989] as follows: "*The problem is we have implicitly made an assumption, but have been prevented from considering what other assumption we made as a side effect of this assumption.*" Technically, the problem arises since in Definition 3.1.1 the mere possibility (or consistency) of a justification allows for the application of a default rule. In particular, the consistency of each justification is only checked separately. Thus, there is no way to account for incompatible sets of justifications.

Again, the class of normal default theories avoids this problem and therefore classical extensions of normal default theories commit to their assumptions. This is because due to the equivalence of the justifications and consequents of normal default rules, the justifications of all applying default rules are contained in the corresponding classical extensions. Thus, provided we have a consistent classical extension, all justifications are jointly consistent. If not, none of the default rules applies.

We have already noted in Section 3.3.2 that the failure to commit to assumptions may also be seen as a cause for the lack of semi-monotonicity. In particular, we have argued that semi-monotonicity holds whenever the continued *individual* consistency of the justifications of applied default rules is preserved. We have referred to this as "weak commitment to assumptions". Accordingly, (strong) commitment to assumptions is obtainable by preserving the continued *joint* consistency of all justifications of applied default rules (see Chapter 4). As a consequence, every default logic that commits to assumptions is also semi-monotonic and, therefore, also guarantees the existence of extensions.

3.3.5 Other problems

Finally, we deal with some peculiarities which arise from encoding defaults as inference rules. In particular, there are some features which one might expect from the perspective that defaults should behave like material implications, except that they are defeasible.

However, we will discuss subclasses avoiding these problems in a separate section, namely Section 3.4. A thorough discussion of these problems can be found in [Besnard, 1989; Delgrande *et al.*, 1992].

Modus tollens or reasoning by contraposition

First, default logic provides a notion of a "defeasible modus ponens". That is, we may somehow infer from A and $\frac{A:B}{B}$ that B holds. However, this does not extend to modus tollens, or *reasoning by contraposition.*

Example 3.3.7 *The default theory*

$$\left(\left\{\frac{A\ :\ B}{B}\right\}, \{\neg B\}\right)$$

has one classical extension: $Th(\{\neg B\})$.

For instance, in standard logic, if we know that $A \to B$ and $\neg B$ are true, then $\neg A$ must also be true, since $A \to B$ is equivalent $\neg B \to \neg A$. As we will see in Section 3.4, we may reason by contraposition, if we treat defaults as "defeasible implications".

Reasoning by cases

Second, default logic provides no means for *reasoning by cases.*

Example 3.3.8 *The default theory*

$$\left(\left\{\frac{A\ :\ B}{B}, \frac{\neg A\ :\ B}{B}\right\}, \emptyset\right)$$

has one classical extension: $Th(\emptyset)$.

Although $A \vee \neg A$ is a tautology, neither A nor $\neg A$ alone are valid, and so neither default rule can be applied.

This problem disappears if we weaken the applicability condition of default rules (see Section 3.4). Note that autoepistemic logic allows for reasoning by cases, as we have seen in Section 2.2.2.

Reasoning about default rules

A final limitation of default logic is that while we can reason with default rules, we cannot reason *about* default rules. In standard first-order logic, for example, if we know that $A \to C$

and $B \rightarrow C$ are true, then $A \vee B \rightarrow C$ must also be true. However, in default logic, there is no explicit connection between the default rules

$$\frac{A : C}{C}, \quad \frac{B : C}{C}$$

and

$$\frac{A \vee B : C}{C}.$$

Nevertheless, we informally know that the third default rule can be applied whenever either of the first two can. That is, for instance, if we know A, then we also know $A \vee B$. Thus, if C can be consistently assumed the first and the third default rule are both applicable.

3.4 Prerequisite-free default theories

We have observed in Section 3.3.5 that default rules do not allow for reasoning by contraposition and for reasoning by cases since the requirement of proving the prerequisite is sometimes too strong. Therefore, Delgrande argues in [1992] in favor of prerequisite-free default theories, since they allow us to retain the properties of standard implications (unless explicitly blocked). The emphasis then shifts to the implication itself, rather than a rule involving a prerequisite and a justification for a conclusion. In what follows, we describe how prerequisite-free default rules allow for reasoning by cases and for reasoning by contraposition. Also, we sketch how prerequisite-free normal default rules allow for reasoning about defaults.

Delgrande has shown in [1992], how general default theories can be transformed into prerequisite-free default theories. Given a general default theory, each of its default rules is transformed into a prerequisite-free default rule as follows:

$$\frac{\alpha : \beta}{\gamma} \qquad \mapsto \qquad \frac{: (\alpha \rightarrow \gamma) \wedge \beta}{\alpha \rightarrow \gamma} \tag{3.3}$$

Independently, the same transformation has been proposed in [Besnard, 1989] for normal default theories. Therein, Besnard has particularly argued that the computational behavior of prerequisite-free normal default theories is very appealing.

Also, we obtain a very simple characterization of classical extensions in the case of prerequisite-free default theories.

Theorem 3.4.1 *Let (D, W) be a prerequisite-free default theory and E a set of formulas. Then,*

$$E \text{ is a classical extension of } (D,W) \text{ iff } E = Th\Big(W \cup \Big\{ \gamma \;\Big|\; \tfrac{:\beta}{\gamma} \in D, \neg\beta \notin E \Big\}\Big).$$

The above characterization reduces the applicability condition of prerequisite-free default rules to the computation of maximally consistent sets of formulas.

Let us consider how prerequisite-free default theories deal with reasoning by contraposition. Therefore, let us transform the default theory given in Example 3.3.7 according to the transformation (3.3). Recall that we have obtained one classical extension $Th(\{\neg B\})$ in Example 3.3.7.

Example 3.4.1 *The default theory*

$$\left(\left\{\frac{: A \rightarrow B}{A \rightarrow B}\right\}, \{\neg B\}\right)$$

has one classical extension: $Th(\{\neg B, \neg A\})$

That is, compared with Example 3.3.7, we additionally obtain the conclusion $\neg A$ by modus tollens.

However, the possibility of reasoning by contraposition is not always desired.[8] Usually, this issue is addressed by semi-normal default rules (cf. [Reiter and Criscuolo, 1981]). For instance, we can block the contraposition of the default rule given in Example 3.4.1, $\frac{:A\rightarrow B}{A\rightarrow B}$, by adding B to the justification. This yields the default rule $\frac{:(A\rightarrow B)\wedge B}{A\rightarrow B}$ which does not allow us to derive $\neg A$ in the presence of $\neg B$ since the justification is inconsistent. Therefore, blocking the contrapositive of an "implication", like $\frac{:\alpha\rightarrow\gamma}{\alpha\rightarrow\gamma}$, is a special case of the above transformation (3.3), where β is γ.

As suggested by the above example, the transformation (3.3) yields more general theories than the original ones. Delgrande shows in [1992] the following.

Theorem 3.4.2 [Delgrande *et al.*, **1992]** *Let (D, W) be a default theory and let*

$$D' = \left\{ \frac{:(\alpha\rightarrow\gamma)\wedge\beta}{\alpha\rightarrow\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D \right\}.$$

If E is a classical extension of (D, W) then there is a classical extension E' of (D', W) such that $E \subseteq E'$.

However, we may also get more classical extensions from transformed default theories [Delgrande *et al.*, 1992].

Example 3.4.2 *The default theory*

$$\left(\left\{ \frac{A:B}{B}, \frac{B:\neg A}{\neg A} \right\}, \{A\} \right)$$

has one classical extension: $Th(\{A, B\})$.

The transformation (3.3) yields the following default theory, which has an additional extension.

Example 3.4.3 *The default theory*

$$\left(\left\{ \frac{:A\rightarrow B}{A\rightarrow B}, \frac{:B\rightarrow\neg A}{B\rightarrow\neg A} \right\}, \{A\} \right)$$

has two classical extensions: $Th(\{A, B\})$ and $Th(\{A, \neg B\})$.

Now, let us look at the problem of reasoning by cases. We transform the default theory given in Example 3.3.8 into the following one.

Example 3.4.4 *The default theory*

$$\left(\left\{ \frac{:A\rightarrow B}{A\rightarrow B}, \frac{:\neg A\rightarrow B}{\neg A\rightarrow B} \right\}, \emptyset \right)$$

has one classical extension: $Th(\{B\})$.

In contrast to Example 3.3.8, in which we got $Th(\emptyset)$ as the only classical extension, we now obtain B by reasoning by cases. This seems to be the more appropriate solution.

Finally, we can reason about a set of prerequisite-free normal default theories. So, for normal prerequisite-free default theories (D, W), we can deductively determine, whether a default rule δ is "subsumed" by other default rules in D. In particular, we can check by means of standard modal logic, whether the extensions of (D, W) and $(D \setminus \{\delta\}, W)$ coincide, without necessarily computing them. We will describe this approach to reasoning about defaults in detail in Section 4.7.

[8] Brewka gives the following example in [1992]: *Computer scientists typically do not know much about default logic.* We probably do not want to conclude the contrapositive default *Who knows much about default logic typically is not a computer scientist.*

3.5 Cumulativity and nonmonotonic lemmas

This section deals with the formal property of cumulativity. We first give a detailed discussion of the failure of cumulativity in classical default logic and illustrate how prerequisite-free C-normal default theories preserve cumulativity whenever we are reasoning skeptically. Second, the proof-oriented notion of a lemma default rule is introduced that accounts for the practical impact of cumulativity: the capability of handling nonmonotonic lemmas.

3.5.1 The failure of cumulativity

Classical default logic lacks cumulativity, which is an important property of any logical calculus [Gabbay, 1985; Makinson, 1989]. Intuitively, cumulativity stipulates that the addition of a theorem to the set of premises does not change the theory under consideration. More formally, a consequence operator *Th* is called *cumulative* iff for arbitrary sets of formulas S and T

$$S \subseteq T \subseteq Th(S) \Longrightarrow Th(S) = Th(T).$$

Clearly, cumulativity holds for any monotonic logic. The property of cumulativity is obviously of theoretical importance. Moreover, it is of great practical relevance. This is because a cumulative consequence operator allows for the use of lemmas needed for reducing computational efforts. Since computation in nonmonotonic logics not only involves deduction but also expensive consistency checks, the need to incorporate lemmas is even greater in nonmonotonic theorem proving than in standard theorem proving.

Since default logic allows for multiple extensions, there are two extreme ways of interpreting the notion of a consequence operation: a credulous one which accepts each extension as a possible set of beliefs; and a skeptical one which accepts only the intersection of all extensions as an acceptable set of beliefs.

Reiter [1980, Theorem 2.6] and Makinson [1989, Observation 9] have shown independently that general default theories satisfy one half of the above cumulativity requirement whenever we are reasoning skeptically. This partial cumulativity property is referred to as *cumulative transitivity* and reads formally as follows:

$$S \subseteq T \subseteq Th(S) \Longrightarrow Th(T) \subseteq Th(S)$$

However, default logic fails to satisfy cumulativity in either way of theory formation. In order to illustrate this, let us look at the "canonical" cumulativity example given by Makinson in [1989].

Example 3.5.1 *The default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right) \tag{3.4}$$

has one classical extension: $Th(\{A\})$.

Adding the nonmonotonic theorem $A \vee B \in Th(\{A\})$ *to the set of facts yields the default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right) \tag{3.5}$$

whose classical extensions are: $Th(\{A\})$ *and* $Th(\{\neg A, B\})$.

Let us examine in detail Example 3.5.1.[9] We see that only the default rule $\frac{:A}{A}$ applies in the case of the classical extension $Th(\{A\})$ of the default theory (3.4). The default rule $\frac{A \vee B : \neg A}{\neg A}$

[9]This example is illustrated in Figure 3.3 on page 46 from a semantical point of view.

applies only if its prerequisite $A \vee B$ is derivable and its justification $\neg A$ is consistent. The only possibility to derive $A \vee B$ in the case of the default theory (3.4) is to apply the default rule $\frac{:A}{A}$. But the justification $\neg A$ is denied by the consequent of the default rule $\frac{:A}{A}$. Hence, the default rule $\frac{A \vee B : \neg A}{\neg A}$ is not applicable.

Since extensions are deductively closed, the only classical extension $Th(\{A\})$ of the default theory (3.4) inevitably contains $A \vee B$. However, changing the default theory (3.4) into (3.5) by adding the nonmonotonic theorem $A \vee B$ eliminates the dependency between the two default rules. Then, the prerequisite $A \vee B$ is derivable without any commitment to the consistency of A. As a consequence, $\neg A$ can be consistently assumed and the default rule $\frac{A \vee B : \neg A}{\neg A}$ becomes applicable and prevents the application of the default rule $\frac{:A}{A}$, so that a second classical extension results.

Thus, regardless of whether or not we are employing a skeptical or a credulous notion of theory formation, in both cases we change the theory under consideration when turning the default theory (3.4) into (3.5).

The above example reveals that cumulativity fails because the implicit consistency assumptions have been lost. Thus, whenever we add a nonmonotonic theorem to the set of premises we ignore its underlying assumptions.[10] Therefore, we can trace back the failure of cumulativity to default logic's inability to be aware of the consistency assumptions underlying a default conclusion. In particular, we have observed how the addition of nonmonotonic theorems to the set of facts can change dependencies between default rules. This may result in new classical extensions which then change the set of conclusions and, therefore, destroy cumulativity.

Obviously, prerequisite-free default theories do not admit such dependencies between default rules. Since they have no prerequisites, all default rules (whose justifications are consistent with an extension) can be applied simultaneously and, hence, independently. [Dix, 1992] shows that prerequisite-free C-normal default theories are cumulative whenever we are reasoning skeptically, as is stated in the following theorem.

Theorem 3.5.1 *Let (D, W) be a prerequisite-free C-normal default theory such that all classical extensions of (D, W) contain α. Then,*

E is a classical extension of (D, W) iff E is a classical extension of $(D, W \cup \{\alpha\})$.

The last result does not extend to prerequisite-free semi-normal default theories as can be illustrated by the following example due to [Dix, 1992].

Example 3.5.2 *The default theory*

$$\left(\left\{\frac{: A \wedge B}{A \wedge B}, \frac{: \neg A}{\neg A}, \frac{: C \wedge \neg D}{C}, \frac{: D \wedge \neg E}{D}, \frac{: E \wedge \neg C}{E}\right\}, \{B \rightarrow \neg C\}\right) \tag{3.6}$$

has one classical extension: $Th(\{A, B, \neg C, E\})$.

Adding the nonmonotonic theorem $B \in Th(\{A, B, \neg C, E\})$ to the set of facts yields the default theory

$$\left(\left\{\frac{: A \wedge B}{A \wedge B}, \frac{: \neg A}{\neg A}, \frac{: C \wedge \neg D}{C}, \frac{: D \wedge \neg E}{D}, \frac{: E \wedge \neg C}{E}\right\}, \{B, B \rightarrow \neg C\}\right) \tag{3.7}$$

whose classical extensions are: $Th(\{A, B, \neg C, E\})$ and $Th(\{\neg A, B, \neg C, E\})$.

[10] In [Brewka, 1991b], this observation has led to the proposal of formulas labelled with their underlying consistency assumptions. This approach is discussed in detail in Section 5.3.

The crux in the previous example lies in the last three default rules. We have seen in Example 3.3.2 that the default theory which is composed of these three default rules and the empty set of facts has no classical extension. In the case of the default theory (3.6), the first default rule $\frac{:A\wedge B}{A\wedge B}$ and the axiom $B \rightarrow \neg C$ allow only for applying the default rule $\frac{:E\wedge\neg C}{E}$ which results in the only classical extensions $Th(\{A, B, \neg C, E\})$. On the other hand, applying the default rule $\frac{:\neg A}{\neg A}$ blocks the default rule $\frac{:A\wedge B}{A\wedge B}$ and therefore does not prevent the joint application of two of the last three default rules. As illustrated in Section 3.3.1, this is a source of incoherence and, therefore, does not allow for classical extensions.

However, turning the default theory (3.6) into (3.7) by adding the skeptical nonmonotonic theorem B to the set of facts results in the monotonic theorem $\neg C$. [11] As a consequence, the default rule $\frac{:E\wedge\neg C}{E}$ is applicable regardless of which of the first two default rules applies. The default rules $\frac{:C\wedge\neg D}{C}$ and $\frac{:D\wedge\neg E}{D}$ are inapplicable. Hence, we obtain one classical extension generated by the first default rule $\frac{:A\wedge B}{A\wedge B}$ and the default rule $\frac{:E\wedge\neg C}{E}$, and another generated by the second default rule $\frac{:\neg A}{\neg A}$ and the default rule $\frac{:E\wedge\neg C}{E}$. Thus, we have obtained a second classical extension, which exhibits that cumulativity does not hold for prerequisite-free semi-normal default theories as long as there is a source of incoherence.

On the whole, there seems to be no way to preserve cumulativity in the case of credulous reasoning. Let us illustrate this by means of the following example.

Example 3.5.3 *The prerequisite-free normal default theory*

$$\left(\left\{\frac{:A}{A}, \frac{:\neg A\wedge C}{\neg A\wedge C}\right\}, \emptyset\right)$$

has two classical extensions: $Th(\{A\})$ and $Th(\{\neg A, C\})$.

Adding the credulous nonmonotonic theorem C to the premises yields the default theory

$$\left(\left\{\frac{:A}{A}, \frac{:\neg A\wedge C}{\neg A\wedge C}\right\}, \{C\}\right)$$

which has also two classical extensions: $Th(\{A, C\})$ and $Th(\{\neg A, C\})$.

This example shows that the addition of credulous nonmonotonic theorems does not preserve existing classical extensions. Hence, cumulativity fails even for the very simple class of prerequisite-free normal default theories in the case of credulous theory formation.

3.5.2 Lemma default rules

As mentioned above, cumulativity is of great practical relevance. This is because cumulative consequence relations allow for the use of lemmas needed for reducing computational efforts. Since computation in nonmonotonic logics does not only involve deduction but also expensive consistency checks, the need to incorporate lemmas is even greater in nonmonotonic theorem proving than in standard theorem proving.

As we have seen in Section 3.5.1, it is necessary to be aware of a conclusion's underlying assumptions if we want to preserve cumulativity. But since classical extensions consist of first-order formulas the question arises how to represent these assumptions.

Inspired by default logic's natural distinction between facts and defaults, we view nonmonotonic lemmas as abbreviations for the corresponding default inferences. Thus, it is natural

[11] Compare this with the case of the only classical extensions of the default theory (3.6) where $\neg C$ was nonmonotonically derivable.

to add them as default rules. As a result, we introduce the notion of a *lemma default rule*.[12] That is, informally, in order to lemmatize[13] a nonmonotonic theorem, we take this theorem along with one of its minimal default proofs and construct the corresponding lemma default rule in a certain way.

But before introducing lemma default rules themselves, we have to account for the notion of a *default proof* in classical default logic. In analogy to [Reiter, 1980, Definition 3], we define a default proof as follows.

Definition 3.5.1 *Let (D, W) be a default theory and let S be a set of formulas. A default proof of ρ in S from (D, W) is a sequence $\langle D_1, \ldots, D_k\rangle$ of sets of default rules where $D_i \subseteq GD_D^S$ for $1 \leq i \leq k$ and $\bigcup_{i=1}^k D_i$ is a minimal set of default rules such that*

1. $W \vdash \mathit{Prereq}(D_1)$,
2. $W \cup \mathit{Conseq}(D_i) \vdash \mathit{Prereq}(D_{i+1})$ *for* $1 \leq i \leq k-1$,
3. $W \cup \mathit{Conseq}(D_k) \vdash \rho$.

Note that the sets of default rules $D_1, \ldots, D_n$ are not necessarily distinct. Also, notice that, given a classical extension E, by compactness and groundedness any formula $\gamma \in E$ has a finite default proof which is itself composed of finite sets of default rules.

Then, we define the lemma default rule for a (default) conclusion as follows.

Definition 3.5.2 *Let (D, W) be a default theory and let E be a classical extension of (D, W). Let $\rho \in E$ and $\langle D_1, \ldots, D_k\rangle$ be a default proof of ρ in E from (D, W). We define a lemma default rule ζ_ρ for ρ as*

$$\zeta_\rho = \frac{: \mathit{Justif}(\delta_1), \ldots, \mathit{Justif}(\delta_n)}{\rho}$$

where $\bigcup_{i=1}^k D_i = \{\delta_1, \ldots, \delta_n\}$.

Observe that we obtain a non-singular (prerequisite-free) default rule. This is due to the fact that we have to preserve the consistency of each justification separately.

The main idea behind lemma default rules is expressed in the following theorem. With it, we have passed the halfway stage to the main result given in Theorem 3.5.3.

Theorem 3.5.2 *Let (D, W) be a default theory and let E and E' be classical extensions of (D, W). Let $\langle D_1, \ldots, D_k\rangle$ be a default proof of ρ in E', and let ζ_ρ be the corresponding lemma default rule for ρ. Then,*

$$\zeta_\rho \in GD^E_{D \cup \{\zeta_\rho\}} \text{ iff } \bigcup_{i=1}^k D_i \subseteq GD^E_D.$$

Theorem 3.5.3 *Let (D, W) be a default theory and let E' be a classical extension of (D, W). Let ζ_ρ be a lemma default rule for $\rho \in E'$. Then,*

$$E \text{ is a classical extension of } (D, W) \text{ iff } E \text{ is a classical extension of } (D \cup \{\zeta_\rho\}, W).$$

[12] In order to avoid redundancy, the present section gives only a brief exposition of the approach taken by lemma default rules, since this approach is discussed in more detail in Section 4.10.

[13] Ie. to introduce a derivable theorem as a lemma by adding it to the set of facts.

Thus, it is now possible to enrich default logic such that it admits the generation and the use of nonmonotonic lemmas without altering the logical formalism as such. We will describe in Section 5.3 a complementary approach which has to extend the language and the consequence relation in order to preserve cumulativity.

Let us illustrate the basic idea by reconsidering Example 3.5.1 and the way the failure of cumulativity is tackled by lemma default rules. In order to lemmatize the default conclusion $A \vee B$ of the default theory (3.4) we have to add the lemma default rule $\frac{:A}{A \vee B}$ to the default rules of the default theory (3.4). This is because the default proof of $A \vee B$ is simply $\langle \{ \frac{:A}{A} \} \rangle$.

Example 3.5.4 *The default theory (3.4)*

$$\left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A} \right\}, \emptyset \right)$$

has one classical extension: $Th(\{A\})$.

Adding the lemma default rule $\frac{:A}{A \vee B}$ for the nonmonotonic theorem $A \vee B \in Th(\{A\})$ to the set of default rules yields the default theory

$$\left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}, \frac{:A}{A \vee B} \right\}, \emptyset \right)$$

which has the same classical extension: $Th(\{A\})$.

As the example illustrates, lemma default rules neither produce any new extensions nor delete any previous extensions.

An extensive discussion of lemma default rules is given in Section 4.10. Therefore, we restrict our discussion at this point and conclude this section with a final remark on the formal property of cumulativity. Cumulativity is a property of consequence relations. Consequence relations themselves are concerned with sets of formulas. As a consequence, the approach taken by lemma default rules does not account for cumulativity strictly according to formal regulations. Rather it provides extra-logical means that change the representation of nonmonotonic conclusions, whenever they become nonmonotonic lemmas. However, we will see in Section 5.3 that this approach has advantages over others which manipulate the formulas as the objects of discourse.

3.6 A semantics for classical default logic

Although classical default logic has intuitively been well understood, it took several years until a model-theoretic semantics was given. As a first step, Lukaszewicz [1985] provides a semantical characterization of normal default theories. The general idea is that every normal default rule can be regarded as a transition from classes of models to classes of models. In other words, "*default logic's semantics can be viewed in terms of restrictions of the set of models of the underlying theory. The first-order theory partially specifies a world, which is further specified by the defaults . Each default can be viewed as extending the world-description by restricting the set of possible worlds assumed to contain the "real" world, at the same time constraining how other defaults may further extend the world-description.* " [Etherington, 1987c, p 496].

In order to account for general default theories and their behavior, Etherington introduced in [1987c] a *preference relation* $\geq_\delta$ between classes of models of W.[14] Intuitively, this relation

[14] If it is clear from the context, we simply speak about models rather than models of W.

captures a default rule's preference for more specialized world descriptions. Unlike other approaches (eg. circumscription) that impose a preference relation on classes of models, here the same is done on the power class.

Formally, the preference relation $\geq_\delta$ is defined as follows.

Definition 3.6.1 *Let $\delta = \frac{\alpha:\beta}{\gamma}$ and Π be a class of first-order interpretations. The order $\geq_\delta$ on 2^Π is defined as follows. For all $\Pi_1, \Pi_2 \in 2^\Pi$*

$$\Pi_1 \geq_\delta \Pi_2$$

holds iff

1. $\forall \pi \in \Pi_2 . \pi \models \alpha$,
2. $\exists \pi \in \Pi_2 . \pi \models \beta$,
3. $\Pi_1 = \{\pi \in \Pi_2 \mid \pi \models \gamma\}$.

That is, a default rule $\frac{\alpha:\beta}{\gamma}$ prefers a class of models Π_1 in which its consequent γ holds over a superclass of models Π_2 where the prerequisite α is true and the justification β is consistent but the consequent is not necessarily satisfied.

The induced order $\geq_D$ is defined as the transitive closure of all orders $\geq_\delta$ such that $\delta \in D$.

Definition 3.6.2 *Let D be a set of default rules and Π a class of first-order interpretations. The order $\geq_D$ on 2^Π is defined as follows. For all $\Pi_1, \Pi_2 \in 2^\Pi$ we have*

$$\Pi_1 \geq_D \Pi_2$$

iff

1. $\exists \delta \in D.\ \Pi_1 \geq_\delta \Pi_2$ *or*
2. $\exists \Pi_3 \in 2^\Pi.\ \Pi_1 \geq_D \Pi_3$ *and* $\Pi_3 \geq_D \Pi_2$.

For normal default theories it is sufficient to take into account the $\geq_D$-maximal elements of $2^{\mathrm{MOD}(W)}$. However, an additional so-called *stability* condition is necessary in order to capture general default theories. The reason is that we have to ensure the satisfiability of each justification of the applied default rules by the resulting class of models.[15]

Definition 3.6.3 *Let (D, W) be a default theory and Π a $\geq_D$-maximal set of first-order interpretations in $2^{\mathrm{MOD}(W)}$.*

Π is called stable for (D, W) iff there is a set of default rules $D' \subseteq D$ such that

1. $\Pi \geq_{D'} \mathrm{MOD}(W)$,
2. $\forall \delta = \frac{\alpha:\beta}{\gamma} \in D'. \exists \pi \in \Pi . \pi \models \beta$.

Then, Etherington shows in [1987c] the following correctness and completeness results.

Theorem 3.6.1 (Correctness) *Let (D, W) be a default theory. If E is a classical extension of (D, W), then $\mathrm{MOD}(E)$ is stable for (D, W).*

Theorem 3.6.2 (Completeness) *Let (D, W) be a default theory. If Π is stable for (D, W), then $\{\alpha \mid \Pi \models \alpha\}$ is a classical extension of (D, W).*

[15] This semantically accounts for the continued consistency of the justifications (cf. Section 3.3.2).

Regarding the semantics, we now see why classical default logic does not commit to assumptions. Together, the second condition of Definition 3.6.1 and 3.6.3 require only one model to satisfy the justification of a considered default rule. But it is not required that there has to be one model satisfying all of the justifications of the default rules used during a derivation. Hence, concerning the justifications, classical default logic preserves only a kind of "distributed consistency".

This becomes obvious, if we take a closer look at the model structure obtained in Example 3.3.6.

Example 3.6.1 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right)$$

has one stable class of models: $\mathrm{MOD}(\{C, D\})$.

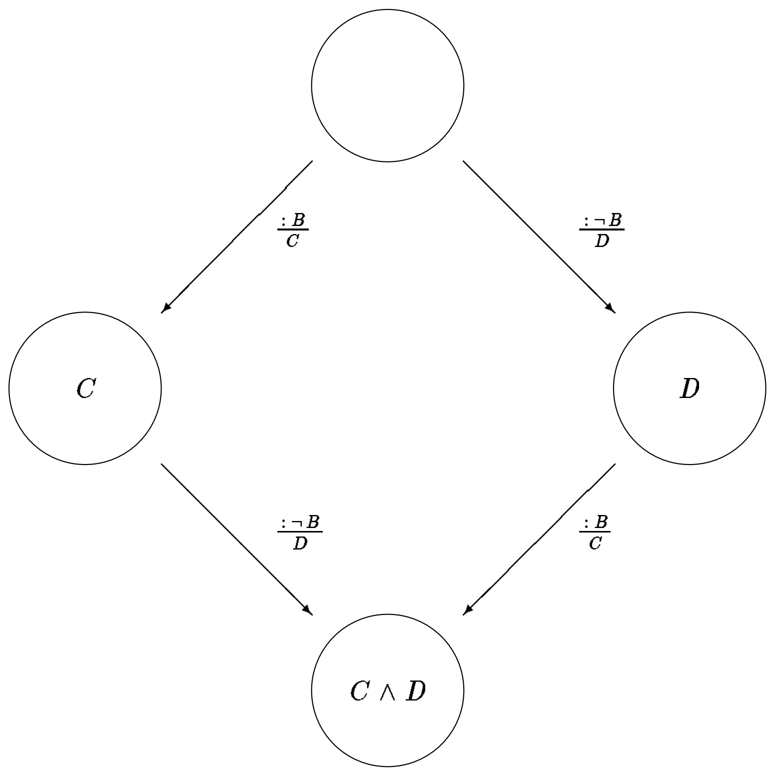

Figure 3.2: Non-commitment to assumptions in classical default logic.

Looking at the order induced by the set of default rules, illustrated in Figure 3.2, we observe one stable class of models $\mathrm{MOD}(\{C, D\})$. However, this class is properly divided into two classes of models: those satisfying B and those falsifying B. Although both default rules apply there is no model satisfying both justifications. In particular, there is no model satisfying the implicit assumptions underlying the default conclusion $C \wedge D$.

Let us also reexamine the failure of cumulativity and its semantical background in order to obtain a better understanding for this failure. Consider again Example 3.5.1, illustrated in Figure 3.3.

Example 3.6.2 *The default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right)$$

has one stable class of models: $\mathrm{MOD}(\{A\})$.

Adding the nonmonotonic theorem $A \vee B \in Th(\{A\})$ *to the set of facts yields the default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right)$$

whose stable classes of models are: $\mathrm{MOD}(\{A\})$ *and* $\mathrm{MOD}(\{\neg A, B\})$.

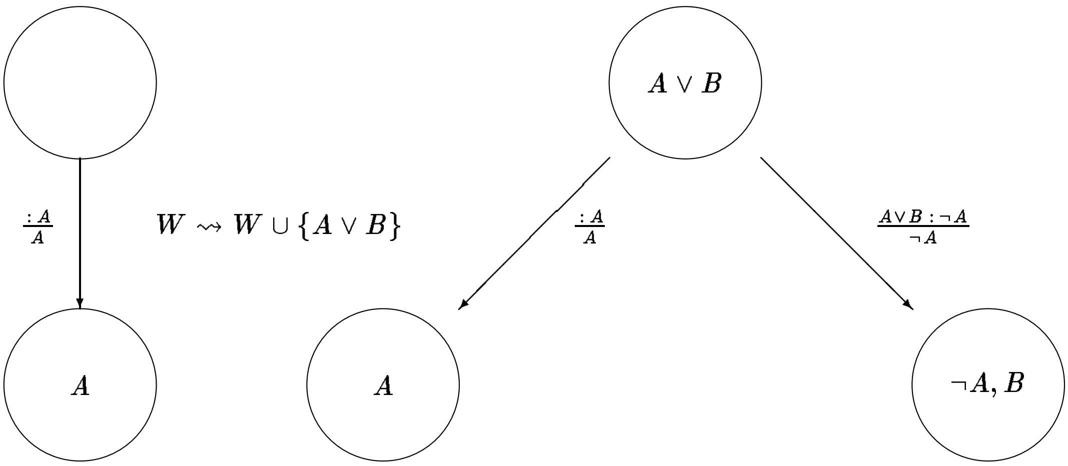

Figure 3.3: The semantical failure of cumulativity in classical default logic.

Regarding Figure 3.3, we observe that adding the nonmonotonic theorem $A \vee B$ to the set of premises of the default theory (3.5) not only enlarges the number of stable classes of models, it also enlarges the "models under consideration". For example, the stable class of models $\mathrm{MOD}(\{A\})$ belonging to the default theory (3.4) is turned into $\mathrm{MOD}(\{A \vee B\})$ by the addition of the nonmonotonic theorem $A \vee B$. Also, the semantical structures change whenever we apply a default rule.

Let us compare this with standard logic. Clearly, a set of axioms W has the same class of models as its deductive closure $Th(W)$. Also, adding a theorem $\alpha \in Th(W) \setminus W$ to W leaves the class of models unchanged. That is, $W \cup \{\alpha\}$ and $Th(W \cup \{\alpha\})$ have the same class of models as W. So, which information is lost when we add nonmonotonic theorems to premises of default theories?

The answer is quite simple. In both previous examples, the counterintuitive results occur because the implicit consistency assumptions have been lost. Classical default logic deals with nonmonotonic theorems but it is not "aware" of the underlying consistency assumptions.

3.7 Conclusion

We have described classical default logic and its properties. In particular, we have discussed its limitations by means of several canonical examples, which will be used throughout this thesis.

In addition, we have described several subsystems of classical default logic which avoid some of the shortcomings. Two somehow orthogonal subsystems seem to stand out. Normal default theories guarantee the existence of classical extensions, and provide semi-monotonicity and orthogonality. Prerequisite-free default theories allow for reasoning by cases and reasoning by contraposition. Prerequisite-free normal default theories — as the common subsystem — additionally guarantees cumulativity and allows for reasoning about default rules.

The reasons for this good behavior of normal and prerequisite-free default theories can be summarized as follows. Normal default rules are distinguished from all others since their respective justifications and consequents are equivalent to each other. Thus, if a normal default rule applies, the consistency of its justification is explicated by adding the consequent of the default rule to the respective extension. As a result, all justifications of applying normal default rules are jointly consistent to each other. As we have argued in Section 3.3.4, this leads to commitment to assumptions, semi-monotonicity and guarantees the existence of extensions. On the other hand, prerequisite-free default theories allow us to retain the properties of standard implications (unless explicitly blocked). The emphasis then shifts to the implication itself, rather than a rule involving a prerequisite and a justification for a conclusion. As a result, prerequisite-free default rules allow for reasoning by cases and for reasoning by contraposition.

We have paid a great deal of attention to the failure of cumulativity, since we were interested in its most important practical consequence: the capability of handling nonmonotonic lemmas. We have addressed this failure by introducing lemma default rules as a general proof-theoretic approach to allow for the use and generation of nonmonotonic lemmas in default logics. We have seen in Section 3.5.1 that it is necessary to be aware of a conclusion's underlying assumptions if we want to preserve cumulativity. Thus, in order to lemmatize a nonmonotonic theorem, we take this theorem along with one of its minimal default proofs from which we extract the assumptions underlying this theorem, and construct the corresponding lemma default rule in a certain way. However, the approach has only been described briefly since it will be further elaborated in Section 4.10.

Finally, we have described a semantical characterization of classical default logic. This has been analyzed and used to further investigate the failure of commitment to assumptions and cumulativity. In particular, we have seen that these counterintuitive results occur because the implicit consistency assumptions have been lost. Classical default logic deals with nonmonotonic theorems but it is not aware of the underlying consistency assumptions.

Chapter 4

Constrained default logic

In this chapter, we present a new variant of default logic which addresses several limitations of the original approach. We introduce the notion of a *constrained extension* and refer to the resulting system as *constrained default logic.* We start by motivating the development of constrained default logic with a brief discussion of some shortcomings encountered in classical default logic. Section 4.2 is devoted to the formal development of constrained default logic. The approach is further elaborated in Section 4.3. Therein, we give the basic properties of the new formalization and show how it addresses the problems encountered in classical default logic. Afterwards, we examine in detail the relationship between constrained default logic and its classical counterpart. Section 4.5 presents the *focused models semantics* which serves as a model-theoretic semantics for constrained default logic and provides useful insights into the enhancements of the underlying approach. In Section 4.6, we show that prerequisite-free default theories preserve cumulativity whenever we are reasoning skeptically. Section 4.7 completes the treatment of prerequisite-free default theories. In Section 4.8, we propose a system called *pre-constrained default logic* by slightly extending the approach taken by constrained default logic. This system allows for predetermined constraints and serves as a basis for further extensions of the new formalization. Section 4.9 shows how priorities among default rules can be incorporated into constrained default logic. The resulting system is called *prioritized constrained default logic.* Section 4.10 introduces lemma default rules for constrained default logic and further elaborates how the approach accounts for the practical impact of cumulativity, namely the adequate use of nonmonotonic lemmas.

4.1 Motivation

Human commonsense reasoning is strongly based on the ability to draw conclusions upon non-verifiable assumptions or simply working assumptions. However, we argue that people do not arbitrarily assume things; rather they keep track of their assumptions and at least verify that they do not contradict each other. No one would justify a conclusion by an assumption as well as its opposite.

However, we have seen in Section 3.3 that classical default logic fails in this respect. As a first example, let us reconsider the default theory given in Example 3.3.6:

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right)$$

As we have seen in Section 3.3.4, the above default theory allows us to conclude $C \wedge D$. This is because in classical default logic both default rules apply, even though they have contradictory justifications.

Since in standard logic one of B or $\neg B$ is false, one of the default conditions cannot hold. Therefore, both default rules should not jointly apply. In particular, the default conclusion $C \wedge D$ is somewhat justified by B and $\neg B$. Clearly, this clashes with the intuition described above.

Also, recall that the foremost aim of any formalism in knowledge representation is to give a formal description of the world. However, the world itself is complete[1] and it, therefore, admits either B or $\neg B$ but not both of them. In this respect, classical default logic fails to be an appropriate model for describing the world.

As a second example take the default theory given in Example 3.3.4:

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{\neg B}\right\}, \emptyset\right)$$

In classical default logic, we obtain one set of beliefs in which B is false. As argued in Section 3.3.2, this seems to be counterintuitive. If we know nothing else originally, then there seems to be no good reason that the first default rule is not applicable. A corresponding argument is given on page 34.

In both cases classical default logic produces conclusions that, intuitively, are stronger than one wants. In particular, we observe that implicit assumptions are not treated in the way described at the start of this section. By virtue of this observation, we strengthen the meaning of justifications in default logic towards unverifiable reasons for believing something. In particular, we require the *set* of justifications used in the specification of an extension to be consistent, rather than each individual justification.

4.2 Constrained default logic

In order to avoid the limitations described in Section 3.3, we introduce the notion of a *constrained extension* and call the resulting system *constrained default logic*. A constrained extension is composed of two sets of formulas E and C, where $E \subseteq C$. The extension E contains all formulas which are assumed to be true; the set of constraints C consists of E and the justifications of all applying default rules. In this approach, we regard the consistency assumptions, ie. the justifications, as constraints on a given extension. This is illustrated in Figure 4.1. There, given a constrained extension (E,C), the extension E is meant to be constrained by C. In particular, the figure illustrates the natural set inclusion between the facts W, their deductive closure $Th(W)$, the extension E and its constraints C. In this respect, the deductive closure of W, namely $Th(W)$, constitutes a lower bound whereas the constraints C constitute an upper bound for our set of beliefs represented by E.

For a default rule $\frac{\alpha : \beta}{\gamma}$ to apply in classical default logic, its prerequisite α must be in E and its justification β has to be consistent with E. In constrained default logic, however, the prerequisite α is proven from the extension E whereas the consistency of the justification β is checked wrt the set of constraints C. Compare Figure 3.1 on page 29 with Figure 4.2 for an illustration of this.[2]

Intuitively, the constraints can be regarded as a *context* established by the premises, the nonmonotonic theorems (ie. all conclusions derived by means of default rules), as well as all underlying consistency assumptions. In this sense, constrained default logic naturally extends the intrinsic context-sensitive character of default rules by distinguishing between our set of

[1] Ie. any proposition is either true or false.

[2] Figure 4.2 is described in detail further below.

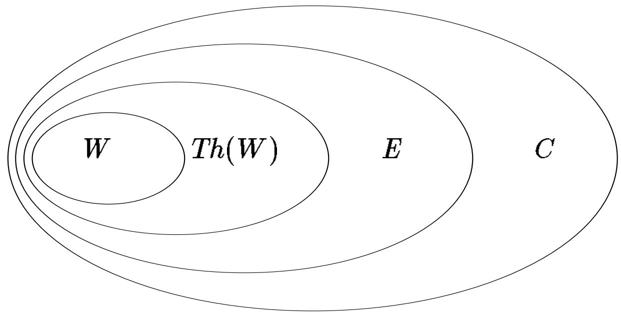

Figure 4.1: A constrained extension (E, C) of a default theory (D, W).

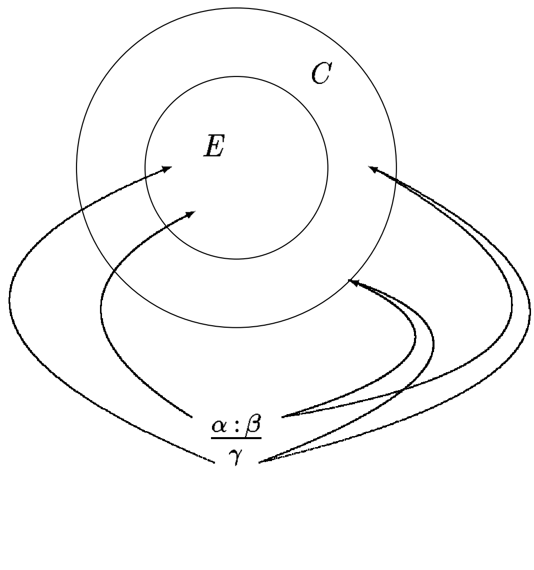

Figure 4.2: The interplay of constrained extensions and default rules.

beliefs, ie. the extension, and the underlying constraints which form a context guiding our beliefs.

Although this slightly complicates the definition of an extension, it also means that rules and extensions are now represented uniformly, in that both consist of a consistency condition along with conclusions based on the consistency conditions.

Definition 4.2.1 *Let (D, W) be a default theory. For any set of formulas T, let $\Upsilon(T)$ be the pair of smallest sets of formulas (S', T') such that*

1. *$W \subseteq S' \subseteq T'$,*
2. *$S' = Th(S')$ and $T' = Th(T')$,*
3. *For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$.*

A pair of sets of formulas (E,C) is a constrained extension of (D, W) iff $\Upsilon(C) = (E,C)$.

The set of constraints is generated by accumulating the justifications from the applied default rules along with the drawn conclusions. Furthermore, the constraints are required to be deductively closed such that they form a superset of the actual extension. The extension itself is constructed in a similar way as in classical default logic (cf. Definition 3.1.1), with the important exception that in order to apply a default rule its justification and its consequent have to be consistent wrt the set of constraints. Thus, each justification has to be jointly consistent with the extension and all other justifications. All this is illustrated once more in Figure 4.2: Given a constrained extension (E,C), a default rule $\frac{\alpha:\beta}{\gamma}$ applies, if its prerequisite α is in the extension E and if its justification β and its consequent γ are consistent with the set of constraints C. If this is the case, the default rule applies by adding the consequent γ to the extension E, whereas, in addition to the consequent, the justification β is added to the set of constraints C.

Notably, compared with Definition 3.1.1, the fixed-point condition relies merely on the constraints. Intuitively, this means that our context of reasoning, T, has to coincide with our set of accumulated constraints, T'. Finally, one should observe that Definition 3.1.1 is still in accord with Reiter's three postulates for an extension (given on page 19). That is, an extension should contain our initial set of facts, it should be deductively closed, and it should contain each consequent of any applicable default rule.

Now, first of all, let us sketch the approach using the simple default theory $(\{\frac{A:B}{C}\}, \{A\})$. Instead of a "flat" extension $Th(\{A, C\})$ as in classical default logic, we now obtain in constrained default logic an extension that is embedded in a context, namely the constrained extension

$$(Th(\{A, C\}), Th(\{A, B, C\})).$$

This is depicted in Figure 4.3.[3]

Although the above example conveys an intuition for the definition of constrained extensions, it remains to be shown that there always exists a pair of smallest sets of formulas satisfying the above requirements. This is done in the next theorem.

Theorem 4.2.1 *The set of all pairs of sets of formulas (S, T) satisfying the conditions 1. to 3. of Definition 4.2.1 is closed under intersection.*

Now, let us reconsider the examples at the start of this chapter. In both examples, we now obtain two constrained extensions.

[3]For simplicity, we label the set of constraints only with the additional constraints.

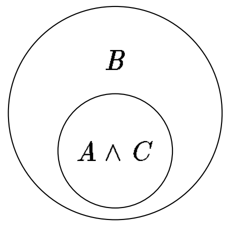

Figure 4.3: The constrained extension $(Th(\{A, C\}), Th(\{A, B, C\}))$.

Example 4.2.1 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right)$$

has two constrained extensions: $(Th(\{C\}), Th(\{C, B\}))$ *and* $(Th(\{D\}), Th(\{D, \neg B\}))$.

That is, we obtain one constrained extension in which C is true and the value of D is unspecified, and another in which D is true and the value of C is unspecified. In the first constrained extension, the constraints consist of the justification and the consequent of the first default rule, C and B. In the second constrained extension, the constraints contain the justification $\neg B$ and the consequent D of the second default rule.

Also in Example 3.3.4, we now obtain two constrained extensions, as shown in the next example.

Example 4.2.2 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{\neg B}\right\}, \emptyset\right)$$

has two constrained extension: $(Th(\{C\}), Th(\{C, B\}))$ *and* $(Th(\{\neg B\}), Th(\{\neg B\}))$.

That is, we obtain one constrained extension in which C is true under the condition that $C \wedge B$ is consistent and another in which $\neg B$ is true. Notice that the two constrained extensions are obtained in correspondence with the reasoning pattern described on page 34.

Thus, in terms of "commitment to assumptions", the extension represents what we believe about the world whereas the constraints tell us what we have committed to in order to adopt our beliefs. Hence, intuitively, an extension is our envisioning of how things are, whereas the context represents additionally our expectations of how things might be. As we have seen by means of the two previous examples, this approach makes constrained default logic commit to assumptions.

We have seen in Section 3.3 that in classical default logic any default theory can be represented by means of a semi-normal default theory. However, this may result in much larger theories than the original ones. Now, in constrained default logic, we obtain a direct correspondence between general and semi-normal default theories as an immediate consequence of Definition 4.2.1:

Corollary 4.2.2 *Let* (D, W) *be a default theory and let*

$$D' = \left\{\tfrac{\alpha : \beta \wedge \gamma}{\gamma} \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D\right\}.$$

Let E *and* C *be sets of formulas. Then,* (E, C) *is a constrained extension of* (D, W) *iff* (E, C) *is a constrained extension of* (D', W).

4.3 Properties of constrained default logic

As in classical default logic (cf. Theorem 3.2.1), we are able to provide an "iterative" and, hence, more intuitive characterization of constrained extensions.

Theorem 4.3.1 *Let (D,W) be a default theory and let E and C be sets of formulas. Define*

$$E_0 = W \qquad \textit{and} \qquad C_0 = W$$

and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ C_{i+1} &= Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \end{aligned}$$

(E,C) is a constrained extension of (D,W) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

The theorem further clarifies the role of the constraints. When computing an extension, reference is made to the previous partial extension E_i, whereas the consistency is checked wrt all constraints, namely C. Consequently, the above characterization is not strictly iterative.

As a consequence, we obtain that constrained extensions are uniquely determined by their sets of constraints.

Corollary 4.3.2 (Uniqueness) *Let (D,W) be a default theory and (E,C) and (E',C') constrained extensions of (D,W). If $C = C'$ then $E = E'$.*

Obviously, the converse of the above corollary does not hold.

Example 4.3.1 *The default theory*

$$\left(\left\{ \frac{:B}{C}, \frac{:\neg B}{C} \right\}, \emptyset \right)$$

has two constrained extensions: $(Th(\{C\}), Th(\{C,B\}))$ and $(Th(\{C\}), Th(\{C,\neg B\}))$.

That is, we obtain two constrained extensions which share the extension $Th(\{C\})$ but which have different sets of constraints, namely $Th(\{C,B\})$ and $Th(\{C,\neg B\})$. However, this example shows that constrained default logic explicates hidden consistency assumptions. As a result of this transparency, constrained extensions reveal different arguments for the same conclusion (see also Section 3.3.3 on classical default logic). For instance, we may conclude C either by assuming that B is consistent or by assuming that $\neg B$ is consistent in Example 4.3.1. We will formally account for this phenomena in Theorem 4.3.9.

Analogously to [Reiter, 1980], we have the following corollaries to Theorem 4.3.1.

Corollary 4.3.3 *Let (D,W) be a default theory and (E,C) a constrained extension of (D,W). Then, we have*

- *C is inconsistent iff E is inconsistent iff W is inconsistent,*
- *if (E,C) is an inconsistent constrained extension (ie. E and C are inconsistent) then (E,C) is the only constrained extension of (D,W).*

Thus, if one of W, E or C is inconsistent, then all of them are inconsistent.

The next result accounts for the *pairwise maximality* of constrained extensions.

Theorem 4.3.4 (Pairwise maximality) *Let (D, W) be a default theory and let (E,C) and (E', C') be constrained extensions of (D, W). Then $E \subseteq E'$ and $C \subseteq C'$ implies $E = E'$ and $C = C'$.*

However, the set of beliefs is not necessarily maximal, as can be seen in the next example [Lukaszewicz, 1988].

Example 4.3.2 *The default theory*

$$\left(\left\{\frac{:B}{A}, \frac{:D}{\neg B}\right\}, \{A\}\right)$$

has two constrained extensions: $(Th(\{A\}), Th(\{A, B\}))$ and $(Th(\{A, \neg B\}), Th(\{A, \neg B, D\}))$.

That is, we may obtain constrained extensions whose actual extensions are included in those of other constrained extensions.

In addition to Theorem 4.3.1, we can characterize constrained extensions by means of their generating default rules, which are defined as follows.

Definition 4.3.1 *Let (D, W) be a default theory and S and T sets of formulas. The set of generating default rules for (S, T) wrt D is defined as*

$$GD_D^{(S,T)} = \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \;\middle|\; \alpha \in S,\ \ T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}.$$

Observe that we do not require (S, T) to be a constrained extension. Thus, our definition is slightly more general than those usually presented (eg. [Reiter, 1980, Definition 2]). In particular, the definition applies to classical default logic in the case of $S = T$.

Now, constrained extensions are characterized unambiguously by their set of generating default rules as is shown next.

Theorem 4.3.5 *Let (E,C) be a constrained extension of a default theory (D, W). We have*

$$\begin{aligned} E &= Th\Big(W \cup Conseq\Big(GD_D^{(E,C)}\Big)\Big), \\ C &= Th\Big(W \cup Conseq\Big(GD_D^{(E,C)}\Big) \cup Justif\Big(GD_D^{(E,C)}\Big)\Big). \end{aligned}$$

Next, we have the following property of groundedness, which distinguishes all variants of default logic from other approaches to default reasoning (eg. autoepistemic logic).

Theorem 4.3.6 (Groundedness) *Let (E,C) be a constrained extension of (D, W). Then, there exists an enumeration $\langle\delta_i\rangle_{i\in I}$ of $GD_D^{(E,C)}$ such that for $i \in I$*

$$W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i).$$

In Section 3.3, we discussed several shortcomings encountered in classical default logic. For instance, we saw that classical default logic does not guarantee the existence of extensions and does not commit to assumptions. We have already shown at the end of the previous section that constrained default logic commits to assumptions (cf. Example 4.2.1 and 4.2.2). In what follows, we deal with the rest of the problems discussed in Section 3.3.

First, we show that constrained default logic enjoys the property of semi-monotonicity. That is, constrained default logic is monotonic wrt the default rules.

Theorem 4.3.7 (Semi-monotonicity) *Let (D, W) be a default theory and D' a set of default rules such that $D \subseteq D'$. If (E,C) is a constrained extension of (D, W), then there is a constrained extension (E', C') of (D', W) such that $E \subseteq E'$ and $C \subseteq C'$.*

Semi-monotonicity implies that constrained extensions are constructible in a truly iterative way by applying one applicable default rule after another. Thus, the consistency of each justification has only to be checked wrt the previous partial set of constraints induced by the facts and all hitherto applied default rules. As argued in Section 3.3.2, the property of semi-monotonicity is of great practical relevance, since it allows for local proof procedures which may discard irrelevant default rules.

Let us return to the default theories used in Example 3.3.3 (to illustrate the failure of semi-monotonicity in classical default logic) and see how these default theories are treated in constrained default logic.

Example 4.3.3 *The default theory*

$$\left(\left\{\frac{:B}{C}\right\},\emptyset\right)$$

has one constrained extension: $(Th(\{C\}), Th(\{B, C\}))$.

Adding the default rule $\frac{:D}{\neg B}$ *yields the default theory*

$$\left(\left\{\frac{:B}{C},\frac{:D}{\neg B}\right\},\emptyset\right)$$

which has two constrained extensions: $(Th(\{C\}), Th(\{B, C\}))$ *and* $(Th(\{\neg B\}), Th(\{\neg B, D\}))$.

Thus, once we have applied the default rule $\frac{:B}{C}$, we preserve the consistency of its justification B by adding it to the set of constraints. As a consequence, the second default rule $\frac{:D}{\neg B}$ is not applicable since it violates the consistency of the constraints accumulated so far.

As an immediate consequence of semi-monotonicity, we obtain that the *existence of constrained extensions* is guaranteed.

Theorem 4.3.8 (Existence of extensions) *Every default theory has a constrained extension.*

Let us also revisit Example 3.3.1, which illustrated that classical default logic does not guarantee the existence of extensions.

Example 4.3.4 *The default theory*

$$\left(\left\{\frac{:\neg A}{A}\right\},\emptyset\right)$$

has one constrained extension: $(Th(\emptyset), Th(\emptyset))$.

That is, the default rule $\frac{:\neg A}{A}$ is not applicable since according to Definition 4.2.1 its justification and its consequent should be consistent. As a result, we obtain a constrained extension, namely $(Th(\emptyset), Th(\emptyset))$.

In classical default logic, the two previous properties hold only for *normal* default theories. Another property which holds only for normal default theories in classical default logic is referred to as orthogonality (cf. Section 3.3.3); this means that two different extensions are always contradictory to each other. A similar property, holds for constrained extensions; we refer to this as *weak orthogonality.*

Theorem 4.3.9 (Weak orthogonality) *Let* (D, W) *be a default theory. If* (E,C) *and* (E',C') *are distinct constrained extensions*[4] *of* (D, W), *then* $C \cup C'$ *is inconsistent.*

[4] According to Corollary 4.3.2, that is $C \neq C'$.

That is, given two different constrained extensions, the constraints of both extensions are contradictory to each other.

Example 4.3.5 *The default theory*

$$\left(\left\{\frac{:\neg B}{C}, \frac{:\neg C}{B}\right\}, \emptyset\right)$$

has two constrained extensions: $(Th(\{C\}), Th(\{C, \neg B\}))$ *and* $(Th(\{B\}), Th(\{B, \neg C\}))$.

Obviously, the two alternative constrained extensions for the same default theory stem from incompatible sets of constraints. In contrast to classical extensions, which hide their underlying consistency assumptions (cf. Section 3.3.3) and, therefore, lack transparency, constrained extensions exhibit these assumptions. In other words, constrained extensions explicate contradictory arguments which lead to alternative but not necessarily contradictory conclusions.

4.4 Constrained versus classical default logic

We characterize here the relationship between classical and constrained default logic.[5] In the previous section, we have seen that constrained default logic extends all properties "classically" possessed by normal default theories to general default theories. The tight relationship between classical and constrained default logic in the case of normal default theories is mirrored by the fact that the approaches coincide in this particular case.

Theorem 4.4.1 *Let* (D, W) *be a normal default theory and* E *a set of formulas. Then,* E *is a classical extension of* (D, W) *iff* (E, E) *is a constrained extension of* (D, W).

Obviously, the above result extends to C-normal default theories (cf. Section 3.3), since they simply impose an additional but fixed consistency condition on the respective extensions.

At first sight, it seems that constrained default logic is weaker than its classical counterpart, ie. that for any classical extension E' there is a constrained extension (E,C) such that $E \subset E'$. To see that this is not the case consider the following two examples.

Example 4.4.1 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg C}{E}, \frac{:\neg D}{F}\right\}, \emptyset\right) \tag{4.1}$$

has one classical extension: $Th(\{C, D\})$.

Example 4.4.2
The default theory (4.1) has three constrained extensions: $(Th(\{C, F\}), Th(\{C, B, F, \neg D\}))$, $(Th(\{D, E\}), Th(\{D, \neg B, E, \neg C\}))$, *and* $(Th(\{E, F\}), Th(\{E, \neg C, F, \neg D\}))$.

The classical extension $Th(\{C, D\})$ is generated by the first two default rules, whereas the last two default rules are blocked. In constrained default logic, however all four default rules contribute to the reasoning process. We obtain one constrained extension including $C \wedge F$ under the constraints $C \wedge F \wedge B \wedge \neg D$. That is, the default rules $\frac{:B}{C}$ and $\frac{:\neg D}{F}$ form the constrained extension. The second constrained extension, asserting $D \wedge E$ while assuming the consistency

[5] Meanwhile, the relationship between classical and constrained default logic has been tightened by the author [Delgrande *et al.*, 1992]. The author.

of $D \wedge E \wedge \neg B \wedge \neg C$, is generated by the default rules $\frac{:\neg B}{D}$ and $\frac{:\neg C}{E}$. Finally, the third constrained extension $(Th(\{E, F\}), Th(\{E, \neg C, F, \neg D\}))$ is generated by the last two default rules $\frac{:\neg C}{E}$ and $\frac{:\neg D}{F}$. On the whole, all three sets of generating default rules in constrained default logic differ from the default rules which led to the only classical extension. As a consequence, none of the extensions obtained in constrained default logic is contained in this classical extension. Therefore, constrained default logic is neither stronger nor weaker than its classical counterpart.

We can describe the relationship between classical and constrained default logic in concrete terms by taking advantage of the justifications of the generating default rules, namely

$$C_E = \left\{ \beta \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E \right\}.^6 \tag{4.2}$$

With it, we obtain the following result.

Theorem 4.4.2 *Let (D, W) be a default theory and let E be a classical extension of (D, W). If $E \cup C_E$ is consistent, then $(E, Th(E \cup C_E))$ is a constrained extension of (D, W).*

Observe that the converse of the above theorem does not hold since classical default logic does not guarantee the existence of extensions. However, if the extensions coincide we have the following relationship.

Theorem 4.4.3 *Let (D, W) be a default theory and let E and C be sets of formulas. If (E,C) is a constrained extension of (D, W) and E is a classical extension of (D, W), then $C \subseteq Th(E \cup C_E)$.*

In order to illustrate that the reversal of Theorem 4.4.3 does not hold, consider again Example 4.3.1. There, we obtain two constrained extensions which have the same extension, $Th(\{C\})$, but different sets of constraints, $Th(\{C, B\})$ and $Th(\{C, \neg B\})$. The default theory in Example 4.3.1 yields the classical extension $Th(\{C\})$ which is obviously identical to the extensions in constrained default logic. Since the default rules $\frac{:B}{C}$ and $\frac{:\neg B}{C}$ generate the classical extension the set of justifications C_E is inconsistent. Thus, we have for both constrained extensions (E,C) that $C \subseteq Th(E \cup C_E)$, but not vice versa.

We have summarized the previous discussion in Table 5.1 on page 89 along with a comparison to other variants of default logic, which will be discussed in Chapter 5.

4.5 The focused models semantics

In order to characterize constrained extensions semantically, we define a preference relation similar to the one given in Section 3.6 for classical default logic [Etherington, 1987c]. Instead of classes of models Π, we consider pairs $(\Pi, \check{\Pi})$ of classes of models such that $\check{\Pi} \subseteq \Pi$ in order to allow for more structured world descriptions. We will refer to these pairs of classes of models as *focused models structures.*

In Section 4.1, we argued that people do not arbitrarily assume things but rather are aware of their assumptions. Moreover, we argue here that people who assume properties also somehow "assume their validity". In other words, they focus on a certain class of models that satisfy their assumptions. Hence, the intuition behind a focused models structure is as follows.

If we view the justifications of default rules as a kind of *working assumptions* the distributed consistency of classical default logic (cf. Section 3.6) is not adequate any longer. This manifests itself primarily in classical default logic's inability to commit to assumptions, as illustrated in

[6] Observe that the membership qualifying property is exactly the third condition in the definition of a classical extension.

Example 3.6.1. Semantically, we instead need to focus on those models satisfying our implicit assumptions. But since we do not require their validity[7], there may exist other models that falsify them and, therefore, somehow "overlap" our focused models regarding our working assumptions. Consequently, we simply impose more structure on the classes of models under consideration, viewing the second component $\breve{\Pi}$ — which is just a subclass of Π — as our focused class of models. The corresponding structure of focused models structures is illustrated in Figure 4.4.

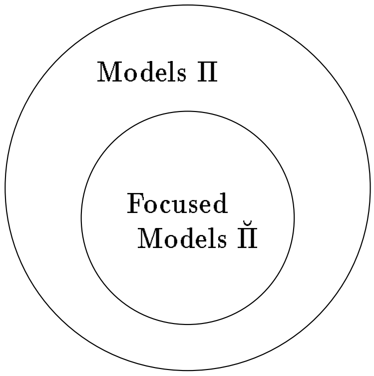

Figure 4.4: A focused models structure $(\Pi, \breve{\Pi})$.

In order to illustrate this briefly, let us reconsider the simple default theory $(\{\frac{A:B}{C}\}, \{A\})$. In Etherington's semantics [1987c] for classical default logic, we characterize the classical extension $Th(\{A, C\})$ by means of a "flat" class of models

$$\Pi = \{\pi \mid \pi \models A, B, C\} \cup \{\pi \mid \pi \models A, \neg B, C\}.$$

Hence, there are as many models satisfying our "working assumption" B as there are models falsifying it. The approach taken by the focused models semantics yields a pair

$$(\Pi, \breve{\Pi}) = (\{\pi \mid \pi \models A, C\}, \{\pi \mid \pi \models A, B, C\})$$

which corresponds to a *structured* class of models including a *focus* which additionally satisfies our implicit assumptions. This is the class of models satisfying A, C, and in particular B. Hence, we admit more structured classes of models by focusing on those models that satisfy our assumptions. The corresponding focused models structure is illustrated in Figure 4.5 (compare with Figure 4.3).[8]

Semantically, a default rule $\frac{\alpha:\beta}{\gamma}$ *prefers* a focused models structure $(\Pi_1, \breve{\Pi}_1)$ to another $(\Pi_2, \breve{\Pi}_2)$ if its prerequisite α is valid in Π_2 and the conjunction of its justification and consequent $\beta \wedge \gamma$ is satisfiable in some focused model in $\breve{\Pi}_2$, and if Π_1 and $\breve{\Pi}_1$ entail the consequent γ (in addition to the previous requirements).

Formally, we achieve all this by defining an order relating the consistency of the justifications with their satisfiability in the focused models.

Definition 4.5.1 *Let $\delta = \frac{\alpha:\beta}{\gamma}$ and Π be a class of first-order interpretations. The order $\succcurlyeq_\delta$ on $2^\Pi \times 2^\Pi$ is defined as follows. For all $(\Pi_1, \breve{\Pi}_1), (\Pi_2, \breve{\Pi}_2) \in 2^\Pi \times 2^\Pi$ we have*

$$(\Pi_1, \breve{\Pi}_1) \succcurlyeq_\delta (\Pi_2, \breve{\Pi}_2)$$

iff

[7] Justifications have only to be consistent.

[8] Again, we label the focused models only with the additional constraints.

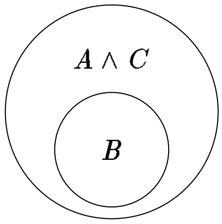

Figure 4.5: The focused models structure $(\mathrm{MOD}(\{A, C\}), \mathrm{MOD}(\{A, B, C\}))$.

1. $\forall \pi \in \Pi_2 . \pi \models \alpha$,
2. $\exists \pi \in \breve{\Pi}_2 . \pi \models \beta \wedge \gamma$,
3. $\Pi_1 = \{\pi \in \Pi_2 \mid \pi \models \gamma\}$,
4. $\breve{\Pi}_1 = \{\pi \in \breve{\Pi}_2 \mid \pi \models \beta \wedge \gamma\}$.

The induced order $\succcurlyeq_D$ is defined as the transitive closure of the union all of orders $\succcurlyeq_\delta$ such that $\delta \in D$.

Definition 4.5.2 *Let D be a set of default rules and Π a class of first-order interpretations. The order $\succcurlyeq_D$ on $2^\Pi \times 2^\Pi$ is defined as follows. For all $(\Pi_1, \breve{\Pi}_1), (\Pi_2, \breve{\Pi}_2) \in 2^\Pi \times 2^\Pi$ we have*

$$(\Pi_1, \breve{\Pi}_1) \succcurlyeq_D (\Pi_2, \breve{\Pi}_2)$$

iff

1. $\exists \delta \in D.\ (\Pi_1, \breve{\Pi}_1) \succcurlyeq_\delta (\Pi_2, \breve{\Pi}_2)$ *or*
2. $\exists (\Pi_3, \breve{\Pi}_3) \in 2^\Pi \times 2^\Pi.\ (\Pi_1, \breve{\Pi}_1) \succcurlyeq_D (\Pi_3, \breve{\Pi}_3)$ *and* $(\Pi_3, \breve{\Pi}_3) \succcurlyeq_D (\Pi_2, \breve{\Pi}_2)$.

For a default theory (D, W), we furthermore define the class of all models of W as Π_W, ie. $\Pi_W = \{\pi \mid \pi \models W\}$. We will refer to the $\succcurlyeq_D$-maximal classes above (Π_W, Π_W) as the *preferred focused models structures* for (D, W).

Compared with [Etherington, 1987c], we have strengthened the notion of consistency by requiring that all justifications and consequents have to be jointly satisfiable by the focused models. In particular, we do not need a stability condition anymore (see Definition 3.6.3). This condition was used in [Etherington, 1987c] in order to ensure the satisfiability of each justification for a given set of default rules and, therefore, to ensure the continued consistency of the justifications of the applying default rules. From a technical point of view, we avoid such a stability condition because we deal with a semi-monotonic default logic.[9] From the viewpoint of the focused models semantics, however, the continued consistency of justifications of already applied default rules is ensured by allowing only those default rules to be applied subsequently which are compatible with the already established focus.

After all, given a preferred focused models structure $(\Pi, \breve{\Pi})$, an extension is formed by all formulas which are valid in Π, whereas the focused models $\breve{\Pi}$ express the constraints surrounding the extension. Accordingly, we have the following correctness and completeness theorem establishing the correspondence between constrained extension and preferred focused models structure for a default theory (D, W).

[9] See Section 3.3.2 for an explanation on how semi-monotonicity is related to the continued consistency of justifications.

Theorem 4.5.1 (Correctness & Completeness) *Let (D,W) be a default theory. Let $(\Pi,\breve{\Pi})$ be a pair of classes of first-order interpretations and E,C deductively closed sets of formulas such that $\Pi = \{\pi \mid \pi \models E\}$ and $\breve{\Pi} = \{\pi \mid \pi \models C\}$. Then, (E,C) is a constrained extension of (D,W) iff $(\Pi,\breve{\Pi})$ is a $\succcurlyeq_D$-maximal element above (Π_W,Π_W).*

As in [Etherington, 1987c], we obtain a simpler semantical characterization in the case of normal default theories.[10] The larger class of models Π collapses to the focused models $\breve{\Pi}$ since normal default rules require their justifications to be valid after they have been shown to be satisfiable.[11]

Looking at Figure 4.6, we see why we obtain two constrained extensions in Example 4.2.1 and therefore two focused models structures as is shown next.

Example 4.5.1 *The default theory*

$$\left(\left\{\frac{:B}{C},\frac{:\neg B}{D}\right\},\emptyset\right)$$

has two preferred focused models structures: $(\mathrm{MOD}(\{C\}),\mathrm{MOD}(\{C,B\}))$ *and* $(\mathrm{MOD}(\{D\}),\mathrm{MOD}(\{D,\neg B\}))$.

Once we have "applied" one of the default rules, the other default rule is not applicable any longer: the focus does not satisfy its justification. Applying one of the default rules does not just require the validity of its consequent; it also makes us focus on its underlying assumption (namely its justification) in order to preserve its satisfiability. For example, adding C under the assumption that B is consistent (by applying the default rule $\frac{:B}{C}$), prohibits us from assuming that $\neg B$ is consistent. Similarly, beginning with the second default rule yields the second constrained extension.

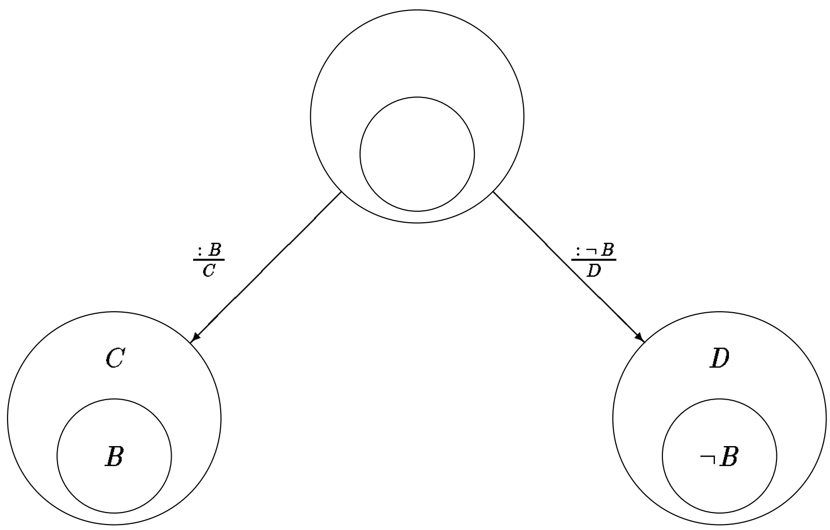

Figure 4.6: Commitment to assumptions in constrained default logic.

Analogously, we can illustrate Example 4.2.2 as done in Figure 4.7.

[10] Recall Theorem 4.4.1.

[11] We will see in Section 5.3 that the focus plays a fundamental role in the case of normal assertional default theories in order to capture semantically the notion of cumulativity.

Example 4.5.2 *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{\neg B}\right\}, \emptyset\right)$$

has two preferred focused models structures: $(\mathrm{MOD}(\{C\}), \mathrm{MOD}(\{C, B\}))$ *and* $(\mathrm{MOD}(\{\neg B\}), \mathrm{MOD}(\{\neg B\}))$.

Initially, we know nothing at all. But once we have "applied" one of the default rules we must take into account its implicit consistency assumption. For example, applying the default rule $\frac{:B}{C}$ makes us focus on those models satisfying B. Thus, the justification of the default rule $\frac{:\neg B}{\neg B}$ is not satisfiable (by the focused models) and the default rule itself is inapplicable.

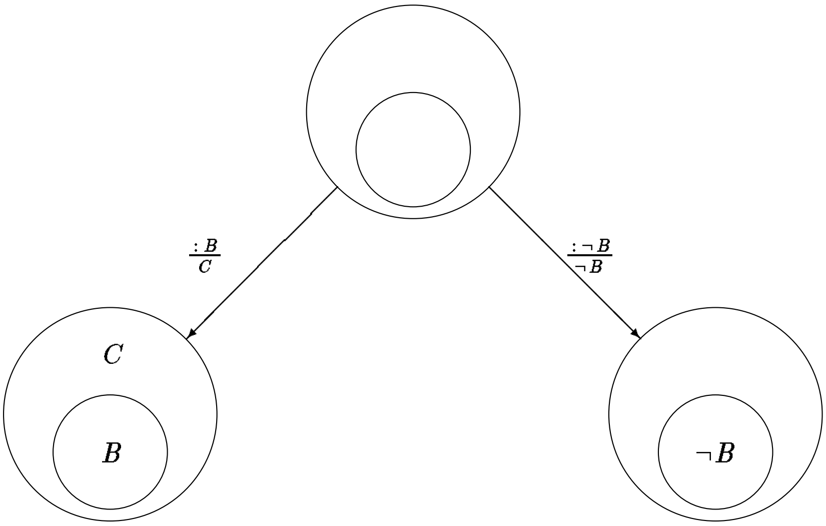

Figure 4.7: Weak commitment to assumptions (or semi-monotonicity) in constrained default logic.

The above illustrations show that the focused models structures semantically account for "commitment to assumptions" (cf. Section 3.3.4). In view of the fact that the focused models semantics captures also Brewka's cumulative default logic (cf. Section 5.3); and the fact that Brewka's variant commits to assumptions as well, we may regard the focused models semantics as a general semantical approach to commitment to assumptions in default logics. In other words, focused models structures account semantically for "awareness of assumptions" by focussing on those models satisfying all implicit assumptions made.

In addition, the semantics supplies us with several insights into the properties of constrained default logic and its extensions, given in Section 4.3. The existence of focused models structures and hence, the existence of constrained extension, is guaranteed since Definition 4.5.1 ensures that $\breve{\Pi}$ never becomes an empty set. The same definition also takes care of semi-monotonicity, since there has to exist a focused model satisfying the prerequisite, consequent, and the justification of an added default rule before it is applied. Weak orthogonality is mirrored by the fact that there never exists a focused model which is shared by two different preferred classes of focused models. In particular, we will see in Section 5.3 that the focused models semantics captures also the property of cumulativity.

An alternative semantics for constrained default logic is given in Section 6.3 by means of Kripke structures, which avoid two-fold semantical structures.

4.6 Cumulativity for prerequisite-free default theories

We have seen in Example 3.5.1 that we can trace back the failure of cumulativity in default logic to its inability to be aware of the consistency assumptions underlying a default conclusion. In particular, we have observed by means of Example 3.5.1 how the addition of nonmonotonic theorems to the set of facts can change dependencies between default rules. This may result in new extensions which then change the set of conclusions and therefore destroy cumulativity.

Obviously, prerequisite-free default theories do not admit such dependencies between default rules. Since they have no prerequisites, all default rules (whose justifications are consistent with an extension) can be applied simultaneously and, hence, independently. Unfortunately, we have seen in Example 3.5.2 that incoherent sets of default rules are another source of the failure of cumulativity. However, normal and C-normal default theories do not admit incoherent default theories in classical default logic. As a consequence, prerequisite-free normal and C-normal default theories enjoy cumulativity.

As regards incoherence, we have shown in Theorem 4.3.8 that the existence of constrained extensions is guaranteed regardless of the considered default theory. This observation explains why general prerequisite-free default theories preserve cumulativity in constrained default logic, as we will show now. For that purpose we first give an alternative characterization for constrained extensions in the case of prerequisite-free default theories.

Theorem 4.6.1 *Let (D, W) be a prerequisite-free default theory and let E, C be sets of formulas. Then, (E,C) is a constrained extension of (D, W) iff*

$$E = Th(W \cup Conseq(D'))$$

$$C = Th(W \cup Conseq(D') \cup Justif(D'))$$

for a maximal set of default rules $D' \subseteq D$ such that $W \cup Conseq(D') \cup Justif(D') \not\vdash \bot$.

Notably, the above definition avoids the usual fixed-point condition, since it reduces the applicability condition of prerequisite-free default rules to the computation of maximally consistent sets of formulas. Therefore, prerequisite-free default rules can be applied independently. This leads to the following theorem which amounts to a cumulativity result for prerequisite-free default theories in the case of skeptical reasoning.

Theorem 4.6.2 *Let (D, W) be a prerequisite-free default theory and let $\alpha \in E'$ for all constrained extension (E', C') of (D, W). Then,*

(E,C) is a constrained extension of (D, W) iff (E,C) is a constrained extension of $(D, W \cup \{\alpha\})$.

We have shown that cumulativity holds for constrained default logic in the case of skeptical reasoning from prerequisite-free default theories. That is, if we let Th_D denote the skeptical consequence operator for a default theory (D, W), we have for a set of formulas W'

$$W \subseteq W' \subseteq Th_D(W) \Longrightarrow Th_D(W) = Th_D(W').$$

As a consequence, cumulativity holds in constrained default logic for a larger class of default theories than in classical default logic. Also, the above result can be easily generalized to default theories which consist of default rules whose prerequisites are monotonically derivable. In this case, all default rules are still independently applicable and, therefore, there are no dependencies among the default rules.

As in classical default logic, there seems to be no way to preserve cumulativity in the case of credulous reasoning. As can be easily verified, Example 3.5.3 carries over to constrained default logic.

Finally, let us deal with cumulative transitivity, which is one half of cumulativity. As mentioned in Section 3.5.1, this property holds for general default theories in classical default logic. As can be expected, this is the case for constrained default logic as well.

Theorem 4.6.3 *Let (E,C) be a constrained extension of a default theory (D,W). If $F \subseteq E$ then (E,C) is also a constrained extension of the default theory $(D, W \cup F)$.*

Besnard describes this property in [1989, p. 50] as follows: "... *[the theorem] can be interpreted as saying that axioms of a default theory enjoy some sort of absorption property wrt conclusions that extensions gather.*" In terms of the skeptical consequence operator Th_D, we have for a default theory (D,W) and a set of formulas W'

$$W \subseteq W' \subseteq Th_D(W) \Longrightarrow Th_D(W') \subseteq Th_D(W).$$

4.7 Prerequisite-free default theories

We have already seen in Section 3.4 that prerequisite-free default theories are very appealing. The omission of the prerequisite has led to a simpler characterization of classical extensions and allows for reasoning by cases and reasoning by contraposition. Clearly, reasoning by cases and reasoning by contraposition carries also over to constrained default logic in the case of prerequisite-free default theories. Moreover, we have shown in Section 4.6 that, in constrained default logic, prerequisite-free default theories avoid fixed-point characterizations for constrained extensions and, most interestingly, they are cumulative in the case of skeptical reasoning.

Another important advantage of prerequisite-free default theories is that we can now reason about a set of default rules. We have already mentioned in Section 3.4 that this is the case for prerequisite-free normal default theories in classical default logic. Fortunately, it turns out that in constrained default logic reasoning about default rules is provided by prerequisite-free default theories in general. To be more precise, we can now decide for a given prerequisite-free default theory (D,W), whether a default rule δ is "subsumed" by other default rules in D. In particular, we can check whether the extensions of (D,W) and $(D \setminus \{\delta\}, W)$ coincide, without necessarily computing them.

For example, if C is not known to be false for a given extension, then the default rules

$$\frac{: A \to C}{A \to C}, \quad \frac{: B \to C}{B \to C}$$

have precisely the same effect as

$$\frac{: A \vee B \to C}{A \vee B \to C}$$

(cf. Section 3.3.5). In order to make this precise, we give the following definition which formalizes the notion of *applicability*.

Definition 4.7.1 **[Delgrande** *et al.*, **1992]** *Let S be a set of formulas. A formula α is applicable wrt S iff $S \cup \{\alpha\}$ is consistent. A default rule δ is applicable wrt S iff $S \cup \{Justif(\delta) \wedge Conseq(\delta)\}$ is consistent.*

The applicability of a formula α is denoted by $\blacklozenge\alpha$.[12] Accordingly, the applicability of a default rule δ is abbreviated by $\blacklozenge(\mathit{Justif}(\delta) \wedge \mathit{Conseq}(\delta))$. Then, Delgrande shows in [1992] the following theorem.

Theorem 4.7.1 **[Delgrande** *et al.*, **1992]** *Let α and β be formulas. Then, we have*

1. $\blacklozenge(\alpha \wedge \beta) \rightarrow \blacklozenge\alpha \wedge \blacklozenge\beta$.
2. $\blacklozenge(\alpha \vee \beta) \leftrightarrow \blacklozenge\alpha \vee \blacklozenge\beta$.
3. $\blacklozenge\alpha \leftrightarrow \blacklozenge\neg\neg\alpha$.
4. $\neg\blacklozenge\alpha \rightarrow \blacklozenge\neg\alpha$.
5. From $\alpha \rightarrow \beta$ infer $\blacklozenge\alpha \rightarrow \blacklozenge\beta$.
6. From $\neg\alpha$ infer $\neg\blacklozenge\alpha$.

Now, we can formally verify the former informal statement dealing with the interdependence between the above three default rules. That is, for a given extension, if the default rules $\frac{:A\rightarrow C}{A\rightarrow C}$ and $\frac{:B\rightarrow C}{B\rightarrow C}$ apply and $\neg C$ is not provable, then the default rule $\frac{:A\vee B\rightarrow C}{A\vee B\rightarrow C}$ applies, too. Formally, we can prove from Theorem 4.7.1 that[13]

$$\vdash \left(\blacklozenge \mathit{Justif}\left(\tfrac{:A\rightarrow C}{A\rightarrow C}\right) \wedge \blacklozenge \mathit{Justif}\left(\tfrac{:B\rightarrow C}{B\rightarrow C}\right) \wedge \blacklozenge C\right) \rightarrow \blacklozenge \mathit{Justif}\left(\tfrac{:A\vee B\rightarrow C}{A\vee B\rightarrow C}\right).$$

Note that this result applies for any default theory and for any (classical or) constrained extension.

Furthermore, Delgrande shows in [1992] that defining $\blacksquare$ as $\neg\blacklozenge\neg$ yields exactly the modal system K (cf. Section 6.2). Therefore, we can use standard modal deduction in order to reason about default rules.

In particular, the capability of reasoning about default rules turns out to be extremely useful in the light of the approach taken by lemma default rules (cf. Section 3.5.2 and 4.10) in order to minimize redundancy. That is, given a prerequisite-free default theory (D, W) and a corresponding (prerequisite-free) lemma default rule δ_ρ we can ask which default rules in D are subsumed in the new default theory $(D \cup \{\delta_\rho\}, W)$.

As a result, we can now regard default reasoning from prerequisite-free default theories as being composed of two distinct and disjoint parts. First, we have the notion of using the default rules to construct an extension. Second, we have additional means of reasoning about a set of default rules (and a set of formulas) to determine, for example, whether or not a particular default rule is "subsumed" by others, and so can be discarded. This division into two parts is basically the approach proposed in [Delgrande, 1988] for default reasoning.

On the whole, it appears that prerequisite-free default theories constitute an important subclass in constrained default logic. In particular, they combine all features of constrained default logic and prerequisite-free default theories. As a consequence, they address all difficulties identified in Section 3.3. Their importance becomes even more apparent in Section 5.4, where their correspondence to Poole's Theorist framework [1988] is established.

[12] In [Delgrande *et al.*, 1992] the operator $\blacklozenge$ is written as *applic*.

[13] Observe that we are dealing with normal default rules, so that it suffices to consider the respective justifications.

4.8 Pre-constrained default logic

In principle, the idea of constraining nonmonotonic theories is quite similar to Poole's [1988] approach to default reasoning (cf. Section 5.4). Although Poole's constraints have to be specified in advance — often to block contraposition or to introduce priorities between defaults — they share the notion of "cutting off undesired theories" with the constraints used in constrained default logic. They also direct the reasoning process but are not part of it.

Clearly, an analogous approach can be taken in order to extend constrained default logic. Therefore, we introduce the notion of a *pre-constrained extension* and call the resulting system *pre-constrained default logic*. The basic idea is to supplement the set of constraints with some kind of *pre-constraints*. The purpose of pre-constraints is to direct the reasoning process by enforcing their consistency. In other words, the context of reasoning becomes predetermined and therefore dominated by some given consistency requirements.

A *pre-constrained default theory* (D, W, C_B) consists of a set of formulas W, a set of default rules D, and a set of formulas C_B representing the set of pre-constraints. In order to ensure that the set of pre-constraints does not introduce any inconsistencies, we require that $W \cup C_B$ being inconsistent implies W being inconsistent. Technically, the only thing to do is to replace the definition of C_0 in Theorem 4.3.1 with $C_0 = W \cup C_B$. This amounts to the following definition.

Definition 4.8.1 *Let (D, W, C_B) be a pre-constrained default theory and let E and C be sets of formulas. Define*

$$E_0 = W \qquad \textit{and} \qquad C_0 = W \cup C_B$$

and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= \mathit{Th}(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ C_{i+1} &= \mathit{Th}(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \end{aligned}$$

(E,C) is a pre-constrained extension of (D, W) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

On the one hand, the step from constrained to pre-constrained default logic is so small that all of the properties of constrained default logic carry over to its pre-constrained counterpart. Also, the focused models semantics is adaptable if we allow initially for non-empty classes of focused models (cf. Section 5.3).

All this becomes even more apparent by the fact that a pre-constrained extension can be computed by means of constrained default logic. The idea is to shift the information given by the pre-constraints to the justifications of the default rules. Therefore, each such justification is supplied with an additional but fixed consistency condition given by the set of pre-constraints.

Theorem 4.8.1 *Let (D, W, C_B) be a pre-constrained default theory and let*

$$D' = \left\{ \tfrac{\alpha:\beta\wedge\hat{C}_B}{\gamma} \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D \right\} \cup \left\{ \tfrac{:\hat{C}_B}{\top} \right\},$$

where $\hat{C}_B$ is the conjunction of all formulas contained in the finite set of pre-constraints C_B. Let E and C be sets of formulas. Then, (E,C) is a pre-constrained extension of (D, W, C_B) iff (E,C) is a constrained extension of (D', W).

The purpose of the synthetic default rule $\frac{:\hat{C}_B}{\top}$ is to add the pre-constraints C_B to the resulting set of constraints C in the case no other default rule applies. Thus, this default rule is redundant in the above relation whenever $GD_D^{(E,C)} \neq \emptyset$.

On the other hand, the use of pre-constraints allows for many interesting enhancements of the original approach. First, notice that pre-constraints normally reduce the number of extensions. This is similar to the use of constraints in Poole's Theorist framework [1988], in the sense of providing means to suppress unwarranted extensions. Let us illustrate this by means of the default theory given in Example 4.4.2. However, unlike Example 4.4.2, we pre-constrain the default theory with C, intuitively, saying that we prefer scenarios in which C can at least be consistently assumed.

Example 4.8.1 *The pre-constrained default theory*

$$\left(\left\{\frac{:B}{C},\frac{:\neg B}{D},\frac{:\neg C}{E},\frac{:\neg D}{F}\right\},\emptyset,\{C\}\right)$$

has one pre-constrained extensions: $(Th(\{C,F\}),Th(\{C,B,F,\neg D\}))$.

In Example 4.4.2, we obtained three constrained extensions: $(Th(\{C,F\}),Th(\{C,B,F,\neg D\}))$, $(Th(\{D,E\}),Th(\{D,\neg B,E,\neg C\}))$, and $(Th(\{E,F\}),Th(\{E,\neg C,F,\neg D\}))$. Thus, by pre-constraining the corresponding default theory, we have exactly eliminated those constrained extensions which do not allow C to be consistently assumed.

Notice that adding a formula to the pre-constraints and to the set of facts makes a big difference. This is because new facts might increase the number of applying default rules and hence lead to larger or even new extensions. In contrast, the addition of pre-constraints never increases the number of applying default rules.

Second, pre-constraints could serve as an instrument to restore cumulativity to default logics. In particular, we have seen in Section 3.5 that the failure of cumulativity in classical default logic stems from its inability to account for the consistency assumptions underlying a default conclusion whenever we are adding it to the set of facts. Thus, the obvious thing to do would be to take the consistency assumptions and add them to the set of pre-constraints. Hence, pre-constraints could serve as a means to accumulate the consistency assumptions underlying any lemmatized default conclusion. We will discuss this idea in more detail in Section 4.10.

Third, we will see in Section 5.4 that pre-constrained default logic is better suited to simulate Poole's approach to default reasoning then ordinary constrained default logic.

Finally, we will employ pre-constrained extensions in Section 4.9 in order to define a prioritized variant of constrained default logic. On the whole, pre-constrained default logic accounts for situations in which we want to restrict our reasoning to certain contexts. In other words, pre-constrained default logic may enforce reasoning under certain consistency assumptions. As a result, pre-constrained default logic turns out to be a simple but powerful extension of constrained default logic.

4.9 Prioritized constrained default logic

Another important practical feature of nonmonotonic formalisms is their capability to incorporate priorities between defaults. In general, the introduction of priorities reduces the number of solutions and therefore often leads to more plausible results. Hence, many formalisms have been extended in order to handle priorities.

Our approach is similar to Brewka's preferred subtheories [Brewka, 1989b] or even McCarthy's prioritized circumscription [McCarthy, 1986] in introducing a hierarchy which serves as a total order on sets of defaults. Also, the approach shares with them the difficulty that the hierarchy has to be determined in advance, since there is no obvious way to extract an appropriate hierarchy from a given set of default rules. This leads to the following definitions.

A *prioritized default theory* $(\tilde{D}, W)$ consists of a set of formulas W and a *sequence* of finite sets of default rules $\tilde{D} = \langle D_0, \ldots, D_m \rangle$ representing the hierarchy of sets of default rules. The default rules in a layer D_i are meant to have a higher priority than those in a layer D_j provided that $j > i$. Then, a *prioritized constrained extension* is defined by building on constrained and pre-constrained extensions as follows.

Definition 4.9.1 *Let $(\tilde{D}, W)$ be a prioritized default theory such that $\tilde{D} = \langle D_0, \ldots, D_m \rangle$ and let E and C be sets of formulas. Define*

(E_0, C_0) to be a constrained extension of (D_0, W)

and for $i \geq 0$

(E_{i+1}, C_{i+1}) to be a pre-constrained extension of

$$\left(\bigcup_{j=0}^{i+1} D_j, W \cup Conseq\left(GD^{(E_i,C_i)}_{\cup_{j=0}^{i} D_j}\right), Justif\left(GD^{(E_i,C_i)}_{\cup_{j=0}^{i} D_j}\right) \right)$$

(E,C) is a prioritized constrained extension of $(\tilde{D}, W)$ iff $(E,C) = (\bigcup_{i=0}^{m} E_i, \bigcup_{i=0}^{m} C_i)$.

Notice that each "layer" admits several constrained extensions which themselves may produce multiple (pre-)constrained extensions. Observe furthermore that by semi-monotonicity every prioritized constrained extension is also a constrained extension but not vice versa. More formally, this amounts to the following corollary.

Corollary 4.9.1 *Let $(\tilde{D}, W)$ be a prioritized default theory such that $\tilde{D} = \langle D_0, \ldots, D_m \rangle$ and let E and C be sets of formulas. If (E,C) is a prioritized constrained extension of $(\tilde{D}, W)$, then (E,C) is a constrained extension of the default theory $(D_1 \cup \ldots \cup D_m, W)$.*

Note that the above corollary does not hold for the prioritized variant of classical default logic proposed in [Brewka, 1989a], since classical default logic does not enjoy semi-monotonicity.

Another interesting point concerning Definition 4.9.1 is that a default rule in D_i may contribute to all partial constrained extensions (E_j, C_j) where $j > i$. Again, this is different from the approach taken in [Brewka, 1989a], where the definition prevents the application of default rules whose prerequisite is derived in a "higher" layer.

In order to illustrate the approach briefly, we consider a prioritized formalization of the default theory given in Example 3.1.1.

Example 4.9.1 *The prioritized default theory*

$$\left(\left\langle \left\{ \frac{:\neg B}{A} \right\}, \left\{ \frac{:\neg A}{B} \right\} \right\rangle, \{A \rightarrow C, B \rightarrow C\} \right)$$

has one prioritized constrained extension: $(Th(\{A, C\}), Th(\{A, C, \neg B\}))$.

Initially, we obtain $(Th(\{A, C\}), Th(\{A, C, \neg B\}))$ as the constrained extension of the default theory

$$\left(\left\{ \frac{:\neg B}{A} \right\}, \{A \rightarrow C, B \rightarrow C\} \right).$$

Then, we get the same pre-constrained extension of the pre-constrained default theory

$$\left(\left\{ \frac{:\neg B}{A}, \frac{:\neg A}{B} \right\}, \{A \rightarrow C, B \rightarrow C\}, Th(\{A, C, \neg B\}) \right).$$

4.10 Nonmonotonic lemmas

We have argued in Section 3.5 that cumulativity is of great practical relevance. We have observed that once a consequence relation is cumulative it allows for the generation of lemmas, which often leads to the reduction of computational efforts. Also, we have seen in Section 3.5.1 that it is necessary to be aware of the assumptions underlying a default conclusion in order to preserve cumulativity. However, constrained extensions, just like classical extensions, consist of first-order formulas and, therefore, the question arises again (cf. Section 3.5.2) which form these lemmas should have in order to represent the aforementioned assumptions.

At first sight, a plausible solution seems to be to employ pre-constrained default logic. To do so, we could add the actual lemma to the facts, and its underlying consistency assumptions to the set of pre-constraints. This solution offers the following advantages. First, the approach simply deals with first-order formulas and thus avoids extra-logical formalisms as lemma default rules, or an extended language as Brewka's [1991b] approach (cf. Section 5.3). Second, the use of pre-constraints eliminates constrained extensions which are inconsistent with the nonmonotonic lemma or even its underlying assumptions (cf. Section 4.8). Of course, this might sometimes be a desired side-effect. As an example, take the situation where consistency checks are very expensive. Then, the motivation for lemmatizing a nonmonotonic theorem might be to lemmatize the consistency assumptions rather then the theorem as such.

However, this solution turns out to have a crucial drawback. Since we draw conclusions in the absence of information, it may happen that new information arises which denies our former default conclusions or their underlying consistency assumptions. So, establishing these default conclusions and their underlying assumptions by adding them to the set of facts and pre-constraints, respectively, forces a "hard" contradiction in the presence of subsequent contradictory information, since the "smooth" default properties of the original conclusion are lost.

Let us see what happens if we lemmatize the nonmonotonic theorem in our canonical cumulativity example (cf. Example 3.5.1) according to the above recipe. Hence, let us consider the pre-constrained counterpart of the default theory (3.4), as given below in Example 4.10.1. Now, lemmatizing the nonmonotonic theorem $A \vee B$ yields a pre-constrained default theory, whose facts contain the default lemma $A \vee B$ and whose pre-constraints contain the consistency assumptions made while deriving it. These are given by the justification A of the default rule used for deriving $A \vee B$.

Example 4.10.1 *The pre-constrained default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset, \emptyset\right)$$

has one pre-constrained extension: $(Th(\{A\}), Th(\{A\}))$.

Adding the nonmonotonic theorem $A \vee B \in Th(\{A\})$ *to the set of facts and its underlying assumption* A *to the set of pre-constraints yields the pre-constrained default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}, \{A\}\right)$$

which has the same pre-constrained extension: $(Th(\{A\}), Th(\{A\}))$.

So far, this seems to be an appropriate solution, since the lemmatization has neither produced any new nor changed any previous extensions. In our example, this is because even though the prerequisite of the default rule $\frac{A \vee B : \neg A}{\neg A}$ is now derivable, its justification is inconsistent with the set of pre-constraints. Consequently, this default rule is blocked and, therefore, does not produce a second pre-constrained extension.

However, we were drawing conclusions in the absence of information. Thus, it may happen that new information arises which denies our lemmatized default conclusion $A \vee B$. For instance, let us add $\neg(A \vee B)$ to the set of facts of the previous pre-constrained default theory:

Example 4.10.2 *The pre-constrained default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{\bot\}, \{A\}\right)$$

has an inconsistent pre-constrained extension.

We see that the addition of the fact $\neg(A \vee B)$ results in an inconsistent set of facts, which leads to an inconsistent extension. This is because neither the lemma itself nor its underlying consistency assumptions are "retractable" any more, after they have been lemmatized. In general, if our lemma was still a nonmonotonic theorem (ie. derived by means of default rules), it would be either "retractable" (since a default rule's justification is not consistent anymore, as in our case) or we would obtain a second constrained extension.

Because of this drawback, we follow the approach taken in Section 3.5 and introduce the notion of a lemma default rule for constrained default logic. Again, the purpose of lemma default rules is to account for the practical impact of cumulativity: the capability of handling nonmonotonic lemmas in order to reduce computational efforts.

As in Section 3.5, we first account for the notion of a default proof by slightly adjusting Definition 3.5.1 to constrained default logic.

Definition 4.10.1 *Let (D, W) be a default theory and let S and T be sets of formulas. A default proof of ρ in (S, T) from (D, W) is a sequence $\langle D_1, \ldots, D_k \rangle$ of sets of default rules where $D_i \subseteq GD_D^{(S,T)}$ for $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} D_i$ is a minimal set of default rules such that*

1. $W \vdash \mathit{Prereq}(D_1)$,
2. $W \cup \mathit{Conseq}(D_i) \vdash \mathit{Prereq}(D_{i+1})$ *for* $1 \leq i \leq k-1$,
3. $W \cup \mathit{Conseq}(D_k) \vdash \rho$.

As in Definition 3.5.1, observe that the sets of default rules $D_1, \ldots, D_n$ are not necessarily distinct. Also, notice that, given a constrained extension (E, C), by compactness and groundedness any formula $\gamma \in E$ has a finite default proof which is itself composed of finite sets of default rules.

Accordingly, we define a conclusion's lemma default rule in constrained default logic as follows.

Definition 4.10.2 *Let (D, W) be a default theory and let (E, C) be a constrained extension of (D, W). Let $\rho \in E$ and $\langle D_1, \ldots, D_k \rangle$ be a default proof of ρ in (E, C) from (D, W). We define a lemma default rule δ_ρ for ρ as*

$$\delta_\rho = \frac{: \bigwedge_{\delta \in D_\rho} \mathit{Justif}(\delta) \wedge \bigwedge_{\delta \in D_\rho} \mathit{Conseq}(\delta)}{\rho}$$

where $D_\rho = \bigcup_{i=1}^{k} D_i$.

As in Section 3.5, we first give the following result, which captures the basic idea behind lemma default rules as an abbreviation of default proofs.

Theorem 4.10.1 *Let (D, W) be a default theory and let (E, C) and (E', C') be constrained extensions of (D, W). Let $\langle D_1, \ldots, D_k \rangle$ be a default proof of ρ in (E', C'), and let δ_ρ be the corresponding lemma default rule for ρ. Then,*

$$\delta_\rho \in GD^{(E,C)}_{D\cup\{\delta_\rho\}} \text{ iff } \bigcup_{i=1}^{k} D_i \subseteq GD^{(E,C)}_{D}.$$

Accordingly, we have the following theorem stating that the addition of lemma default rules does not alter the constrained extensions of a given default theory.

Theorem 4.10.2 *Let (D, W) be a default theory and let (E', C') be a constrained extension of (D, W). Let δ_ρ be a lemma default rule for $\rho \in E'$. Then,*

(E,C) is a constrained extension of (D, W) iff (E,C) is a constrained extension of $(D \cup \{\delta_\rho\}, W)$.

Again, the approach provides a simple solution for generating and using nonmonotonic lemmas. Also, it clarifies the notion of nonmonotonic lemmas by distinguishing between them and their original theorems. Whenever we lemmatize a conclusion, we change its representation into a default rule and add it to the default rules of a considered default theory.[14]

Let us look again at the canonical cumulativity example given in Example 3.5.1. Analogous to Example 3.5.4, the default proof of the default conclusion $A \vee B$ is simply $\langle\{\frac{:A}{A}\}\rangle$. Accordingly, the lemma default rule for $A \vee B$ is $\frac{:A}{A\vee B}$.

Example 4.10.3 *The default theory (3.4)*

$$\left(\left\{\frac{:A}{A}, \frac{A\vee B : \neg A}{\neg A}\right\}, \emptyset\right)$$

has one constrained extension: $(Th(\{A\}), Th(\{A\}))$.

Adding the lemma default rule $\frac{:A}{A\vee B}$ for the nonmonotonic theorem $A \vee B \in Th(\{A\})$ to the set of default rules yields the default theory

$$\left(\left\{\frac{:A}{A}, \frac{A\vee B : \neg A}{\neg A}, \frac{:A}{A\vee B}\right\}, \emptyset\right)$$

which has the same constrained extension: $(Th(\{A\}), Th(\{A\}))$.

The last example is illustrated in Figure 4.8. There, we see that the addition of the lemma default rule $\frac{:A}{A\vee B}$ for the proposition $A \vee B$ yields the same constrained extension and no others. Although there are still two ways to derive $A \vee B$ (as the prerequisite of the default rule $\frac{A\vee B:\neg A}{\neg A}$; compare with Example 3.5.1), both of them rely on the consistency of A and, therefore, prevent the application of the default rule $\frac{A\vee B:\neg A}{\neg A}$.

Now, let us consider what happens in the presence of subsequent contradictory information. The default theory in the following example results from adding $\neg(A \vee B)$ to the set of facts of the default theory obtained in Example 4.10.3 after lemmatizing the nonmonotonic theorem $A \vee B$. Notice that the same procedure has led to an inconsistent extension in Example 4.10.2.

Example 4.10.4 *The default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A\vee B : \neg A}{\neg A}, \frac{:A}{A\vee B}\right\}, \{\neg(A\vee B)\}\right)$$

has one constrained extension: $(Th(\{\emptyset\}), Th(\{\emptyset\}))$.

In this case, we do not obtain an inconsistent extension. This is because adding $\neg(A \vee B)$ in the presence of the lemma default rule $\frac{:A}{A\vee B}$ just blocks the lemma default rule (along with all other default rules) and does not harm the reasoning process itself. We see that the lemma default

[14]A comparison between lemma default rules and an approach taken by Brewka in [1991b] is given in Section 5.3.

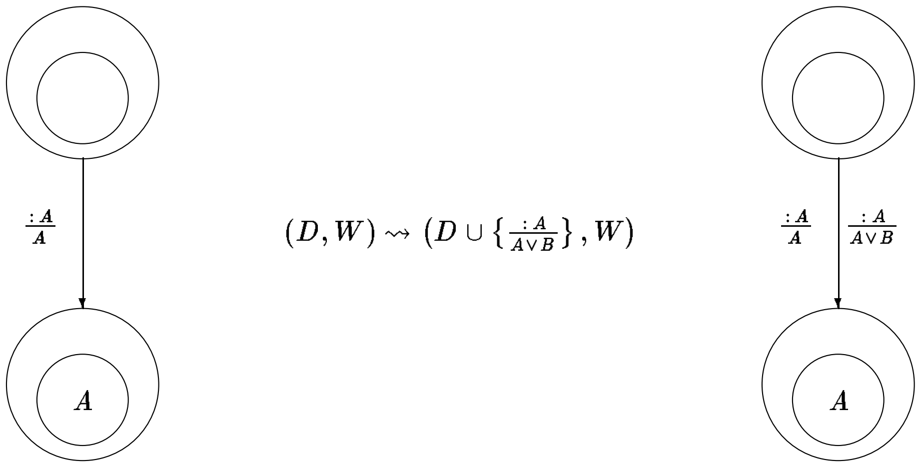

Figure 4.8: How lemma default rules cope with the failure of cumulativity in terms of focused models structures.

rule is retractable and, therefore, preserves the smooth default properties of the original default conclusion.

An obvious question is, whether the justification of a lemma default rule constitutes a minimal condition for Theorem 4.10.1 and 4.10.2. Clearly, the justifications of the default rules taken from a default proof are necessary. In order to see that this extends to the corresponding consequents, consider the following example.

Example 4.10.5 *The default theory*

$$\left(\left\{\frac{:A}{C}, \frac{C:B}{D}, \frac{:\neg C}{\neg C}\right\}, \emptyset\right)$$

has two constrained extensions: $(Th(\{C, D\}), Th(\{A, B, C, D\}))$ *and* $(Th(\{\neg C\}), Th(\{\neg C\}))$.

Let us consider the nonmonotonic theorem D which is in the first but not in the second constrained extension. The only default proof of D is

$$\left\langle \left\{\frac{:A}{C}\right\}, \left\{\frac{C:B}{D}\right\}\right\rangle. \tag{4.3}$$

Assume we add the default rule $\frac{:A \wedge B}{D}$ as the "lemma default rule" for D. Obviously, the justification of this "lemma default rule" consists merely of the justifications of the default rules in (4.3). As shown in Example 4.10.6, we then change the second constrained extension $(Th(\{\neg C\}), Th(\{\neg C\}))$ into $(Th(\{\neg C, D\}), Th(\{\neg C, D, A, B\}))$, since the putative "lemma default rule" becomes applicable.

Example 4.10.6 *The default theory*

$$\left(\left\{\frac{:A}{C}, \frac{C:B}{D}, \frac{:\neg C}{\neg C}, \frac{:A \wedge B}{D}\right\}, \emptyset\right)$$

has two constrained extensions: $(Th(\{C, D\}), Th(\{A, B, C, D\}))$ *and* $(Th(\{\neg C, D\}), Th(\{\neg C, D, A, B\}))$.

Clearly, the change of the second constrained extension is impossible if we additionally require the consistency of consequents of the default rules in (4.3), namely C and D. Hence, the addition of the correct lemma default rule

$$\frac{:(A \wedge B) \wedge (C \wedge D)}{D}$$

for the (credulous) nonmonotonic theorem D does neither change any constrained extensions nor result in any new ones. This is illustrated in the next example.

Example 4.10.7 *The default theory*

$$\left(\left\{\frac{:A}{C}, \frac{C:B}{D}, \frac{:\neg C}{\neg C}, \frac{:(A \wedge B) \wedge (C \wedge D)}{D}\right\}, \emptyset\right)$$

has two constrained extensions: $(Th(\{C, D\}), Th(\{A, B, C, D\}))$ *and* $(Th(\{\neg C\}), Th(\{\neg C\}))$.

That is, we obtain the same constrained extensions as in Example 4.10.5.

In addition, the above examples illustrate another advantage of the approach taken by lemma default rules. Namely, since the addition of lemma default rules does not change the extensions of an initial default theory, lemma default rules allow for the use of skeptical as well as credulous nonmonotonic lemmas. That is, we can also introduce lemmas for default conclusions which do not belong to all extensions. As we will see in Section 5.3, this is not the case for the approach taken in [Brewka, 1991b].

What has been achieved? One of the original postulates of nonmonotonic formalisms was to "jump to conclusions" in the absence of information. But since the computation of nonmonotonic conclusions involves not only deduction but also expensive consistency checks, the need to incorporate lemmas is even greater in nonmonotonic theorem proving than in standard theorem proving. Hence, nonmonotonic lemmas can be seen as a step in this direction. This becomes obvious by means of Theorem 4.3.1: it is possible to jump to a conclusion ρ normally derived in layer E_k by skipping all previous layers E_0 to E_{k-1} and solely applying the (prerequisite-free) lemma default rule.

Let us look at a simplified default proof of a nonmonotonic theorem ρ consisting of a chain of default rules

$$\left\langle\left\{\frac{\alpha_0 : \beta_0}{\gamma_0}\right\}, \ldots, \left\{\frac{\alpha_i : \beta_i}{\gamma_i}\right\}, \ldots, \left\{\frac{\alpha_n : \beta_n}{\rho}\right\}\right\rangle$$

such that $W \vdash \alpha_0$, $W \cup \{\gamma_i\} \vdash \alpha_{i+1}$ for $0 \leq i < n$ and $W \cup \{\gamma_{n-1}\} \vdash \rho$. Normally, proving ρ from scratch requires n proofs and n consistency checks. Each consistency check involves the justification as well as the consequent of each default rule. By comparison, applying the corresponding lemma default rule requires *no* proofs since lemma default rules are prerequisite-free. The effort of checking consistency reduces to *one* consistency check. But although the justification of the lemma default rule contains all justifications and consequents of previously applied default rules we have the advantage that their joint consistency has already been proven.

4.11 Conclusion

We have developed constrained default logic in order to rectify several limitations of classical default logic. We have introduced the notion of a constrained extension which explicates the context-sensitive nature of default logic. This is done by distinguishing between our set of beliefs, the extension, and the underlying constraints which form a context guiding our beliefs. Accordingly, we have strengthened the applicability condition for default rules such that a default rule is applicable iff its prerequisite is provable from the extension and the conjunction of its justification and consequent is consistent with the set of constraints.

As a result, constrained default logic has many desirable properties: the existence of constrained extensions is guaranteed, constrained default logic is semi-monotonic, all constrained extensions of a given default theory are weakly orthogonal to each other, and constrained default logic commits to assumptions. Also, we have investigated the relationship between classical and constrained default logic. We have shown that the systems coincide in the case of normal default theories. In addition, we have given a general criterion which indicates when extensions coincide in both systems.

Furthermore, we have introduced the focused models semantics as semantical underpinnings for constrained default logic. We have seen that the class of focused models provides a natural semantical counterpart to the constraints in a constrained extension. The approach does not require stability conditions as required in [Etherington, 1987c] (cf. Section 3.6); in addition it supplies us with useful semantical insights into the enhancements of the underlying approach.

An important subsystem of constrained default logic is given by the restriction to prerequisite-free default theories. We have illustrated that they allow for reasoning by cases, reasoning by contraposition, as well as reasoning about default rules [Delgrande *et al.*, 1992]. Notably, we have proven that prerequisite-free default theories are cumulative. Note also that reasoning about default rules and cumulativity hold only for prerequisite-free normal default theories in classical default logic. As a by-product, we have obtained a non-fixed-point characterization for constrained extensions of prerequisite-free default theories. Also, we have described variations of constrained default logic which allow us to incorporate pre-constraints and priorities.

Finally, we have introduced lemma default rules for constrained default logic in order to obtain facilities to generate and to use nonmonotonic lemmas. We have argued in favor of default rules for representing nonmonotonic lemmas. This has two major advantages. First, lemma default rules are "retractable", in that inconsistencies in the presence of subsequent but contradictory information are avoided. Second, lemma default rules do not harm the reasoning process from given default theories. That is, all extensions remain the same after lemmatizing an arbitrary theorem. This approach is contrasted with that of [Brewka, 1991b] in Section 5.3.

Chapter 5

Other variants of default logic

This chapter contains an extensive study of the relationships between the various derivatives of classical default logic. After briefly surveying the evolution of classical default logic, we employ constrained default logic as an instrument for comparing the various approaches. First, we discuss in Section 5.2 a variant of default logic proposed by Łukaszewicz in [1988] and investigate its relationship to constrained default logic. We refer to this variant as justified default logic. Second, we elaborate in Section 5.3 on the relationship between constrained default logic and Brewka's cumulative default logic [1991b]. Moreover, we provide the first semantical characterization of Brewka's variant by means of the focused models semantics. Furthermore, we compare Brewka's approach to restore cumulativity to default logic with that of lemma default rules. Last but not least, we deal with Poole's approach to default reasoning [1988] which turns out to be a proper subsystem of constrained default logic. We conclude this chapter by comparing the properties of these variants of classical default logic.

5.1 The evolution of default logic

Default logic has evolved during the last decade. Two well-known approaches are Łukaszewicz' *justified default logic* [1988] and Brewka's *cumulative default logic* [1991b]. Historically, Łukaszewicz' approach can be regarded as an ancestor of Brewka's variant. Also, cumulative default logic shares most of the properties of justified default logic. However, they differ in the way they enforce their results. Łukaszewicz attached sets of formulas to extensions whereas Brewka labelled formulas with sets of formulas. Thus, both employ constraints but differ basically in the location they put them. Now, constrained default logic turns out to be an amalgamation of both approaches. Therefore, it is well suited as an instrument for comparing the descendents of classical default logic. See Figure 5.1 for an illustration of this evolutionary process.

In order to facilitate the treatment of the various approaches, we concentrate in this chapter on how far each of them commits to assumptions. We consider the following default theory.

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg C \wedge \neg D}{E}\right\}, \emptyset\right) \tag{5.1}$$

This default theory serves as an indicator of how far each variant of default logic commits to assumptions. Here, the term "commitment to assumptions" is understood in a broader sense such that it also subsumes the notion of semi-monotonicity as "weak commitment to assumptions" (cf. Section 3.3.2 and 3.3.4). Technically, the above default theory combines several potential conflicts which will reveal the degree of commitment to assumptions for each considered default logic. As we have observed in Example 3.3.6 and 4.2.1, the first two default

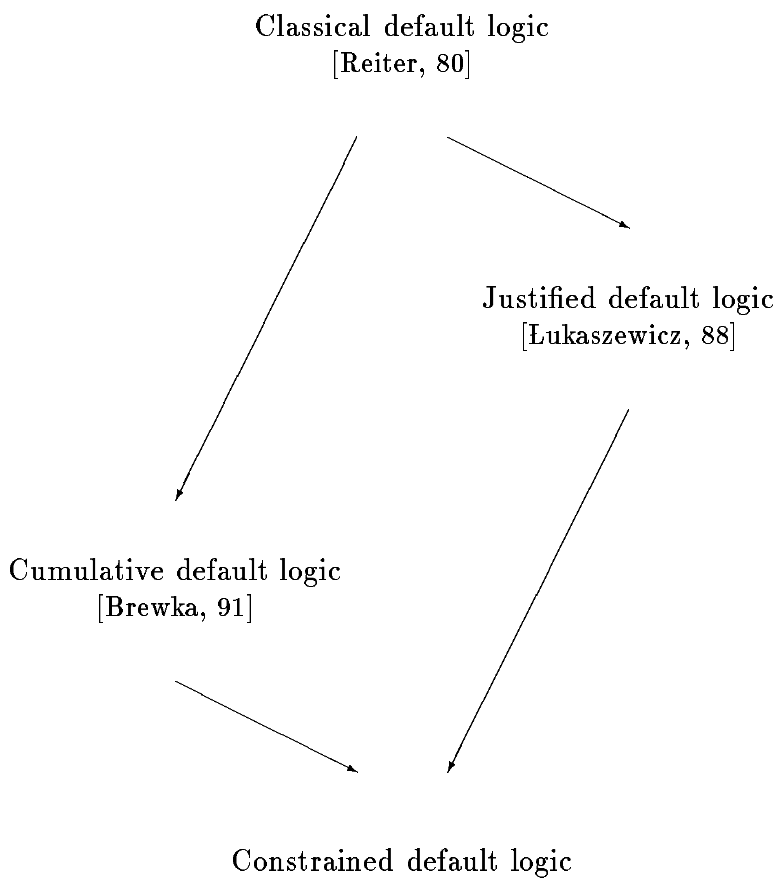

Figure 5.1: From classical towards constrained default logic.

rules indicate whether a given system "strongly commits to assumptions", ie. whether it detects inconsistencies among the set of justifications. The fact of whether or not the third default rule applies indicates whether or not the considered variant is semi-monotonic; or in other words whether it "weakly commits to assumptions".

As we have already seen in Example 3.3.6, classical default logic does not commit to assumptions. Obviously, this extends to the default theory (5.1). Namely, classical default logic does not detect the inconsistency between the justifications of the first two default rules.

Example 5.1.1 *The default theory (5.1) has one classical extension:*

- $Th(\{C, D\})$.

Analogously to Example 3.3.6, the first two default rules apply, although they have contradicting justifications. In this case, the default rule $\frac{:\neg C \wedge \neg D}{E}$ is blocked since $\neg C \wedge \neg D$ cannot be consistently assumed in the presence of C and D. Notice that, due to the failure of semi-monotonicity in classical default logic, the last default rule cannot contribute to any classical extension (see also Example 3.3.4). Therefore, classical default logic does not even weakly commit to assumptions.

In contrast, constrained default logic commits to assumptions and we obtain three constrained extensions (cf. Example 4.2.1). This is because all of the aforementioned inconsistencies are detected.

Example 5.1.2 *The default theory (5.1) has three constrained extensions:*

- $(Th(\{C\}), Th(\{C, B\}))$,
- $(Th(\{D\}), Th(\{D, \neg B\}))$,
- $(Th(\{E\}), Th(\{E, \neg C, \neg D\}))$.

For example, applying the default rule $\frac{:\neg C \wedge \neg D}{E}$ allows us to conclude E but additionally forces us to preserve the consistency of $\neg C \wedge \neg D$ with all other conclusions and their underlying consistency assumptions. Thus, neither of the other two default rules is applicable. Similarly, we can describe the construction of the two other constrained extensions.

5.2 Justified default logic

Łukaszewicz [1988] modified default logic in order to guarantee the existence of extensions and semi-monotonicity for general default theories. Similar to constrained default logic, he attaches constraints to extensions in order to strengthen the applicability condition of default rules. Formally, a *justified extension*[1] is defined as follows.

Definition 5.2.1 *Let (D, W) be a default theory. For any pair of sets of formulas (S, T), let $\Psi(S, T)$ be the pair of smallest sets of formulas S', T' such that*

1. $W \subseteq S'$,
2. $Th(S') = S'$,
3. *For any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $\forall \eta \in T \cup \{\beta\}.\ S \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta \in T'$.

[1] Originally, Łukaszewicz called his extensions *modified* extensions.

A set of formulas E is a justified extension[2] *of (D,W) wrt to a set of formulas J iff $\Psi(E,J) = (E,J)$.*

As constrained extensions, justified extensions are composed of two sets of formulas E and J. In order to illustrate this briefly, we consider the simple default theory $(\{\frac{A:B}{C}\},\{A\})$. This default theory yields the justified extension $(Th(\{A,C\}),\{B\})$. As in constrained default logic, the first set constitutes the actual extension, whereas the second one imposes some constraints on this extension.

However, we observe two major differences by comparing Definition 5.2.1 with Definition 4.2.1. First, we notice that Łukaszewicz employs a weaker consistency check than constrained default logic. A default rule $\frac{\alpha:\beta}{\gamma}$ applies if all justifications of other applying default rules are consistent with the considered extension E and γ; and if additionally γ and β are consistent with E.[3] Second, we observe that the set of constraints J merely consists of the justifications of applied default rules. The constraints have to be neither deductively closed nor consistent. All this prevents justified default logic from strongly committing to assumptions, as is shown below. That is, even though justified default logic detects inconsistencies between consequents and justifications, it ignores inconsistencies among the justifications of the applying default rules.

Example 5.2.1 *The default theory (5.1) has two justified extension:*

- $(Th(\{C,D\}),\{B,\neg B\})$,
- $(Th(\{E\}),\{\neg C \wedge \neg D\})$.

As in classical default logic, the first justified extension is generated by the default rules $\frac{:B}{C}$ and $\frac{:\neg B}{D}$, which have contradicting justifications. Thus, the extension is justified by an inconsistent set of constraints. The second justified extension stems from the fact that justified default logic is semi-monotonic: Assume we have applied the default rule $\frac{:\neg C\wedge\neg D}{E}$. In order to apply the default rule $\frac{:B}{C}$, say, its consequent C must be consistent with $\neg C \wedge \neg D$. Obviously, this is not the case and the default rule is inapplicable. For the same reason, the default rule $\frac{:\neg B}{D}$ is not applied. In particular, the property of semi-monotonicity implies also the existence of justified extensions [Łukaszewicz, 1988].

Since Łukaszewicz is primarily interested in avoiding inconsistencies between justifications and consequents of individual default rules, he neglects inconsistencies among the constraints. So, even though the set of constraints, J, is consistent, it might be inconsistent together with the extension, E, or even the set of premises, W. As an example, consider the following one.

Example 5.2.2 *The default theory*

$$\left(\left\{\frac{:B}{A},\frac{:D}{C}\right\},\{\neg B \vee \neg D\}\right)$$

has one justified extension: $(Th(\{\neg B \vee \neg D, A, C\}),\{B,D\})$.

Obviously, the set of constraints $\{B,D\}$ is inconsistent with the set of facts $\{\neg B \vee \neg D\}$.

Nevertheless, we have the following relationship between the two globally-constrained approaches in the case of no such inconsistencies.

Theorem 5.2.1 *Let (D,W) be a default theory and E a justified extension of (D,W) wrt J. If $E \cup J$ is consistent then $(E, Th(E \cup J))$ is a constrained extension of (D,W).*

[2] Sometimes, we simply denote a justified extension by a pair (E,J).

[3] Observe that omitting γ in the last part of the condition meets exactly the consistency requirement of classical default logic.

Thus, provided that the set of justifications, J, is consistent with the extension, E, we obtain the same extension in justified and constrained default logic.

At first sight, the set of examples given by means of the default theory (5.1) suggests that the stronger the consistency check of a default logic the more extensions are obtained. Indeed, the default theory (5.1) has one classical, two justified, and three constrained extensions. However, this increasing number of extensions is not a universal principle. Let us reconsider the default theory (4.1) given in Example 4.4.1.

Example 5.2.3 *The default theory (4.1)*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg C}{E}, \frac{:\neg D}{F}\right\}, \emptyset\right)$$

has four justified extensions: $(Th(\{C,D\}),\{B,\neg B\})$, $(Th(\{C,F\}),\{B,\neg D\})$, $(Th(\{D,E\}),\{\neg B,\neg C\})$, *and* $(Th(\{E,F\}),\{\neg C,\neg D\})$.

In Section 4.4, we illustrated by means of the default theory (4.1) that constrained default logic is neither stronger nor weaker than its classical counterpart. In particular, we obtained one classical and three constrained extensions. However, we obtain four justified extensions for the default theory (4.1). The reason for this phenomenon is as follows. Since justified default logic is semi-monotonic (or "weakly commits to assumptions"), it allows for the application of each default rule. However, since it discards inconsistencies among the justifications of applying default rules it is not strong enough to exclude the combination of the default rules $\frac{:B}{C}$ and $\frac{:\neg B}{D}$. In other words, although "weak commitment" guarantees the individual consistency of each justification, it does not prevent inconsistent sets of justifications.

On the whole, justified default logic allows for the application of more default rules than constrained default logic as is shown next.

Theorem 5.2.2 *Let (D,W) be a default theory and (E,C) be a constrained extension of (D,W). Then, there is a justified extension (E',J') of (D,W) such that $E \subseteq E'$ and $C \subseteq Th(E' \cup J')$.*

Moreover, whenever the actual extensions coincide we have the following relationship.

Theorem 5.2.3 *Let (D,W) be a default theory and let E,C, and J be sets of formulas. If (E,C) is a constrained extension of (D,W) and E is a justified extension of (D,W) wrt J then $C \subseteq Th(E \cup J)$.*

An argument analogous to the one following Theorem 4.4.3 illustrates that the converse of Theorem 5.2.3 does not hold.

Semantically, Łukaszewicz characterized justified extensions in [1988] by means of pairs (Π, J), where Π is a class of first-order interpretations and J is a set of formulas. Then, such a "preferred" pair characterizes a justified extension E wrt J iff Π is the class of all models of E. Clearly, the occurrence of sets of formulas inside semantical structures is unfortunate. However, this is due to the fact that justified extensions admit inconsistent set of constraints which also prevents us from applying the focused models semantics in order to capture justified default logic semantically. We will see in Section 6.4 how justified default logic can be captured by purely model-theoretic means.

5.3 Cumulative default logic

Brewka [1991b] adds commitment to assumptions and cumulativity to default logic also by strengthening the applicability condition for default rules and making the reasons for believing something explicit. But in order to keep track of implicit assumptions, he introduced so-called *assertions*, ie. formulas labelled with the set of justifications and consequents of the default rules which were used for deriving them. Intuitively, assertions represent formulas together with the reasons for believing them.

Definition 5.3.1 *Let $\alpha, \gamma_1, \ldots, \gamma_m$ be formulas. An assertion ξ is any expression of the form $\langle \alpha, \{\gamma_1, \ldots, \gamma_m\}\rangle$, where $\alpha = Form(\xi)$ is called the asserted formula and the set $\{\gamma_1, \ldots, \gamma_m\} = Supp(\xi)$ is called the support of α.*[4]

To guarantee the proper propagation of the supports, Brewka had to extend in [1991b] the standard inference relation as follows.

Definition 5.3.2 *Let S be a set of assertions and let $\widehat{Th}$ denote the assertional consequence operator. Then, $\widehat{Th}(\mathcal{S})$ is the smallest set of assertions such that*

1. $\mathcal{S} \subseteq \widehat{Th}(\mathcal{S})$,
2. *if* $\xi_1, \ldots, \xi_n \in \widehat{Th}(\mathcal{S})$ *and* $Form(\xi_1), \ldots, Form(\xi_n) \vdash \gamma$, *then* $\langle \gamma, Supp(\xi_1) \cup \ldots \cup Supp(\xi_n)\rangle \in \widehat{Th}(\mathcal{S})$.

An *assertional default theory* becomes a pair $(D, \mathcal{W})$, where D is a set of default rules and $\mathcal{W}$ is a set of assertions. An assertional default theory $(D, \mathcal{W})$ is called *well-based* if $Form(\mathcal{W}) \cup Supp(\mathcal{W})$ is consistent. Then, an *assertional extension* is defined as follows.

Definition 5.3.3 *Let $(D, \mathcal{W})$ be an assertional default theory. For any set of assertions $\mathcal{S}$, let $\Omega(\mathcal{S})$ be the smallest set of assertions $\mathcal{S}'$ such that*

1. $\mathcal{W} \subseteq \mathcal{S}'$,
2. $\widehat{Th}(\mathcal{S}') = \mathcal{S}'$,
3. *For any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{S}'$ *and* $Form(\mathcal{S}) \cup Supp(\mathcal{S}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{S}'$.

A set of assertions $\mathcal{E}$ is an assertional extension for $(D, \mathcal{W})$ iff $\Omega(\mathcal{E}) = \mathcal{E}$.

Comparing the last definition with that of constrained extensions, we see that the justifications and consequents of applied default rules are recorded locally to the default conclusions. In order to illustrate this briefly, we consider the simple assertional default theory $\left(\{\frac{A:B}{C}\}, \{\langle A, \emptyset\rangle\}\right)$. This assertional default theory yields the assertional extension $\widehat{Th}(\{\langle A, \emptyset\rangle, \langle C, \{B, C\}\rangle\})$. We see that the default conclusion C is labelled with a certain set of constraints.

However, a closer look reveals that the applicability conditions for a default rule $\frac{\alpha:\beta}{\gamma}$ in constrained and cumulative default logic require both the joint consistency of its justification β and its consequent γ with the set of justifications and consequents of all other applying default rules. Therefore, assertional extensions share the notion of "joint consistency" with constrained extensions — but in a distributed way. Thus, while cumulative default logic deals with "formulas with constraints", constrained default logic deals with constrained extensions.

As with constrained default logic, cumulative default logic strongly commits to assumptions.

Example 5.3.1 *The assertional default theory (5.1) has three assertional extensions:*

[4]The two projections extend to sets of assertions in the obvious way. The projection *Supp* is also used to denote the support of an asserted formula, eg. $\langle \alpha, Supp(\alpha)\rangle$.

- $\widehat{Th}(\{\langle C, \{B, C\}\rangle\})$,
- $\widehat{Th}(\{\langle D, \{\neg B, D\}\rangle\})$,
- $\widehat{Th}(\{\langle E, \{\neg C \wedge \neg D, E\}\rangle\})$.

Let us examine the assertional extension containing the assertion $\langle D, \{\neg B, D\}\rangle$. This assertion is derived by applying the default rule $\frac{:\neg B}{D}$. In order to apply another default rule the corresponding justification and consequent have to be consistent with $\{D\} \cup \{\neg B, D\}$, ie. the asserted formula and the support of the assertion $\langle D, \{\neg B, D\}\rangle$. As can be easily verified, none of the two other default rules is applicable. So, once we have derived a conclusion, we are aware of its underlying assumptions. Therefore, cumulative default logic prevents the derivation of conclusions which contradict previously derived conclusions or their underlying consistency assumption.

In all, cumulative and constrained default logic are very close to each other. In order to characterize this relation directly, we give kind of an equivalence result between our formulation and that of [Brewka, 1991b].

Theorem 5.3.1 *Let (D, W) be a default theory and $(D, \mathcal{W})$ the assertional default theory, where $\mathcal{W} = \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\}$. Then, if (E,C) is a constrained extension of (D, W) then there is an assertional extension $\mathcal{E}$ of $(D, \mathcal{W})$ such that $E = Form(\mathcal{E})$ and $C = Th(Form(\mathcal{E}) \cup Supp(\mathcal{E}))$; and, conversely if $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ then $(Form(\mathcal{E}), Th(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ is a constrained extension of (D, W).*

Observe that we get a one-to-one correspondence between the "real" extensions, namely $E = Form(\mathcal{E})$. However, the constraints of a constrained extension correspond to the deductive closure of the supports and the asserted formulas of the extension. Thus, we can map assertional extensions onto constrained extensions only modulo equivalent sets of supports.

Also, notice that Theorem 5.3.1 establishes a relationship between constrained extensions and assertional extensions of assertional default theories $(D, \mathcal{W})$ which have a non-supported set of assertional facts, ie. $Supp(\mathcal{W}) = \emptyset$. However, a corresponding relationship between pre-constrained extensions and assertional extensions in the case of supported sets of assertional facts can be given in the obvious way.

Because of this closeness, cumulative default logic shares several properties with constrained default logic: the existence of assertional extensions is guaranteed, cumulative default logic is semi-monotonic and all assertional extensions of a given assertional default theory are weakly orthogonal to each other (ie. the supports of two distinct assertional extensions are always contradictory to each other). In particular, cumulative default logic is cumulative.

Theorem 5.3.2 **[Brewka, 1991b]** *Let $(D, \mathcal{W})$ be an assertional default theory. If $(D, \mathcal{W})$ has an assertional extension containing $\langle\alpha, Supp(\alpha)\rangle$, then $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ containing $\langle\alpha, Supp(\alpha)\rangle$ iff $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W} \cup \{\langle\alpha, Supp(\alpha)\rangle\})$.*

This is illustrated by means of the following example (cf. Example 3.5.1 and 4.10.3).

Example 5.3.2 *The assertional default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right) \tag{5.2}$$

has one assertional extension: $\widehat{Th}(\{\langle A, \{A\}\rangle\})$.

Adding the assertion $\langle A \vee B, \{A\}\rangle \in \widehat{Th}(\{\langle A, \{A\}\rangle\})$ to the set of assertional facts yields the assertional default theory

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{\langle A \vee B, \{A\}\rangle\}\right) \tag{5.3}$$

which has the same assertional extensions: $\widehat{Th}(\{\langle A, \{A\}\rangle\})$.

The case of the first assertional extension is analogous to that of the first classical extension in Example 3.5.1. There, classical default logic allows for the derivation of A and $A \vee B$. In contrast to this, cumulative default logic allows us to derive the assertions $\langle A, \{A\}\rangle$ and $\langle A \vee B, \{A\}\rangle$.

Let us detail the case of the second assertional extension. We observe that changing the assertional default theory (5.2) into (5.3) by adding the assertion $\langle A \vee B, \{A\}\rangle$ amounts to the addition of a nonmonotonic theorem together with its underlying consistency assumptions. Although the prerequisite of the default rule $\frac{A \vee B : \neg A}{\neg A}$ is derivable from the assertional facts (compare with Example 3.5.1) the default rule remains inapplicable, since its justification $\neg A$ is inconsistent with the support of the assertion $\langle A \vee B, \{A\}\rangle$. Notice that this is similar to the case of the assertional default theory (5.2) where the justification $\neg A$ is denied by the consequent of the default rule $\frac{: A}{A}$ (which has to be applied in order to derive the prerequisite of the default rule $\frac{A \vee B : \neg A}{\neg A}$).

On the whole, we see how easily Brewka restores cumulativity in [1991b] by means of assertions. However, cumulative default logic lacks a clear model-theoretic semantics since those proposed for classical default logic (eg. [Etherington, 1987c]) are not applicable here. Thus, the question arises how the "syntactic sugar" introduced by labelled formulas can be realized semantically.

Fortunately, it turns out that cumulative default logic can be captured by means of the focused models semantics.[5] An assertional extension $\mathcal{E}$ is characterized by a preferred focused models structure $(\Pi, \breve{\Pi})$ such that all asserted formulas of $\mathcal{E}$, $\mathit{Form}(\mathcal{E})$, are valid in Π and additionally all supports, $\mathit{Supp}(\mathcal{E})$, are valid in $\breve{\Pi}$.

Let $\Pi_{\mathcal{W}}$ be the class of all models of $\mathit{Form}(\mathcal{W})$ and let $\breve{\Pi}_{\mathcal{W}}$ be the class of all models of $\mathit{Form}(\mathcal{W}) \cup \mathit{Supp}(\mathcal{W})$. Then, we can characterize the exact relationship between assertional extensions and preferred focused models structures as follows.

Theorem 5.3.3 (Correctness & Completeness) *Let* $(D, \mathcal{W})$ *be an assertional default theory and let* $(\Pi, \breve{\Pi})$ *be a pair of classes of first-order interpretations.*

If $\mathcal{E}$ *is an assertional extension of* $(D, \mathcal{W})$ *then* $(\mathrm{MOD}(\mathit{Form}(\mathcal{E})), \mathrm{MOD}(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E})))$ *is a* $\succcurlyeq_D$*-maximal element above* $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$.

If $(\Pi, \breve{\Pi})$ *is a* $\succcurlyeq_D$*-maximal element above* $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$ *then there is an assertional extension* $\mathcal{E}$ *of* $(D, \mathcal{W})$ *such that* $\Pi = \{\pi \mid \pi \models \mathit{Form}(\mathcal{E})\}$ *and* $\breve{\Pi} = \{\pi \mid \pi \models \mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E})\}$.

Notably, we do not obtain a simpler semantical characterization for normal assertional default theories in general (compare with the case of constrained default logic). In cumulative default logic, the focus is necessary in the case of normal assertional default theories which have a supported set of assertional facts $\mathcal{W}$, ie. $\mathit{Supp}(\mathcal{W})$ is non-empty. In particular, this is the case whenever derived assertions are added to the assertional facts. Let us demonstrate this by means of the assertional default theories given in Example 5.3.2.

Example 5.3.3 *The assertional default theory (5.2) has one preferred focused models structure:* $(\mathrm{MOD}(\{A\}), \mathrm{MOD}(\{A\}))$.

Adding the assertion $\langle A \vee B, \{A\}\rangle \in \widehat{Th}(\{\langle A, \{A\}\rangle\})$ *to the set of assertional facts of the assertional default theory (5.2) yields the assertional default theory (5.3) which has the same preferred focused models structure:* $(\mathrm{MOD}(\{A\}), \mathrm{MOD}(\{A\}))$.

The above example — and with it the way Example 5.3.2 is accomplished semantically — is illustrated in Figure 5.2. There, we observe first that in the case of the assertional default theory

[5] Originally, the focused models semantics was proposed in order to capture cumulative default logic [Schaub, 1991a].

(5.3) the focus $\breve{\Pi}_{\mathcal{W}}$ of the focused models structure $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$ corresponding the assertional facts $\{\langle A \vee B, \{A\}\rangle\}$ differs from the whole set of models $\Pi_{\mathcal{W}}$. Second, we see why the default rule $\frac{A \vee B \,:\, \neg A}{\neg A}$ is blocked, although we have added the assertion $\langle A \vee B, \{A\}\rangle$ to the premises: Asserting $A \vee B$ by focusing on those models satisfying A does not allow $\neg A$ to be consistently assumed. Intuitively speaking, we are aware of the underlying consistency assumptions.

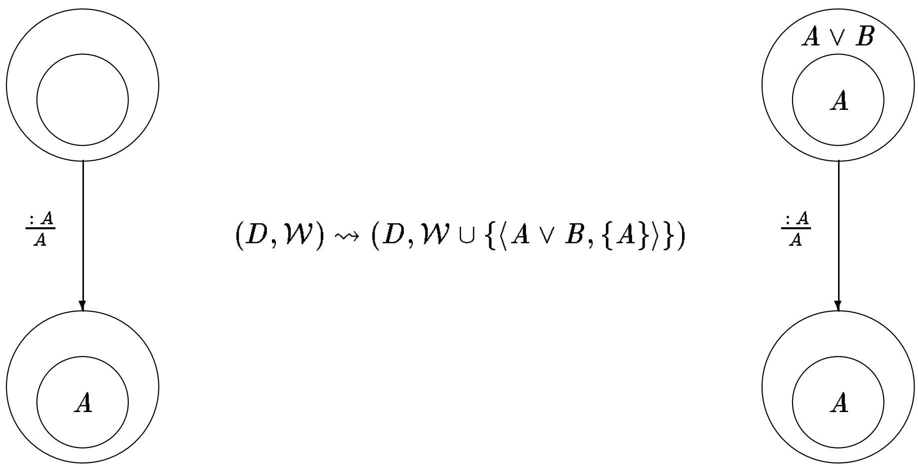

Figure 5.2: Cumulativity in cumulative default logic in terms of focused models structures.

An obvious question arising in the context of cumulativity is, how the approach taken by cumulative default logic is related to that of lemma default rules as introduced in Section 4.10. First, the major difference between the addition of assertions to the assertional facts and the addition of lemma default rules to the set of default rules is that once we have added an assertion to the premises it is not "retractable" any more whenever an inconsistency arises. As an example, take the assertional default theory (5.3) obtained after lemmatizing the assertion $\langle A \vee B, \{A\}\rangle$. Now, adding $\langle \neg(A \vee B), \emptyset\rangle$ yields a hard contradiction since $\mathit{Form}(\langle A \vee B, \{A\}\rangle) \cup \mathit{Form}(\langle \neg(A \vee B), \emptyset\rangle) \vdash \bot$. Consider the following example.

Example 5.3.4 *The assertional default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B \,:\, \neg A}{\neg A}\right\}, \{\langle A \vee B, \{A\}\rangle, \langle \neg(A \vee B), \emptyset\rangle\}\right)$$

has an inconsistent assertional extension.

Thus, the smooth default properties of the original default conclusion have been lost. The same phenomenon has been observed in Example 4.10.2 in the case of pre-constrained default logic. However, adding $\neg(A \vee B)$ in the presence of the lemma default rule $\frac{:A}{A \vee B}$ just blocks the default rule and does not harm the reasoning process itself, as we have seen in Example 4.10.4. Consequently, the addition of assertions [Brewka, 1991b] is stronger than that of lemma default rules.

Second, the approach taken by cumulative default logic guarantees only the continued existence of assertional extensions containing the assertion itself. All extensions inconsistent with the asserted formula or even its support are eliminated after its addition. Consequently, the generation of credulous nonmonotonic lemmas may eliminate incompatible assertional extensions. Hence, let us consider an adaptation of Example 4.10.5.

Example 5.3.5 *The assertional default theory*

$$\left(\left\{\frac{:A}{C}, \frac{C:B}{D}, \frac{:\neg C}{\neg C}\right\}, \emptyset\right)$$

has two assertional extensions: $\widehat{Th}(\{\langle C, \{A, C\}\rangle, \langle D, \{A, B, C, D\}\rangle\})$ *and* $\widehat{Th}(\{\langle \neg C, \{\neg C\}\rangle\})$.

However, adding the assertion $\langle D, \{A, B, C, D\}\rangle$ as a credulous nonmonotonic lemma (which is solely contained in the first assertional extensions) results in the elimination of the second assertional extension, as is shown next (compare with Example 4.10.7).

Example 5.3.6 *The assertional default theory*

$$\left(\left\{\frac{:A}{C}, \frac{C:B}{D}, \frac{:\neg C}{\neg C}\right\}, \{\langle D, \{A, B, C, D\}\rangle\}\right)$$

has one assertional extension: $\widehat{Th}(\{\langle C, \{A, C\}\rangle, \langle D, \{A, B, C, D\}\rangle\})$.

In Example 5.3.5, the default rule $\frac{:\neg C}{\neg C}$ has generated the second assertional extension. However, this default rule is blocked in Example 5.3.6, since its justification (or its consequent) is inconsistent with the support of the assertion $\langle D, \{A, B, C, D\}\rangle$.

In contrast, lemma default rules preserve all extensions and therefore their purpose is more an abbreviation of default proofs in order to improve the computational efforts.

We have seen by means of Theorem 5.3.1 that constrained and cumulative default logic are very close to each other. However, since constrained extensions consist of first-order formulas they do not run into the "floating conclusions" problem [Brewka *et al.*, 1991] that arises whenever we want to reason skeptically by intersecting several extensions. Hence, let us reconsider an adaptation of Example 3.1.1.

Example 5.3.7 *The assertional default theory*

$$\left(\left\{\frac{:\neg B}{A}, \frac{:\neg A}{B}\right\}, \{\langle A \to C, \emptyset\rangle, \langle B \to C, \emptyset\rangle\}\right)$$

has two assertional extensions: $\widehat{Th}(\{\langle A, \{\neg B, A\}\rangle, \langle C, \{\neg B, A\}\rangle\})$ *and* $\widehat{Th}(\{\langle B, \{\neg A, B\}\rangle, \langle C, \{\neg A, B\}\rangle\})$.

Reasoning skeptically, we cannot draw any conclusion about C. Although C is asserted in both extensions the corresponding supports differ and, hence, also the assertions as such are different and do not belong to the intersection.[6]

Let us take a look at the corresponding default theory and its constrained extensions:

Example 5.3.8 *The default theory*

$$\left(\left\{\frac{:\neg B}{A}, \frac{:\neg A}{B}\right\}, \{A \to C, B \to C\}\right)$$

has two constrained extensions: $(Th(\{A, C\}), Th(\{A, C, \neg B\}))$ *and* $(Th(\{B, C\}), Th(\{B, C, \neg A\}))$.

[6] The "floating conclusions" problem has been attacked in [Brewka, 1991a] by computing first one set of assertions containing all assertional extensions and then filtering out the respective assertional extensions.

Reasoning skeptically by intersecting the above extensions and set of constraints yields the following set of skeptical conclusions: $Th(\{A \vee B, C\})$ in the context of $Th(\{C, \neg(A \leftrightarrow B)\})$. Hence, we obtain C as a skeptical conclusion.[7]

The crux in the two previous examples lies in the possibility of introducing the exclusive disjunction $\neg(A \leftrightarrow B)$ as a constraint on the skeptical theorem C. Using assertions we cannot apply any kind of deduction to the supports — apart from considering them when checking consistency. But, by encoding the underlying consistency assumptions as a context guiding our beliefs, we have the whole deductive machinery of standard logic at hand. Consequently, both of the above sets of constraints contain the proposition $\neg(A \leftrightarrow B)$.

5.4 The Theorist approach

A superficially different approach to default reasoning has been taken by Poole in [1988]. Default reasoning is regarded as a process of theory formation and defaults are considered to be hypotheses used to form a consistent set of formulas representing the world.

In this approach, knowledge is represented by three sets of formulas. A *Theorist system* is a triple $(\mathcal{F}, \Delta, \mathcal{C})$ where

- $\mathcal{F}$ is a consistent set of formulas, called the *facts*,
- Δ is a set of formulas, called the *possible hypotheses*[8], and
- $\mathcal{C}$ is a set of formulas, called the *constraints*.[9]

Default reasoning is effectively reduced to hypothetical reasoning. That is, any possible hypothesis may be assumed as long as it is consistent with the facts and the constraints. Therefore, Poole introduces in [1988] the notion of a *scenario* and that of an *explanation* from a given Theorist system.

Definition 5.4.1 *A scenario of $(\mathcal{F}, \Delta, \mathcal{C})$ is a set $\mathcal{F} \cup \Lambda$ where $\Lambda \subseteq \Delta$ such that $\mathcal{F} \cup \Lambda \cup \mathcal{C}$ is consistent.*

A *Theorist extension* is the deductive closure of a maximal (wrt Λ) scenario.

Definition 5.4.2 *A formula α is explainable from $(\mathcal{F}, \Delta, \mathcal{C})$ iff there is a scenario, $\mathcal{F} \cup \Lambda$, such that $\mathcal{F} \cup \Lambda \vdash \alpha$.*

For default reasoning, Δ contains material implications representing default statements. Let us illustrate this with our *Larissa* example given in Section 2.1. For example, the default theory (2.7) corresponds to the following Theorist system $(\mathcal{F}, \Delta, \mathcal{C})$, where

$$\mathcal{F} = \left\{ \begin{array}{l} \mathsf{child}(Larissa), \\ \mathsf{has\text{-}toothache}(Larissa) \end{array} \right\}$$

$$\Delta = \left\{ \begin{array}{l} \mathsf{child}(Larissa) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(Larissa), \\ \mathsf{has\text{-}toothache}(Larissa) \rightarrow \neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa) \end{array} \right\}$$

[7] Observe that constrained extensions are not closed under intersection.

[8] As in default logic, open hypotheses are regarded as schemata representing the set of all their ground instances. Therefore, we shall deal with (closed) formulas.

[9] In what follows, we expect $\mathcal{C}$ to be consistent with $\mathcal{F}$.

and $\mathcal{C} = \emptyset$. Then, $\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$ is explainable by means of the hypothesis

$$\mathsf{child}(Larissa) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(Larissa),$$

whereas $\neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$ is explainable from the hypothesis

$$\mathsf{has\text{-}toothache}(Larissa) \rightarrow \neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa).$$

Accordingly, we obtain two scenarios.

Moreover, Poole introduces in [1988] the concept of "naming" defaults. If α is a default then α is named with n_α, where n_α is a predicate symbol not appearing in $\mathcal{F}, \Delta$, or $\mathcal{C}$. With named defaults, Δ contains only the names of defaults and $\mathcal{F}$ contains formulas of the form $n_\alpha \rightarrow \alpha$, for each name n_α. For instance, the above example may be written as,

$$\mathcal{F} = \left\{ \begin{array}{l} \mathsf{child}(Larissa), \\ \mathsf{has\text{-}toothache}(Larissa), \\ n_{c\rightarrow l} \rightarrow (\mathsf{child}(Larissa) \rightarrow \mathsf{likes\text{-}ice\text{-}cream}(Larissa)), \\ n_{t\rightarrow\neg l} \rightarrow (\mathsf{has\text{-}toothache}(Larissa) \rightarrow \neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)) \end{array} \right\}$$

$$\Delta = \left\{ \begin{array}{l} n_{c\rightarrow l}, \\ n_{t\rightarrow\neg l} \end{array} \right\}$$

and $\mathcal{C} = \emptyset$, where $n_{c\rightarrow l}$ and $n_{t\rightarrow\neg l}$ are "naming" the defaults.

Constraints are mostly stated to specify blocking conditions for defaults, eg. in order to avoid contraposition or unwanted transitivities in the case of interacting defaults. A constraint of the form $v \rightarrow \neg n$ states that the default named n is not applicable when condition v is true. For example, the constraint

$$\mathsf{teething}(Larissa) \rightarrow \neg n_{t\rightarrow\neg l}$$

blocks the default $\mathsf{has\text{-}toothache}(Larissa) \rightarrow \neg\mathsf{likes\text{-}ice\text{-}cream}(Larissa)$ by stating that this default is not applicable when *"Larissa is teething"*, ie. $\mathsf{teething}(Larissa)$ is true.

Poole has shown in [1988] that naming defaults does not affect what is explainable, provided that there are no constraints. However, this is not the case in the presence of constraints. Jackson demonstrates in [Delgrande *et al.*, 1992] that when there are constraints, naming defaults does affect what is explainable.

Example 5.4.1 *The Theorist system*

$$(\{A \rightarrow B, A \rightarrow C\}, \{A, B\}, \{C \rightarrow \neg B\})$$

has one scenario: Th$(\{A \rightarrow B, A \rightarrow C, B\})$.

Now, naming the two defaults in the last example yields the following Theorist system.[10]

Example 5.4.2 *The Theorist system*

$$(\{A \rightarrow B, A \rightarrow C, n_A \rightarrow A, n_B \rightarrow B\}, \{n_A, n_B\}, \{C \rightarrow \neg n_B\})$$

has two scenarios: Th$(\{A \rightarrow B, A \rightarrow C, n_B, B\})$ *and Th*$(\{A \rightarrow B, A \rightarrow C, n_A, A\})$.

[10]Therein, also each constraint of the form $v \rightarrow \neg\alpha$, is replaced by $v \rightarrow \neg n_\alpha$. This translation is not explicitly stated in [Poole, 1988] but seems obvious from the examples given there.

In Example 5.4.1, without naming, $\mathcal{F}$[11] $\cup \{A\}$ is not a scenario because $\mathcal{F} \cup \{A\} \vdash \neg(C \rightarrow \neg B)$. With naming (see Example 5.4.2), however, A is explainable using the hypothesis n_A. Intuitively, it seems that A should be explainable, so named defaults should be used when there are constraints.

Now, we turn to the relationship of Theorist to default logic. Despite their different view, there is a close connection between the two approaches. In particular, we will see that extensions in restricted default logics correspond to Theorist extensions, or equivalently, that explanation in Theorist corresponds to membership in extensions of certain default theories, and vice versa.

Poole himself has shown in [1988] that Theorist systems without constraints are equivalent to prerequisite-free normal default theories in classical default logic. That is, Theorist systems of the form $(\mathcal{F}, \Delta, \emptyset)$ correspond to default theories of the form

$$\left(\left\{ \frac{:\gamma}{\gamma} \,\middle|\, \gamma \in \Delta \right\}, \mathcal{F}\right),$$

and vice versa. Because of Theorem 4.4.1, this equivalence extends to constrained default logic.

The interesting case is, however, Theorist with constraints. Independently, [Brewka, 1991c] and [Dix, 1992] have shown that Theorist systems with constraints are equivalent to prerequisite-free C-normal default theories in classical default logic. That is, Theorist systems of the form $(\mathcal{F}, \Delta, \mathcal{C})$ correspond to default theories of the form

$$\left(\left\{ \frac{:\gamma \wedge \hat{\mathcal{C}}}{\gamma} \,\middle|\, \gamma \in \Delta \right\}, \mathcal{F}\right),$$

and vice versa, where $\hat{\mathcal{C}}$ is the conjunction of all formulas in a finite set of constraints $\mathcal{C}$.

Interestingly, there are three ways for establishing a correspondence between Theorist systems with constraints and restricted default theories in constrained default logic. First of all, notice that the equivalence between Theorist systems with constraints and prerequisite-free C-normal default theories carries over to constrained default logic, since classical and constrained extension coincide in the case of C-normal default theories (cf. Theorem 4.4.1).

The next example shows a Theorist system with constraints along with its translation into a prerequisite-free C-normal default theory.

Example 5.4.3 *The Theorist system*

$$\left(\left\{ \begin{array}{l} A, B, \\ n_{A\rightarrow C} \rightarrow (A \rightarrow C), \\ n_{B\rightarrow\neg C} \rightarrow (B \rightarrow \neg C) \end{array} \right\}, \left\{ \begin{array}{l} n_{A\rightarrow C}, \\ n_{B\rightarrow\neg C} \end{array} \right\}, \{C \rightarrow \neg n_{B\rightarrow\neg C}\}\right) \tag{5.4}$$

yields the default theory

$$\left(\left\{ \begin{array}{l} \frac{:n_{A\rightarrow C}\wedge(C\rightarrow\neg n_{B\rightarrow\neg C})}{n_{A\rightarrow C}}, \\ \frac{:n_{B\rightarrow\neg C}\wedge(C\rightarrow\neg n_{B\rightarrow\neg C})}{n_{B\rightarrow\neg C}} \end{array} \right\}, \left\{ \begin{array}{l} A, B, \\ n_{A\rightarrow C} \rightarrow (A \rightarrow C), \\ n_{B\rightarrow\neg C} \rightarrow (B \rightarrow \neg C) \end{array} \right\}\right)$$

So, we obtain two extensions from the Theorist system (5.4) as well as its corresponding default theory. One contains $n_{A\rightarrow C}$ and C, and the other contains $n_{B\rightarrow\neg C}$ and $\neg C$.

Second, Theorist systems with constraints turn out to be equivalent to prerequisite-free normal default theories in pre-constrained default logic (cf. Section 4.8).

[11] Here, $\mathcal{F}$ stands for the set of facts in Example 5.4.1.

Theorem 5.4.1 *Let W, E, C, C_B and Δ be sets of formulas and let*

$$D = \left\{ \frac{:\beta}{\beta} \;\middle|\; \beta \in \Delta \right\}.$$

Then, (E, C) is a pre-constrained extension of (D, W, C_B) iff E is a Theorist extension of (W, Δ, C_B).

The following example shows the Theorist system (5.4) along with its translation into pre-constrained default logic.

Example 5.4.4 *The Theorist system (5.4) yields the pre-constrained default theory*

$$\left(\left\{ \begin{array}{l} \frac{:n_{A\to C}}{n_{A\to C}}, \\ \frac{:n_{B\to\neg C}}{n_{B\to\neg C}} \end{array} \right\}, \left\{ \begin{array}{l} A, B, \\ n_{A\to C} \to (A \to C), \\ n_{B\to\neg C} \to (B \to \neg C) \end{array} \right\}, \{C \to \neg n_{B\to\neg C}\} \right)$$

As in Example 5.4.3, we obtain two extensions from the above pre-constrained default theory, namely one containing $n_{A\to C}$ and C and another containing $n_{B\to\neg C}$ and $\neg C$.

However, we observe that the above transformation into pre-constrained default logic is closer to the Theorist system (5.4) than the one given in Example 5.4.3 (into ordinary constrained default logic). As in Theorist, the constraints are kept separately from the default rules by means of pre-constraints. This seems to be preferable to the approach taken by C-normal default theories, where the constraints are duplicated at the justifications of the default rules.

Finally, Jackson proves in [Delgrande *et al.*, 1992][12] the correspondence between Theorist systems with constraints and prerequisite-free semi-normal default theories in constrained default logic. A main motivation of this approach is to show that the naming of defaults in Theorist can be avoided in the corresponding default theories. As a result, Jackson gives a translation of Theorist systems into default theories that eliminates these names.

In the remainder of this section, we follow [Delgrande *et al.*, 1992] and assume that defaults are named and that: (i) Δ consists only of the names of defaults; (ii) the only formulas in $\mathcal{F}$ containing names are of the form $n \to \beta$; (iii) and all formulas in $\mathcal{C}$ are of the form $\neg\gamma \to \neg n$, where n is the name of the default.[13] These assumptions reflect how Theorist is used in practise.[14] In addition, one has to prohibit the degenerate case in which the constraints permanently block the application of certain defaults. Therefore, we only consider Theorist systems in which there is no name n such that $\mathcal{F} \cup \mathcal{C} \vdash \neg n$ (see [Delgrande *et al.*, 1992] for further details).

Then, a Theorist system $(\mathcal{F}, \Delta, \mathcal{C})$ is translated into a prerequisite-free default theory by using the function *Tr* defined as follows.

$$\begin{array}{rcl} Tr(\mathcal{F}, \Delta, \mathcal{C}) & = & (D, W) \\ \text{where} & & \\ W & = & \{f \mid f \in \mathcal{F} \text{ and } f \text{ does not contain any default name}\} \\ D & = & \left\{ \frac{:\beta\wedge\gamma}{\beta} \;\middle|\; n \in \Delta, n \to \beta \in \mathcal{F} \text{ and } \neg\gamma \to \neg n \in \mathcal{C} \right\} \end{array}$$

The following example shows how the Theorist system (5.4) is translated into constrained default logic [Delgrande *et al.*, 1992].

[12] Originally published in [Delgrande and Jackson, 1991].

[13] Observe that for any name n, we may have either $\neg\gamma \to \neg n \in \mathcal{C}$ or $\bot \to \neg n \in \mathcal{C}$.

[14] Thus, the following assumption make the correspondence easier to prove without affecting the power of Theorist: Any formula α containing a default name is considered to be nonexplainable, although there may be a scenario $\mathcal{F} \cup \Lambda$ such that $\mathcal{F} \cup \Lambda \vdash \alpha$.

Example 5.4.5 *The Theorist system (5.4) yields the default theory*

$$\left(\left\{\frac{: A \to C}{A \to C}, \frac{: (B \to \neg C) \wedge \neg C}{(B \to \neg C)}\right\}, \{A, B\}\right)$$

As in Example 5.4.3, we obtain two extensions from the above default theory. However, notice that even though we obtain one extension containing C and another containing $\neg C$ neither of them contains any names as in Example 5.4.3 and 5.4.4. In addition, we observe by comparing the above example with the two previous ones that the translation *Tr* yields the most compact representation of the Theorist system (5.4).

The following theorem shows that anything explainable from the Theorist system will occur in some extension of the corresponding default theory.

Theorem 5.4.2 [**Delgrande** *et al.*, **1992**] *A formula α is explainable from $(\mathcal{F}, \Delta, \mathcal{C})$ iff there exists a constrained extension (E, C) of $Tr(\mathcal{F}, \Delta, \mathcal{C})$ such that $\alpha \in E$.*

This relationship between Theorist and constrained extensions implies that the focused models semantics developed for constrained default logic can be used for Theorist. In addition, the logic for reasoning about defaults, described in Section 4.7, can be applied to Theorist.

The inverse of the *Tr* function translates a prerequisite-free semi-normal default theory to a Theorist system. The inverse of *Tr* is defined as:

$$\begin{array}{rcl} Tr^{-1}(D, W) & = & (\mathcal{F}, \Delta, \mathcal{C}) \\ \text{where} & & \\ \mathcal{F} & = & W \cup \left\{ n_\beta \to \beta \;\middle|\; \frac{:\beta \wedge \gamma}{\beta} \in D \right\} \\ \Delta & = & \left\{ n_\beta \;\middle|\; \frac{:\beta \wedge \gamma}{\beta} \in D \right\} \\ \mathcal{C} & = & \left\{ \neg\gamma \to \neg n_\beta \;\middle|\; \frac{:\beta \wedge \gamma}{\beta} \in D \right\} \end{array}$$

and each n_β is a new name for a default rule $\frac{:\beta \wedge \gamma}{\beta} \in D$.

Theorem 5.4.3 [**Delgrande** *et al.*, **1992**] *Let (D, W) be a prerequisite-free semi-normal default theory. A formula α is in E for some constrained extension (E, C) of (D, W) iff α is explainable from $Tr^{-1}(D, W)$.*

As a consequence, the implementations for Theorist can be used to determine if a formula α is in some constrained extension of a prerequisite-free default theory.

5.5 Discussion

In view of the above results, we can make use of the central role of constrained default logic and obtain as corollaries the corresponding relationships between cumulative default logic on one side and classical and justified default logic on the other. The same applies obviously in the case of prerequisite-free default theories. There, we obtain as corollaries the relationships between Poole's approach and all considered derivatives of classical default logic.

On the whole, constrained default logic seems to be closer to cumulative default logic than to justified default logic. Although Łukaszewicz also attaches constraints to extensions, he employs a weaker consistency condition. Similar to classical default logic, justifications need only to be separately consistent with an extension at hand. In particular, this is mirrored by the notion of commitment since cumulative and constrained default logic commit to assumptions, whereas classical and justified default logic do not. Since additionally every classical extension is also a

	Classical default logic	Constrained default logic	Justified default logic	Cumulative default logic
Maximality	G	N	N	N
Pairwise maximality	—	G	G	G
Existence	N	G	G	G
Semi-monotonicity	N	G	G	G
Orthogonality	N	N	N	N
Weak orthogonality	—	G	N	G
Commitment	N	G	N	G
Cumulativity	PfN	Pf	(Pf)	G
Reasoning by cases	Pf	Pf	Pf	Pf
Contraposition	Pf	Pf	Pf	Pf
Reasoning about defaults	PfN	Pf	(PfN)	(Pf)
Skeptical reasoning	G	G	G	W

G	$\cong$	general default theories
N	$\cong$	(C-)normal default theories
Pf	$\cong$	prerequisite-free default theories
PfN	$\cong$	prerequisite-free (C-)normal default theories
W	$\cong$	default theories without default rules

Table 5.1: The variants of classical default logic.

justified extension (cf. [Lukaszewicz, 1988]), justified default logic seems to be closer to classical default logic than to its two constrained descendents.

But constrained default logic also differs from its constrained relatives in employing a deductively closed set of constraints. With this, it neither discards inconsistencies among the constraints nor runs into the "floating conclusions" problem. However, all default logics coincide in the case of normal default theories, ie. we obtain the same extensions (modulo constraints and supports).

We have summarized the previous comparison in Table 5.1. There we give the discussed variants of default logic along with the properties they possess. For completeness, we have also included results which were not explicitly stated here. Others were omitted since they do not refer to entire subsystems of default logic. For instance, we have suppressed the fact that semi-normal default theories which are ordered in a certain way guarantee the existence of extensions [Etherington, 1987a; Zhang and Marek, 1990]. A dash indicates that the considered property is meaningless for the respective default logic. Items in parentheses indicate yet unproven conjectures.

5.6 Conclusion

We have extensively investigated the relationships among the major variants of default logic. In particular, we have benefited from the central role of constrained default logic among the constrained variants of default logic in order to establish the relationships between classical[15] [Reiter, 1980], justified [Łukaszewicz, 1988], assertional [Brewka, 1991b], Theorist [Poole, 1988], and constrained extensions. Furthermore, we have given criteria which indicate when extensions of different default logics coincide.

We benefit from the relationship between Theorist and constrained default logic in two ways. On the one hand, the Theorist implementations can be used to compute constrained extensions of prerequisite-free default theories. On the other hand, the logic for reasoning about defaults (cf. Section 4.7) as well as the focused models semantics can be used for Theorist.

Moreover, we have provided the first semantical characterization of Brewka's cumulative default logic by means of the focused models semantics. The class of focused models has provided a natural semantical counterpart to the supports used to label assertions. The focused models semantics supplied us with several semantical insights into the way cumulative default logic deals with the notions of cumulativity and commitment to assumptions. This gives us another indication that the focused models semantics may be considered as a general semantical approach to commitment to assumptions in default logics.

Also, we have compared the approach taken by lemma default rules with that of assertions. By using lemma default rules, we have avoided labelled formulas as objects of discourse. Since lemma default rules provide an extra-logical approach, the propagation of justifications and consequents of applied default rules is avoided. We merely look at the default proofs and, hence, regard the assumptions underlying a conclusion only by need. Moreover, lemma default rules have the advantage that they are somehow retractable such that later inconsistencies by subsequent information are avoided. Also, lemma default rules allow for the usage of credulous as well as skeptical nonmonotonic lemmas, since they preserve all extensions of a given default theory. In all, the purpose of lemma default rules is more an abbreviation of default proofs in order to improve the computational efforts than to account for the formal property of cumulativity.

[15] This has been done in Section 4.4.

Chapter 6

Possible worlds semantics for default logics

This chapter presents a uniform semantical framework for default logics in terms of Kripke structures. The possible worlds approach provides a simple but meaningful instrument for comparing existing default logics in a unified setting. The semantics is introduced in Section 6.2 by means of constrained default logic. In addition, it easily deals with cumulative default logic. In Sections 6.3 and 6.4, the semantics is extended to classical as well as justified default logic. The possible worlds approach remedies several difficulties encountered in former proposals aiming at individual default logics. Notably, it provides the first pure model-theoretic semantics for justified default logic. Since the semantical framework is presented from the perspective of "commitment to assumptions" (see Section 3.3.4), we also obtain a very natural modal interpretation of the notion of commitment.

6.1 Motivation

We have seen in the previous chapters that recent research on default logic has produced many derivatives of Reiter's original formalism. A common feature of all of these variants is their use of constraints, either on formulas, as in cumulative default logic, or on sets of formulas, as in justified and constrained default logic. In other words, all of the descendants of classical default logic employ more "structure" in order to achieve their desired results. In a similar way, Etherington's semantics for classical default logic has been extended in order to account for the additional syntactical structures. As a result, two-fold semantics were proposed whose second component was intended to capture the enriched structure in default logics.

Although the elements of these two-fold semantics are standard first-order interpretations, splitting the semantical characterizations of the extension and its underlying constraints might appear to be artificial. On the other hand, Kripke structures (cf. [Bowen, 1979]) provide a means to establish relations between first-order interpretations: A Kripke structure has a distinguished world, the "actual" world, and a set of worlds accessible from it (each world is associated with a first-order interpretation). As a consequence, a first aim of this work is to avoid two-fold semantics by characterizing extensions in default logics by means of Kripke structures, thereby somehow absorbing the additional syntactical structures used in each variant of classical default logic. In fact, this approach turns out to be very general, so that we obtain a uniform semantical framework for comparing existing default logics in a unified setting.

The idea is roughly as follows. In default logics, our beliefs consist of the conclusions given by the applying default rules, and the constraints on our beliefs stem from the justifications provided by the same default rules. Accordingly, the intuition behind our semantics is very natural and easy to understand: The actual worlds of a considered class of Kripke structures exhibits what we believe and the accessible worlds exhibit what constraints we have imposed upon our beliefs. Hence, the actual world is our envisioning of how things are and, therefore, characterizes an extension, whereas the surrounding worlds additionally deal with the constraints and, therefore, provide a context in which that envisioning takes place.

Let us put this in more concrete terms by means of constrained default logic. In constrained default logic, a set of constraints C is attached to an extension E. Given a constrained extension (E,C) and a Kripke structure $\mathfrak{m}$, we require that the actual world be a model of the extension, E, and demand that each world accessible from the actual world be a model of the constraints, C. That is, $\mathfrak{m} \models E \wedge \Box C$.[1]

First, we present a semantical characterization of constrained and assertional extensions in terms of Kripke structures. Also, we show how our possible worlds semantics captures classical and justified default logic. We can then easily compare default logics and characterize the differences between them. In particular, the semantics reveals that all of the various default logics employ constraints (induced by the consequents and justifications of applied default rules) but differ basically in the extent to which the constraints are taken into account. Since this extent is directly related to the notion of "commitment to assumptions", we also obtain a very natural semantical characterization of this notion in the context of default logics.

6.2 A modal characterization of constrained default logic

In Section 4.5, we characterized a constrained extension (E,C) by means of focused models structures $(\Pi, \breve{\Pi})$, which are pairs of classes of first-order interpretations. The first class, Π, characterizes E and the second class, $\breve{\Pi}$, characterizes C. However, the focused models structures suggest that the ordering induced by a default rule has a modal nature with the corresponding semantical approach being based on Kripke structures. Intuitively, a pair $(\Pi, \breve{\Pi})$ is to be rendered as a class $\mathfrak{M}$ of Kripke structures such that Π is captured by the actual worlds in $\mathfrak{M}$ and $\breve{\Pi}$ by the accessible worlds in $\mathfrak{M}$. Ie. consider a non-modal formula α: it is valid in Π iff α is valid in $\mathfrak{M}$ and it is valid in $\breve{\Pi}$ iff $\Box\alpha$ is valid in $\mathfrak{M}$.

Correspondingly, the counterpart to a maximal focused models structure is a class $\mathfrak{M}$ of Kripke structures such that

$$(\{\alpha \text{ non-modal } \mid \mathfrak{M} \models \alpha\}, \{\alpha \text{ non-modal } \mid \mathfrak{M} \models \Box\alpha\})$$

forms a constrained extension of a default theory under consideration. As always, the first set establishes the extension whereas the second set characterizes its constraints.

We follow the definitions in [Bowen, 1979] of a Kripke structure (called K-model in the sequel) as a quadruple $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, where Ω is a non-empty set (also called a set of worlds), $\omega_0 \in \Omega$ a distinguished world, $\mathcal{R}$ a binary relation on Ω (also called the accessibility relation) and $\mathcal{I}$ is a function which defines a first-order interpretation $\mathcal{I}_\omega$ for each $\omega \in \Omega$. As usual, a K-model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$ is such that the domain of $\mathcal{I}_\omega$ is a subset of the domain of $\mathcal{I}_{\omega'}$ whenever $(\omega, \omega') \in \mathcal{R}$.

Formulas in K-models are interpreted using a language enriched in the following way: in a K-model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, for each $\omega \in \Omega$, the first-order interpretation $\mathcal{I}_\omega$ is extended so that for each $e \in D_\omega$ (the domain of $\mathcal{I}_\omega$), a constant $\overline{e}$ is introduced, letting $\mathcal{I}_\omega(\overline{e}) = e$. In every world ω, each term is mapped into an element of D_ω as follows: $\mathcal{I}_\omega(f(t_1, \ldots, t_n)) = (\mathcal{I}_\omega(f))(\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n))$, $n \geq 0$.

[1]Given a set of formulas S let $\Box S$ stand for $\wedge_{\alpha \in S} \Box\alpha$.

Given a K-model $\mathfrak{m} = \langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, the modal entailment relation $\omega \models \alpha$ (in $\mathfrak{m}$) is defined by recursion on the structure of α:

$$
\begin{array}{lll}
\omega \models P(t_1, \ldots, t_n) & \text{iff} & (\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n)) \in \mathcal{I}_\omega(P) \\
\omega \models \neg\alpha & \text{iff} & \omega \not\models \alpha \\
\omega \models \alpha \vee \beta & \text{iff} & \omega \models \alpha \text{ or } \omega \models \beta \\
\omega \models \forall x\ \alpha[x] & \text{iff} & \omega \models \alpha[\overline{e}] \text{ for all } e \in D_\omega \\
\omega \models \Box\alpha & \text{iff} & \omega' \models \alpha \text{ whenever } (\omega, \omega') \in \mathcal{R}
\end{array}
$$

We write $\mathfrak{m} \models \alpha$ if $\omega_0 \models \alpha$ (in $\mathfrak{m}$). This means that $\mathfrak{m}$ is a model of α. We denote classes of K-models by $\mathfrak{M}$. We extend the modal entailment relation $\models$ to classes of K-models $\mathfrak{M}$ and write $\mathfrak{M} \models \alpha$ to mean that each element in $\mathfrak{M}$ (ie. each K-model) entails α.

In order to characterize constrained extensions semantically, we now define a family of strict partial orders on classes of K-models. As in Section 3.6 and 4.5, given a default rule δ, its application conditions and the result of applying it are captured by an order $\succ_\delta$ as follows.

Definition 6.2.1 *Let $\delta = \frac{\alpha : \beta}{\gamma}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be distinct classes of K-models. We define $\mathfrak{M} \succ_\delta \mathfrak{M}'$ iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box(\gamma \wedge \beta)\}$$

and

1. $\mathfrak{M}' \models \alpha$
2. $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$

Given a set of default rules D, the strict partial order $\succ_D$ amounts to the union of the strict partial orders $\succ_\delta$ as follows.

Definition 6.2.2 *Let D be a set of default rules and $\mathfrak{M}$ a class of K-models. The order $\succ_D$ on $2^{\mathfrak{M}}$ is defined as follows. For all $\mathfrak{M}', \mathfrak{M}'' \in 2^{\mathfrak{M}}$ we have*

$$\mathfrak{M}' \succ_D \mathfrak{M}''$$

iff there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of some $D' \subseteq D$ such that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for some sequence $\langle \mathfrak{M}_i \rangle_{i \in I}$ of subclasses of $\mathfrak{M}''$ satisfying $\mathfrak{M}'' = \mathfrak{M}_0$ and $\mathfrak{M}' = \bigcap_{i \in I} \mathfrak{M}_i$.

For a default theory (D, W), we furthermore define the class of K-models associated with W as $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models \gamma \wedge \Box\gamma, \gamma \in W\}$[2] and refer to $\succ_D$-maximal classes of K-models above $\mathfrak{M}_W$ as the *preferred* classes of K-models wrt (D, W).

As for modal logic, observe that the K-models define the modal system K. This makes sense because the only property needed is distributivity for the modal operator $\Box$ to ensure that the constraints are deductively closed.

As a reminder, we give below the axiom schema (K) and inference rule (NEC) which must be added to a standard first-order system in order to obtain K:

$$(K) \qquad \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$$

$$(NEC) \qquad \frac{\alpha}{\Box\alpha}$$

[2]If it is clear from the context, we simply write $\mathfrak{M}_W$ without explicitly stating the respective elements of W.

With K, we have chosen the most general modal logic, since it imposes no restrictions on the accessibility relation. Notably, the accessibility relation must not be reflexive, since then the strict separation between actual and accessible worlds would be destroyed. However, [Delgrande, 1992] favors modal logic *K45* in order to draw comparisons with other semantics, eg. [Levesque, 1984], at the cost of introducing additional constraints on the accessibility relation.

The choice of Condition 2 in Definition 6.2.1 is also worth discussing. At first glance, it seems more adequate to require $\mathfrak{M}' \not\models \neg\Box(\gamma \wedge \beta)$ since we want to add K-models entailing $\Box(\gamma \wedge \beta)$. This is because Condition 2 in Definition 6.2.1, namely $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$, does not a priori exclude $\mathfrak{M}' \models \neg\Box(\gamma \wedge \beta)$. We illustrate why this condition is needed by means of the next example.

Example 6.2.1 *The default theory*

$$\left(\left\{\frac{:A}{A}\right\}, \{\neg A\}\right)$$

has one preferred class of K-models: $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models \neg A \wedge \Box\neg A\}$.

With $\mathfrak{M}_W \models \neg A$, we also have $\mathfrak{M}_W \models \Box\neg A$. But using the condition $\mathfrak{M}' \not\models \neg\Box A$ would not prevent the "application" of the only default rule.

Notice that Condition 2 in Definition 6.2.1 is equivalent to

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond(\gamma \wedge \beta). \tag{6.1}$$

That is, the consistency condition in constrained default logic corresponds semantically to the requirement that there is a K-model which has some accessible world that satisfies $\gamma \wedge \beta$.

In the following examples, we show how preferred classes of K-models can characterize constrained extensions. First, we illustrate the main idea and so look at the simple the default theory $(\{\frac{A:B}{C}\}, \{A\})$. As we have seen in Section 4.2, this default theory yields the constrained extension $(Th(\{A, C\}), Th(\{A, B, C\}))$.

In order to characterize this semantically, we have to find the corresponding preferred class of K-models, ie. a $\succ_D$-maximal class of K-models above $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models A \wedge \Box A\}$. Since $\mathfrak{M}_W \models A$ (establishing Condition 1) it remains to ensure Condition 2 of Definition 6.2.1, namely $\mathfrak{M}_W \not\models \Box\neg(C \wedge B)$. Obviously, this is also satisfied and we obtain a $\succ_{\{\frac{A:B}{C}\}}$-greater class of K-models

$$\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models A \wedge \Box A \wedge C \wedge \Box(C \wedge B)\}.$$

Clearly, $\mathfrak{M}$ constitutes the only preferred class. Thus, the actual worlds of our preferred K-models satisfy the formulas of the extension $Th(\{A, C\})$ whereas the surrounding worlds additionally fulfill the constraints, namely $Th(\{A, B, C\})$.

In order to have a comprehensive example throughout this chapter, let us revisit the default theory (5.1) discussed in Chapter 5. In Example 5.1.2, we have seen that the default theory (5.1)

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has three constrained extensions: $(Th(\{C\}), Th(\{B, C\})), (Th(\{D\}), Th(\{\neg B, D\}))$, and $(Th(\{E\}), Th(\{\neg D, \neg C, E\}))$. The way the underlying process is accomplished semantically is described below.

Example 6.2.2 *The default theory (5.1) has three preferred classes of K-models:*

- $\{\mathfrak{m} \mid \mathfrak{m} \models C \wedge \Box(B \wedge C)\}$,
- $\{\mathfrak{m} \mid \mathfrak{m} \models D \wedge \Box(\neg B \wedge D)\}$,
- $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box(\neg D \wedge \neg C \wedge E)\}$.

The above example is illustrated in Figure 6.1 by means of a directed acyclic graph representing the orders induced by the default theory (5.1) according to Definition 6.2.1. The nodes constitute classes of K-models which are described by means of some canonical K-models. The arcs are labelled with default rules in order to indicate the corresponding order. The root stands for the class of K-models associated with the facts, namely $\mathfrak{M}_W$, whereas each leaf represents a preferred class of K-models.

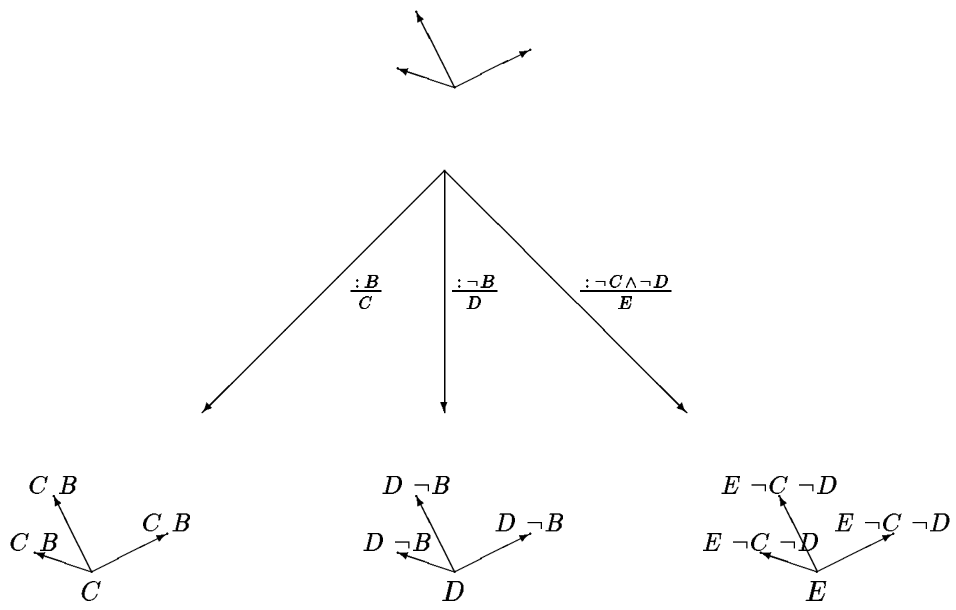

Figure 6.1: Commitment in constrained default logic.

Now, let us examine Example 6.2.2 in detail. $\mathfrak{M}_W$ is the class of all K-models and clearly, we have $\mathfrak{M}_W \not\models \Box\neg(C \wedge B)$, $\mathfrak{M}_W \not\models \Box\neg(D \wedge \neg B)$, and $\mathfrak{M}_W \not\models \Box\neg(E \wedge \neg D \wedge \neg C)$. Therefore, all of the default rules are potentially "applicable".

Let us detail the case of the first preferred class of K-models, say $\mathfrak{M}$. We obtain a $\succ_{\{\frac{:B}{C}\}}$-greater class

$$\mathfrak{M} \models C \wedge \Box(C \wedge B).$$

In order to show that there is a $\succ_{\{\frac{:B}{C}, \frac{:\neg B}{D}\}}$-greater class, we would have to show that $\mathfrak{M} \not\models \Box\neg(D \wedge \neg B)$. But since $\Box(C \wedge B) \models \Box B$, we have $\mathfrak{M} \models \Box(B \vee \neg D)$, which prevents us from "applying" the second default rule. Analogously, we do not obtain a $\succ_{\{\frac{:B}{C}, \frac{:\neg D \wedge \neg C}{E}\}}$-greater class.

The last example shows how our semantics copes with incoherent default theories. Consider the default theory given in Example 3.3.1:

Example 6.2.3 *The default theory*

$$\left(\left\{\frac{:\neg A}{A}\right\}, \emptyset\right)$$

has one preferred class of K-models: $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models \top\}$.

$\mathfrak{M}_W$ is the class of all K-models. But since $\mathfrak{M}_W \models \Box\neg(A \wedge \neg A)$ Condition 2 of Definition 6.2.1 is falsified and, therefore, $\mathfrak{M}_W$ is also the only preferred class. Obviously, this class corresponds to the only constrained extension of the above default theory obtained in Example 4.3.4, $(Th(\emptyset), Th(\emptyset))$.

An interesting point concerning Definition 6.2.1 is that finding a non-empty $\mathfrak{M} \subseteq \mathfrak{M}'$ such that $\mathfrak{M} \models \Box(\gamma \wedge \beta)$ whenever $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$ might appear to be impossible. Hence, we give the next theorem.

Theorem 6.2.1 *The empty class of K-models is never preferred wrt (D, W) whenever W is consistent.*

As a corollary we obtain that the existence of constrained extensions is guaranteed.

The notion of a preferred class of K-models illustrated above is put into a precise correspondence with constrained extensions in the following theorem.

Theorem 6.2.2 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K-models and E, C deductively closed sets of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$. Then,*

$$(E,C) \textit{ is a constrained extension of } (D, W) \textit{ iff } \mathfrak{M} \textit{ is a } \succ_D\textit{-maximal class above } \mathfrak{M}_W.$$

Then our possible worlds approach amounts to the focused models semantics presented in Section 4.5: the first-order interpretations associated with the accessible worlds take over the role of the focused models.

Corollary 6.2.3 *Let (D, W) be a default theory, $(\Pi, \breve{\Pi})$ a $\succcurlyeq_D$-maximal focused models structure above $(\{\pi \mid \pi \models W\}, \{\pi \mid \pi \models W\})$ and $\mathfrak{M}$ a preferred class of K-models wrt (D, W). Then, for α, β non-modal*

$$\Pi \models \alpha \textit{ iff } \mathfrak{M} \models \alpha \textit{ and } \breve{\Pi} \models \beta \textit{ iff } \mathfrak{M} \models \Box\beta.$$

In view of the above corollary, observe that a preferred class of K-models contains "more" different actual worlds than accessible ones. The reason is that focused models structures $(\Pi, \breve{\Pi})$ have the inclusion property $\breve{\Pi} \subseteq \Pi$.

How does our semantics reflect the notion of commitment? As already pointed out, the intuition behind our semantics is very natural and easy to understand: The actual world of a K-model captures what we believe and the surrounding worlds capture what commitments we have allowed to adopt our beliefs. Therefore, our semantics reflects the notion of commitment through modal necessity: the commitments correspond to formulas whose necessity holds.

Since we have shown in Theorem 5.3.3 that the focused models semantics captures cumulative default logic, Theorem 6.2.2 and Corollary 6.2.3 establish a possible worlds semantics for cumulative default logic as is shown next.

Theorem 5.3.1 shows that if (E,C) is a constrained extension of (D, W) then there is an assertional extension $\mathcal{E}$ of $(D, \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\})$ such that $E = Form(\mathcal{E})$ and $C = Th(Form(\mathcal{E}) \cup Supp(\mathcal{E}))$ and conversely, if $\mathcal{E}$ is an assertional extension of $(D, \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\})$ then $(Form(\mathcal{E}), Th(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ is a constrained extension of (D, W). Consequently, our possible worlds semantics also characterizes cumulative default logic:

Theorem 6.2.4 (Correctness & Completeness) *Let $(D, \mathcal{W})$ be an assertional default theory. Let $\mathfrak{M}_{\mathcal{W}}$ be the class of all K-models of $\{v \wedge \Box\eta \mid v \in \mathit{Form}(\mathcal{W}), \eta \in \mathit{Supp}(\mathcal{W})\}$. Then, there exists a set of assertions $\mathcal{E}$ which is an assertional extension of $(D, \mathcal{W})$ such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models \mathit{Form}(\mathcal{E}) \wedge \Box \mathit{Supp}(\mathcal{E})\}$ iff $\mathfrak{M}$ is a preferred class of K-models above $\mathfrak{M}_{\mathcal{W}}$.*

In the context of cumulative default logic, naturally the question arises how the notion of cumulativity can be characterized by our possible worlds semantics. In the case of constrained default logic, the failure of cumulativity was tackled (in Section 4.10) by means of lemma default rules. As we have seen in Section 5.3, the major difference between the addition of assertions to the facts and the addition of lemma default rules to the set of default rules is that once we have added an assertion to the premises it is not retractable any more whenever an inconsistency arises. Thus, the addition of assertions is stronger than that of lemma default rules. Adding an assertion to the premises eliminates all extensions inconsistent with the asserted formula or even its support. On the contrary, lemma default rules preserve all extensions and, therefore, their purpose is more an abbreviation of default proofs.

How can those differences be envisioned by our semantics? Assume we have a constrained extension (E, C) and the corresponding assertional extension $\mathcal{E}$. Whenever we have a theorem $\rho \in E$ and a minimal set of default rules $D_\rho \subseteq GD_D^{(E,C)}$ which has been used to derive ρ, there exists as well an assertion $\xi_\rho \in \mathcal{E}$, where

$$\xi_\rho = \langle \rho, \textstyle\bigcup_{\delta \in D_\rho} \{\mathit{Justif}(\delta), \mathit{Conseq}(\delta)\}\rangle.$$

For a complement, the corresponding lemma default rule is

$$\delta_\rho = \frac{: \bigwedge_{\delta \in D_\rho} \mathit{Justif}(\delta) \wedge \mathit{Conseq}(\delta)}{\rho}.$$

Take a default theory (D, W) and its assertional counterpart $(D, \mathcal{W})$, where $\mathcal{W} = \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\}$. Looking at cumulative default logic, we stipulate (by adding the assertion ξ_ρ to $\mathcal{W}$) that all preferred classes of K-models entail the formula

$$\rho \wedge \Box\Big(\rho \wedge \textstyle\bigwedge_{\delta \in D_\rho} \mathit{Justif}(\delta) \wedge \mathit{Conseq}(\delta)\Big). \tag{6.2}$$

In constrained default logic the addition of the lemma default rule δ_ρ to the set of default rules only demands the expression (6.2) to be entailed by those preferred classes of K-models, to whose generation the lemma default rule has contributed. That is, we enforce the entailment of (6.2) only for all preferred classes of K-models $\mathfrak{M}$ for which $\mathfrak{M} \succ_{GD_D^{(E,C)} \cup \{\delta_\rho\}} \mathfrak{M}_W$ holds.

6.3 A modal characterization of classical default logic

The possible worlds approach to default logic presented above turns out to be very general. The first evidence of this arises from the fact that the above semantical characterization carries over easily to classical default logic. Indeed, the analogue to Definition 6.2.1 can be defined as follows.[3]

Definition 6.3.1 *Let $\delta = \frac{\alpha:\beta}{\gamma}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be distinct classes of K-models. We define $\mathfrak{M} >_\delta \mathfrak{M}'$ iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta\}$$

and

[3] Given a set of formulas S let $\Diamond S$ stand for $\wedge_{\alpha \in S} \Diamond\alpha$.

1. $\mathfrak{M}' \models \alpha$

2. $\mathfrak{M}' \not\models \Box\neg\beta$

The order $>_D$ is defined analogously to that in Section 6.2.

Even though classical default logic does not employ explicit constraints, there is a natural counterpart given by the justifications of the generating default rules over a set of formulas E as given in equation (4.2). That is,

$$C_E = \left\{\beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}.$$

We obtain a semantical characterization which yields a one-to-one correspondence between consistent extensions and non-empty $>_D$-preferred classes of K-models (an inconsistent extension trivially corresponds to $\mathfrak{M}_W$ being preferred while being empty).

Theorem 6.3.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K-models and E be a deductively closed set of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$. Then,*

E is a consistent classical extension of (D, W) iff $\mathfrak{M}$ is a $>_D$-maximal non-empty class above $\mathfrak{M}_W$.

Comparing Definition 6.3.1 with Definition 6.2.1, we observe two basic differences by reflecting on the fact that constrained default logic employs a stronger consistency check than classical default logic. For one thing, the second condition on $\mathfrak{M}'$ is weakened such that only β instead of $\gamma \wedge \beta$ is required to be satisfied by some accessible world of some K-model in $\mathfrak{M}'$. This becomes clear by comparing the following formulation of Condition 2 in Definition 6.3.1

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond\beta \tag{6.3}$$

with the one given in (6.1). For another thing, Definition 6.3.1 requires $\Diamond\beta$ to be valid in $\mathfrak{M}$ whereas Definition 6.2.1 requires $\Box\beta$ to be valid in $\mathfrak{M}$. Stated otherwise, the possible worlds semantics for classical extensions requires only *some* accessible world satisfying the justification β whereas the semantics for constrained default logic requires *all* accessible worlds to satisfy β.

The conclusion is that from the perspective of commitment, constrained extensions adopt their beliefs by committing to all consequents and all justifications of applied default rules whereas classical default logic commits to consequents taken together but only to justifications taken separately.

Let us now return to default theory (5.1).

Example 6.3.1 *The default theory (5.1)*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has one preferred class of K-models:

- $\{\mathfrak{m} \mid \mathfrak{m} \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond\neg B\}$.

Then, the above class of K-models, say $\mathfrak{M}'$, corresponds to the only classical extension, $Th(\{C, D\})$, of default theory (5.1).[4] How $\mathfrak{M}'$ is obtained from $\mathfrak{M}_W$ is illustrated in Figure 6.2. According to the conventions introduced for Figure 6.1 on page 95, $\mathfrak{M}_W$ is represented by the root of the directed acyclic graph, whereas $\mathfrak{M}'$ is given by means of a corresponding leaf.

[4] Cf. Example 5.1.1.

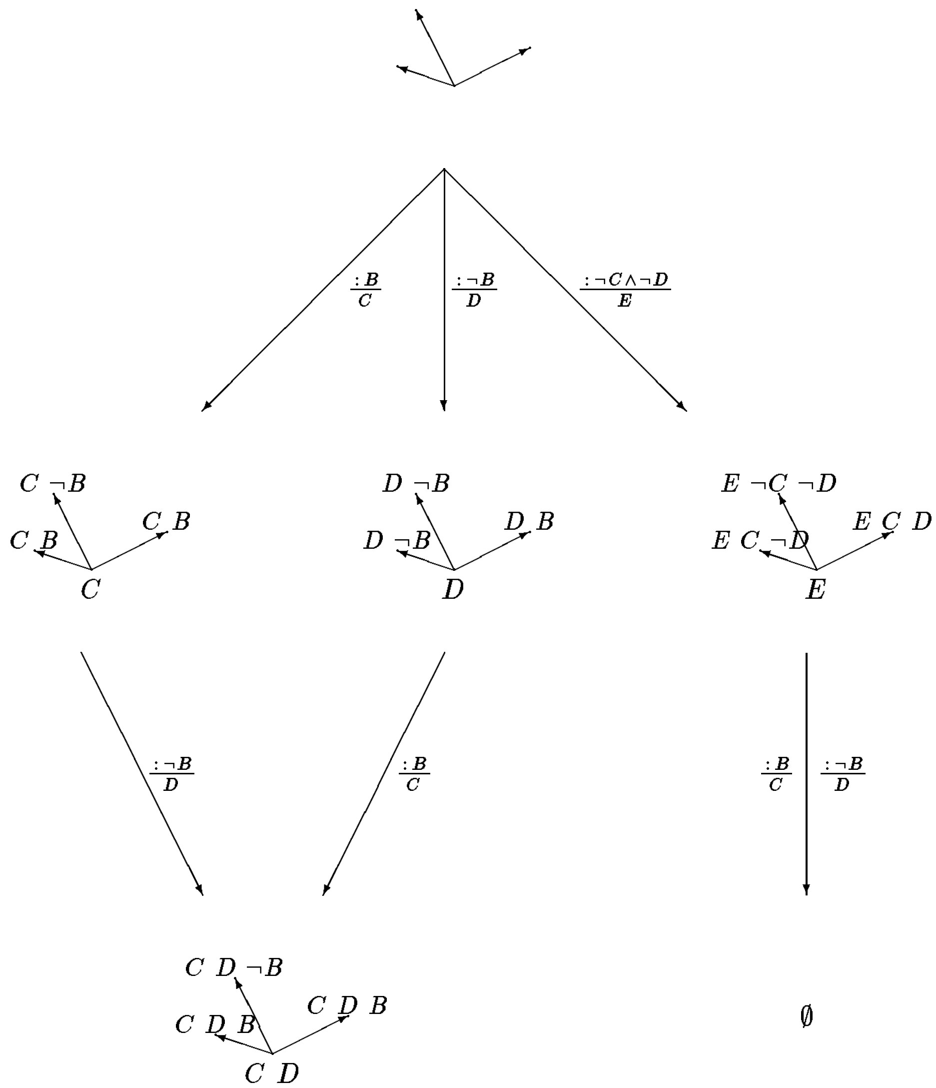

Figure 6.2: Commitment in classical default logic.

Now, let us examine Example 6.3.1 in detail. $\mathfrak{M}_W$ is the class of all K-models and clearly, we have $\mathfrak{M}_W \not\models \Box\neg B$, $\mathfrak{M}_W \not\models \Box\neg(\neg B)$, and $\mathfrak{M}_W \not\models \Box\neg(\neg D \wedge \neg C)$. That is, all of the default rules are potentially "applicable".

From $\mathfrak{M}_W$ we can construct a class of K-models $\mathfrak{M}$ such that $\mathfrak{M} >_{\left\{\frac{:B}{C}\right\}} \mathfrak{M}_W$ and

$$\mathfrak{M} \models C \wedge \Box C \wedge \Diamond B.$$

Accordingly, we can also construct a class of K-models $\mathfrak{M}'$ such that $\mathfrak{M}' >_{\left\{\frac{:B}{C},\frac{:\neg B}{D}\right\}} \mathfrak{M}_W$ and

$$\mathfrak{M}' \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond\neg B.$$

But it is impossible to obtain a class $\mathfrak{M}''$ such that $\mathfrak{M}'' >_{\left\{\frac{:B}{C},\frac{:\neg B}{D},\frac{:\neg D\wedge\neg C}{E}\right\}} \mathfrak{M}_W$ since $\mathfrak{M}' \models \Box\neg(\neg D \wedge \neg C)$. Therefore, $\mathfrak{M}'$ is a non-empty preferred class of K-models.

From $\mathfrak{M}_W$, selecting first the third default rule leads to a $>_{\left\{\frac{:\neg D\wedge\neg C}{E}\right\}}$-greater class $\dot{\mathfrak{M}}$ such that

$$\dot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C).$$

From $\dot{\mathfrak{M}}$ we can construct a class of K-models $\ddot{\mathfrak{M}}$ such that $\ddot{\mathfrak{M}} >_{\left\{\frac{:\neg D\wedge\neg C}{E},\frac{:B}{C}\right\}} \mathfrak{M}_W$ and

$$\ddot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C) \wedge C \wedge \Box C \wedge \Diamond B.$$

So, $\ddot{\mathfrak{M}}$ is the empty set of K-models because $\Diamond(\neg D \wedge \neg C) \models \Diamond\neg C$ and $\Box C \wedge \Diamond\neg C \models \bot$.

In contrast to constrained default logic (cf. Theorem 6.2.1), the possible worlds semantics for classical default logic admits the empty set of K-models above some non-empty $\mathfrak{M}_W$. This is the case whenever a default rule is applied whose consequent contradicts the justification of some default rule which is itself applied. In particular, this reflects the failure of semi-monotonicity in classical default logic (whereas constrained default logic enjoys semi-monotonicity).

Also, characterizing extensions in default logic strictly by non-empty $>_D$-maximal elements above $\mathfrak{M}_W$ avoids post-filtering mechanisms such as the *stability* criterion [Etherington, 1987c] given in Definition 3.6.3. The purpose of the stability criterion is to ensure the satisfiability of each justification for a given set of default rules. In other words, the stability criterion guarantees the continued consistency[5] of the justifications of the applying default rules. In contrast, we ensure the continued consistency of justifications by requiring the validity of $\Diamond\beta$ in all classes of K-models preferred by a default rule $\frac{\alpha:\beta}{\gamma}$. As a consequence, whenever an incoherent default theory arises, our characterization yields an empty set of K-models. For example, we have seen in Example 3.3.1 that the incoherent default theory $(\{\frac{:\neg A}{A}\}, \emptyset)$ has no classical extension. Obviously, $\mathfrak{M}_W$ is the class of all K-models. Clearly, $\mathfrak{M}_W \not\models \Box A$ but the resulting class $\{\mathfrak{m} \in \mathfrak{M}_W \mid \mathfrak{m} \models A \wedge \Box A \wedge \Diamond\neg A\}$ is obviously empty. In contrast, Etherington's semantics yields the class of models of A which is obviously unstable.

The above discussion reveals another difference between Etherington's approach and the one taken by means of possible worlds. Namely, the possible worlds approach accounts for the implicit constraints, C_E, used in classical default logic. In fact, we have seen in the preceding paragraph that we obtain an empty class of K-models in Example 3.3.1 because the modal expressions accounting for the implicit consistency condition, $\Diamond\neg A$, and the one for the result of applying the default rule, $A \wedge \Box A$, were incompatible with each other.

Finally, let us examine the failure of cumulativity in classical default logic. In Section 6.2, we have characterized by means of a modal expression the solutions preserving cumulativity.

[5] Cf. Section 3.3.2.

Taking the expression given in (6.2), but dropping the requirement of joint consistency, yields the following modal expression for classical default logic:

$$\rho \wedge \Box\rho \wedge \Diamond \mathit{Justif}(D_\rho) \tag{6.4}$$

where ρ is contained in a classical extension E of a default theory (D, W) and $D_\rho \subseteq GD_D^E$ is a set of default rules used to derive ρ.

Let us look at the canonical cumulativity example given in Example 3.5.1. Consider the default theory (3.5) obtained after adding $A \vee B$ (so that we are considering $\rho = A \vee B$). This default theory (see below) has two classical extensions which are $\mathit{Th}(\{A\})$ and $\mathit{Th}(\{\neg A, B\})$.

Example 6.3.2 *The default theory (3.5)*

$$\left(\left\{\frac{: A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right)$$

has two preferred classes of K-models $\mathfrak{M}$ and $\mathfrak{M}'$ where

$$\mathfrak{M} \models (A \vee B) \wedge \Box(A \vee B) \wedge \neg A \wedge \Box\neg A$$

and

$$\mathfrak{M}' \models (A \vee B) \wedge \Box(A \vee B) \wedge A \wedge \Box A.$$

As illustrated in Section 3.5, cumulativity fails in Example 3.5.1 since the default theory (3.4) comes up with a second classical extension $\mathit{Th}(\{\neg A, B\})$. The semantical characterization of this classical extension yields a preferred class of K-models $\mathfrak{M}$ which is $>_{\left\{\frac{A \vee B : \neg A}{\neg A}\right\}}$-greater than $\mathfrak{M}_W$ such that $\mathfrak{M} \models (A \vee B) \wedge \Box(A \vee B) \wedge \neg A \wedge \Box\neg A$.

Since $D_\rho = \left\{\frac{: A}{A}\right\}$, $\mathfrak{M}$ does obviously not entail our above modal expression (6.4). That is,

$$\mathfrak{M} \not\models (A \vee B) \wedge \Box(A \vee B) \wedge \Diamond A.$$

The entailment of the expression (6.4) in all preferred classes of K-models $\mathfrak{M}$ such that $\mathfrak{M} >_{GD_D^E \cup \{\zeta_\rho\}} \mathfrak{M}_W$ can be enforced through the corresponding lemma default rule ζ_ρ for classical default logic (cf. Section 3.5.2). That is, given ρ and $D_\rho = \{\delta_1, \ldots, \delta_n\} \subseteq GD_D^E$, we have according to Definition 3.5.2:

$$\zeta_\rho = \frac{: \mathit{Justif}(\delta_1), \ldots, \mathit{Justif}(\delta_n)}{\rho}.$$

6.4 A modal characterization of justified default logic

Further evidence for the generality of our approach is that it can easily capture justified default logic. Indeed, the analogue to Definition 6.2.1 and 6.3.1 can be defined as follows.

Definition 6.4.1 *Let $\delta = \frac{\alpha : \beta}{\gamma}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be distinct classes of K-models. We define $\mathfrak{M} \rhd_\delta \mathfrak{M}'$ iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta\}$$

and

1. $\mathfrak{M}' \models \alpha$

2. $\mathfrak{M}' \not\models \Box\neg\beta \vee \Diamond\neg\gamma$

The order $\rhd_D$ is defined analogously to that in Section 6.2.

Compared to the order $>_\delta$ given for classical default logic, the only difference is that the condition $\mathfrak{M}' \not\models \Box\neg\beta$ has become $\mathfrak{M}' \not\models \Box\neg\beta \vee \Diamond\neg\gamma$, that is, $\mathfrak{M}' \not\models \neg(\Box\gamma \wedge \Diamond\beta)$. Again, this becomes apparent by regarding Condition 2 in Definition 6.4.1, namely

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond\beta \wedge \Box\gamma. \tag{6.5}$$

In classical default logic, there has to be a K-model which has some accessible world satisfying β (see (6.3) above). In justified default logic, however, all accessible worlds of such a K-model additionally have to satisfy γ.

Indeed, the definition reveals the fact that the same constraints implicitly used in classical default logic (in the form of C_E) are explicitly attached to justified extensions (in the form of J, see Definition 5.2.1) and, moreover, are considered when checking consistency. That is, semantically classical and justified default logic account for the justifications of the applied default rules in form of the modal propositions $\Diamond\beta$, which are entailed by $>_\delta$- and $\rhd_\delta$-greater classes of K-models. However, in classical default logic these modal constraints are discarded when checking consistency.

Łukaszewicz has shown in [1988] that justified default logic guarantees the existence of extensions. Semantically, it is obvious that requiring $\mathfrak{M}' \not\models \neg(\Box\gamma \wedge \Diamond\beta)$ and adding those K-models entailing $\Box\gamma \wedge \Diamond\beta$ makes it impossible to obtain the empty set of K-models (hence the analogue to Theorem 6.2.1 trivially holds). Łukaszewicz has also shown that his variant enjoys semi-monotonicity. In fact, "applying" a default rule $\frac{\alpha:\beta}{\gamma}$ enforces all $\rhd_D$-greater classes of K-models $\dot{\mathfrak{M}}$ to entail $\Box\gamma \wedge \Diamond\beta$. Therefore, a later "application" of a default rule $\frac{\alpha':\beta'}{\gamma'}$ whose consequent γ' contradicts β (eg. $\gamma' = \neg\beta$) is prohibited since its "application" requires $\dot{\mathfrak{M}} \not\models \Box\neg\beta' \vee \Diamond\neg\gamma'$.

Analogously to classical default logic, Definition 6.4.1 only requires $\Diamond\beta$ to be valid in $\mathfrak{M}$, which is not enough for justified default logic to commit to assumptions. In Example 5.2.1, we have seen that the default theory (5.1) has two justified extensions, $Th(\{C, D\})$ wrt $\{B, \neg B\}$ and $Th(\{E\})$ wrt $\{\neg D \wedge \neg C\}$.

Example 6.4.1 *The default theory (5.1)*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has two preferred classes of K-models:

- $\{\mathfrak{m} \mid \mathfrak{m} \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond\neg B,\}$
- $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C)\}$

As in the preceding sections, we have illustrated the last example by means of some canonical K-models. This is done in Figure 6.3 according to the conventions introduced for Figure 6.1 and 6.2.

The first preferred class of K-models is obtained analogously to that in Example 6.3.1. That is, we obtain a preferred class

$$\mathfrak{M}' \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond\neg B.$$

Also, selecting the third default rule first leads to a class $\dot{\mathfrak{M}} \rhd_{\left\{\frac{:\neg D \wedge \neg C}{E}\right\}} \mathfrak{M}_W$ such that

$$\dot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C).$$

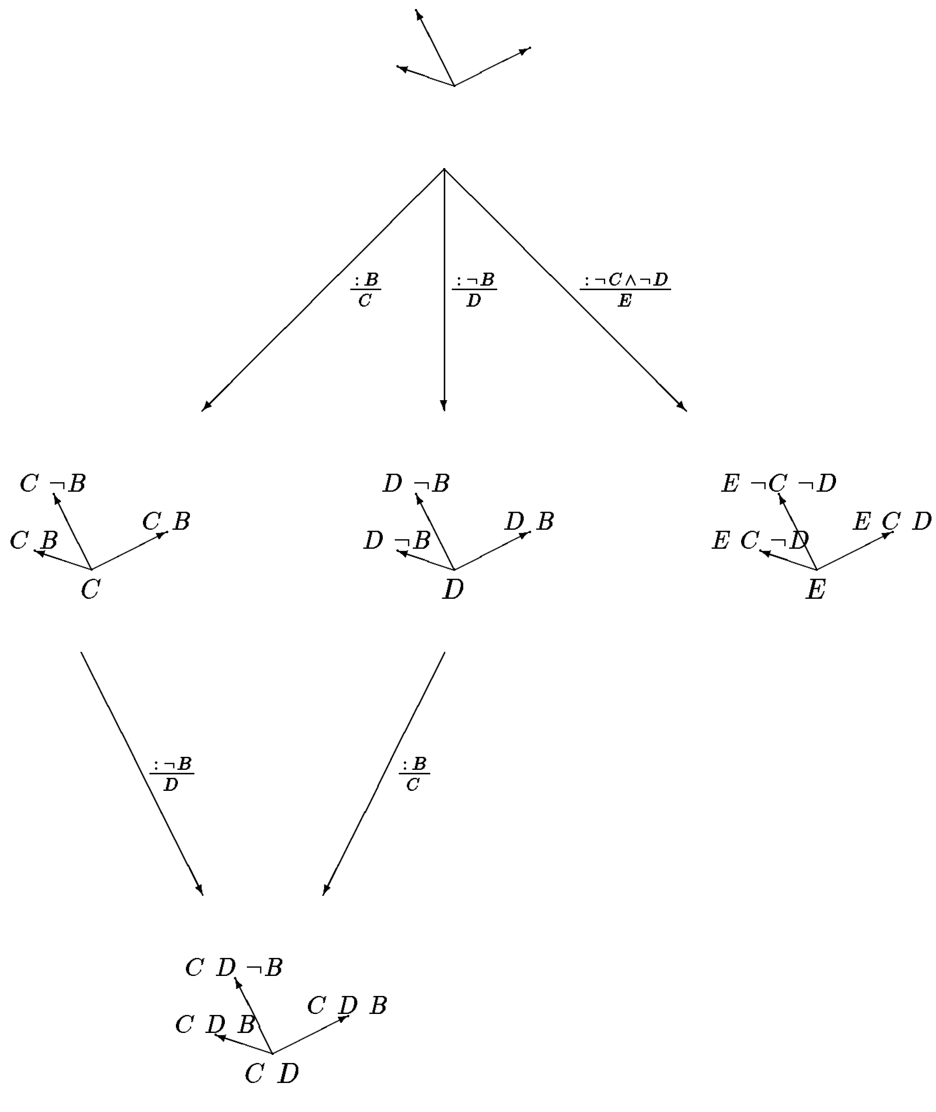

Figure 6.3: Commitment in justified default logic.

Since we have $\dot{\mathfrak{M}} \models \Diamond\neg C$ and $\dot{\mathfrak{M}} \models \Diamond\neg D$ none of the other default rules is "applicable". Therefore, $\dot{\mathfrak{M}}$ is a (non-empty) preferred class.

Similar to the case of classical default logic, there is a natural account of constraints attached to a set of formulas E justified by J: the justifications of the generating default rules over E and J, which are simply

$$C_{(E,J)} = \left\{\beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \forall\eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot\right\}.^{6}$$

Then, a correctness and completeness result holds as in the former sections.

Theorem 6.4.1 (Correctness & Completeness) *Let (D,W) be a default theory. Let $\mathfrak{M}$ be a class of K-models, E a deductively closed set of formulas, and J a set of formulas such that $J = C_{(E,J)}$ and $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$. Then,*

E is a justified extension of (D,W) wrt J iff $\mathfrak{M}$ is a $\rhd_D$-maximal class above $\mathfrak{M}_W$.

The equality $J = C_{(E,J)}$ simply states that the implicit constraints $C_{(E,J)}$ and the explicit constraints J coincide.

Notably, our possible worlds semantics is the first semantical characterization of justified default logic which is purely model-theoretic. In [1988], Łukaszewicz had to characterize justified extension by means of pairs (Π, J), where Π is a class of first-order interpretations and J is a set of formulas. The reason Łukaszewicz did so is that justified default logic allows for inconsistent sets of individually consistent constraints (so that the focused models semantics cannot be adapted there).

Finally, a remark concerning Definition 6.2.1 and 6.4.1 is appropriate. Let us compare the respective consistency condition, (6.1) and (6.5). We observe that the condition in constrained default logic requires that there is a K-model which has some accessible world satisfying $\gamma \wedge \beta$. In contrast, we are faced with a stronger requirement in justified default logic: there has to be a K-model whose accessible worlds all satisfy γ and some accessible world satisfies β. At first glance, this seems to be counterintuitive since constrained default logic has a stronger consistency condition than justified default logic (compare Definition 4.2.1 and 5.2.1). However, consistency or satisfiability are always relative to a given set of formulas or class of models, respectively. In fact, we consider a much more restricted class of K-models $\mathfrak{M}'$ in (6.1) in constrained default logic than in (6.5) in justified default logic. Given a set of default rules D' such that $\mathfrak{M}' \succ_{D'} \mathfrak{M}_W$ and $\mathfrak{M}' \rhd_{D'} \mathfrak{M}_W$, we have $\mathfrak{M}' \models W \wedge \mathit{Conseq}(D') \wedge \Box(W \wedge \mathit{Conseq}(D') \wedge \mathit{Justif}(D'))$ in constrained default logic, whereas we encounter a less restricted class of K-models in justified default logic, namely $\mathfrak{M}' \models W \wedge \mathit{Conseq}(D') \wedge \Box(W \wedge \mathit{Conseq}(D')) \wedge \Diamond \mathit{Justif}(D')$. As a consequence, we have to employ a stronger satisfiability condition in justified default logic, which is given in (6.5).

6.5 Conclusion

We have presented a uniform semantical framework for various default logics in terms of Kripke structures. That is, we have first introduced a possible worlds semantics for constrained default logic and we have proved that it also captures cumulative default logic. Then, we have provided a simple modification to that semantics in order to characterize Reiter's classical default logic and, in turn, Łukaszewicz' justified default logic.

No other semantics for any default logic offers this generality. Moreover, the approach remedies several difficulties encountered in former proposals aiming at individual default logics. First, the

[6] Observe that the membership qualifying property is exactly the third condition in the definition of a justified extension.

approach avoids post-filtering mechanisms such as the stability condition required in [Etherington, 1987c] (and [Łukaszewicz, 1988]). In contrast, our semantics characterizes extensions strictly by maximal classes of K-models. Second, the approach avoids two-fold semantical structures such as focused models structures or so-called frames [Łukaszewicz, 1988]. For a complement, our possible worlds semantics allows us to characterize extensions in a very homogeneous way since the increasing structure in default logics is captured by means of possible worlds. Third, the approach provides the first semantical characterization of justified default logic which is purely model-theoretic. In [1988], Łukaszewicz characterized the (possibly inconsistent) set of constraints of justified extensions by sets of formulas. Again, this is accomplished by means of possible worlds in our approach.

By adopting the perspective of commitment, we have not only gained a clear criterion on that notion itself but also provided a very natural modal interpretation by which existing default logics can be compared in a simple but very substantial and meaningful manner. In particular, the semantics has revealed that all the various default logics employ constraints but differ in the extent to which the constraints are considered when checking consistency. Notably, in terms of modalities we have to "switch from $\Diamond$ to $\Box$" whenever we want to preserve "commitment to assumptions". That is, our semantics reflects this notion of commitment through modal necessity: the "strong" commitments (as in constrained and cumulative default logic) correspond to formulas whose necessity holds, whereas the "weak" commitments (as in justified default logic) and "non" commitments (as in classical default logic) correspond to formulas whose possibility holds.

Finally, let us compare our possible worlds semantics with the focused models semantics developed in Section 4.5. Of course, it would be unfair to compare a whole semantical framework with a particular semantics. Therefore, we restrict ourselves to the case of constrained default logic. We have seen in Section 6.2 that the first-order interpretations associated with the accessible worlds in the possible worlds approach correspond to the focused models used in focused models approach. As argued above, the possible worlds approach is advantageous from a mathematical point of view since it allows us to characterize constrained extensions in a homogeneous way. In particular, it avoids two-fold semantical structures as used in the focused models semantics. However, the focused models semantics seems to be advantageous from a cognitive point of view [Schlieder, 1992] since it reflects the notion of a *"mental model"* found in psychology and cognitive science [Johnson-Laird, 1983].

Chapter 7

Conclusion

In this thesis, we have presented several innovations in the field of default logic. We have developed constrained default logic in order to overcome the problems encountered in the original approach and subsequent variants. Constrained extensions play a fundamental role in constrained default logic. The novel idea has been to explicate the context-sensitive nature of default logic by distinguishing between the set of beliefs given by an extension and the underlying constraints which form a context guiding our beliefs. The approach has clear semantical foundations and remedies the problems encountered in the original approach in an arguably simpler way than other proposals. Aside from the property of cumulativity, constrained default logic possesses all desirable properties which could be expected from general default theories. Cumulativity is preserved in the case of prerequisite-free default theories in constrained default logic. This is a larger cumulative fragment of default theories than obtained in classical default logic. Furthermore, we have extended constrained default logic in order to allow for pre-constraints and priorities among default rules.

Moreover, constrained default logic has served as a bridge between the various derivatives of default logic. We have related classical default logic and its variants to constrained default logic in order to clarify the relationships among these approaches. It appears that cumulative default logic is closer to constrained default logic than justified and classical default logic. As a result, we have given several criteria for the coincidence of extensions of different variants of default logic.

A semantical counterpart to constrained default logic has been given by the focused models semantics. This semantical approach has turned out to be very general. In fact, it has provided the first semantical characterization for cumulative default logic. The approach has supplied us with useful semantical insights into the enhancements obtained in constrained and cumulative default logic. In both cases, the class of focused models has given a natural semantical counterpart to the additional syntactical structure found in the respective systems.

Moreover, we have succeeded in giving a uniform semantical framework for default logics in terms of Kripke structures. No other semantics for any default logic offers this generality. A key advantage of our possible worlds approach is that it provides a simple but meaningful instrument for comparing existing default logics in a unified setting. For instance, the semantics has revealed that all of the various default logics employ constraints but differ in the extent to which the constraints are considered when checking consistency. This has been accomplished by means of a modal criterion which indicates the degree of "commitment to assumptions" found in each default logic. Apart from its unique generality, the approach also remedies several difficulties encountered in former proposals aiming at individual default logics.

Finally, we have provided a general approach to incorporate nonmonotonic lemmas into default logics. This has emerged from cumulativity as its most important practical impact. The approach has been successfully applied to classical and constrained default logic. We have demonstrated that the approach provides a versatile instrument for generating and using non-monotonic lemmas. In this respect, it has appeared to be advantageous over the approach taken by assertions.

Open questions and future work

Two major topics have not been addressed. First, we have not dealt with complexity issues. However, we believe that the overall complexity of default logic also applies to constrained default logic. For instance, [Gottlob, 1992] shows that credulous reasoning in classical default logic is complete for the class Σ_2^P of the polynomial hierarchy, while skeptical reasoning is complete for the dual class Π_2^P.

Second, we have dealt neither with algorithmic nor with implementational details. Although we can employ Theorist algorithms and implementations [Poole, 1988] in order to compute constrained extensions of prerequisite-free default theories, appropriate algorithms for computing constrained extensions in general remain to be designed. However, it seems that the proof-theory developed in [Reiter, 1980] for normal default theories can be extended in a straightforward way to general default theories in the case of constrained default logic. This problem is currently being investigated in [Rothschild, 1993]. In addition, the pool-based connection calculus [Neugebauer and Schaub, 1991] is extended in order to account for query-answering in constrained default logic.

Also, the conjectures indicated in Table 5.1 are not verified yet. That is, it has to be shown whether cumulativity is preserved for prerequisite-free default theories in justified default logic, whether justified default logic allows for reasoning about default rules in the case of prerequisite-free normal default theories, and whether cumulative default logic allows for reasoning about default rules in the case of prerequisite-free assertional default theories. Furthermore, it remains to be shown which kind of lemma default rule is appropriate for justified default logic.

But there are still other avenues to explore. A prime candidate for future research seems to be the possible worlds semantics introduced in Chapter 6. The semantics has supplied us with a uniform semantical framework for default logics. Therefore, it should be possible to combine the various default logics in a uniform syntactical framework which allows us to specify default rules of different types. [1] Also, the semantics gives a clear criteria for the application of default rules in each variant. Hence, it is reasonable to expect that one can reason about default rules in a larger fragment than those described in Section 3.4 and 4.7. Moreover, it is yet an unsolved problem whether there is a logical system that allows for axiomatizing default theories. As a prime candidate the semantics strongly suggests an intuitionistic modal logic in which the modal part captures the applicability conditions for default rules whereas the intuitionistic part captures the strict partial orders. The last three issues are currently being investigated together with Philippe Besnard and Dov Gabbay. Another interesting question is whether one can express all properties discussed in the previous chapters (eg. semi-monotonicity) by means of modal criteria. Also, it seems worth looking for other variants of default logic inside the possible worlds framework.

Another candidate for future research is the idea of a context itself. We have employed this idea in a straightforward way. However, there may be alternative approaches dealing with more structured (or even partial) contexts in order to allow for a reduction of computational efforts.

[1] Meanwhile, a preliminary version of this has been appeared [Besnard and Schaub, 1993a]. The author.

In particular, this would lead to more sophisticated consistency conditions.

Appendix A

Theorems

In the sequel, we will refer to some definitions and results due to other authors on which we draw on in the following chapters. We give these results for the reader's convenience. The corresponding proofs can be found in the indicated literature.

Other results seemed to be irrelevant for the presentation in the previous chapters and were, therefore, postponed to this point. Their proofs are given along with the stated result.

Before we proceed, however, a remark concerning most of the proofs in the remaining chapters is appropriate. The case where the set of facts W is inconsistent is almost always trivial in any variant of default logic. This is because the inconsistency of W implies in each variant of default logic that there is only one extension given by the set of wff. In this case no default rules are applicable.

A.1 Theorems on justified default logic

Theorem A.1.1 [**Łukaszewicz, 1988**] *Let (D, W) be a default theory and let E and J be sets of formulas. Define*

$$E_0 = W \qquad \text{and} \qquad J_0 = \emptyset$$

and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\} \\ J_{i+1} &= \quad J_i \quad \cup \left\{ \beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\} \end{aligned}$$

(E, J) is a justified extension of (D, W) iff $(E, J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$.

Definition A.1.1 [**Risch, 1992**] *Let (D, W) be a default theory and S and T sets of formulas. The set of generating default rules for (S, T) wrt D is defined as*

$$GD_D^{(S,T)} = \left\{ \tfrac{\alpha:\beta}{\gamma} \in D \;\middle|\; \alpha \in S,\ \forall \eta \in T \cup \{\beta\}.\ S \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}.$$

Theorem A.1.2 [**Risch, 1992**] *Let (E, J) be a justified extension of a default theory (D, W). We have*

$$\begin{aligned} E &= Th\Big(W \cup Conseq\Big(GD_D^{(E,J)}\Big)\Big) \\ J &= Justif\Big(GD_D^{(E,J)}\Big) \end{aligned}$$

Theorem A.1.3 [**Risch, 1992**](**Groundedness**) *Let (E, J) be a justified extension of a default theory (D, W). Then, there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,J)}$ such that for $i \in I$*

$$W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i).$$

Theorem A.1.4 [**Łukaszewicz, 1988**](**Semi-monotonicity**) *Let (D, W) be a default theory and D' a set of default rules such that $D \subseteq D'$. If (E, J) is a justified extension of (D, W), then there is a justified extension (E', J') of (D', W) such that $E \subseteq E'$ and $C \subseteq J'$.*

As a corollary, we obtain the following one.

Corollary A.1.5 *Let (D, W) be a default theory and D' a set of default rules such that $D \subseteq D'$. If (E, J) is a justified extension of (D, W), then there is a justified extension (E', J') of (D', W) such that $GD_D^{(E,J)} \subseteq GD_{D'}^{(E',J')}$.*

A.2 Theorems on cumulative default logic

Theorem A.2.1 [**Brewka, 1991b**] *Let $(D, \mathcal{W})$ be an assertional default theory and let $\mathcal{E}$ be a set of assertions. Define*

$$\mathcal{E}_0 = \mathcal{W}$$

and for each $i \geq 0$

$$\begin{aligned}\mathcal{E}_{i+1} \quad = \quad & \widehat{Th}(\mathcal{E}_i) \cup \{\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \mid \tfrac{\alpha : \beta}{\gamma} \in D, \\ & \qquad \langle \alpha, Supp(\alpha) \rangle \in \mathcal{E}_i, Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}\end{aligned}$$

$\mathcal{E}$ is an assertional extension iff $\mathcal{E} = \bigcup_{i=0}^{\infty} \mathcal{E}_i$.

Definition A.2.1 *Let $(D, \mathcal{W})$ be an assertional default theory and let $\mathcal{F}$ be a set of assertions. The set of generating default rules for $\mathcal{F}$ wrt D is defined as*

$$GD_D^{\mathcal{E}} = \left\{ \tfrac{\alpha : \beta}{\gamma} \;\middle|\; \langle \alpha, Supp(\alpha) \rangle \in \mathcal{F}, Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}.$$

Theorem A.2.2 *Let $\mathcal{E}$ be an assertional extension of an assertional default theory $(D, \mathcal{W})$. We have*

$$\mathcal{E} = \widehat{Th}\Big(\mathcal{W} \cup \Big\{ \langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \;\Big|\; \tfrac{\alpha : \beta}{\gamma} \in GD_D^{\mathcal{E}},\ \langle \alpha, Supp(\alpha) \rangle \in \mathcal{E} \Big\}\Big).$$

According to the definition of the assertional consequence operator $\widehat{Th}$ we obtain the following corollary.

Corollary A.2.3 *Let $\mathcal{E}$ be an assertional extension of an assertional default theory $(D, \mathcal{W})$. We have*

$$Form(\mathcal{E}) = Th(Form(\mathcal{W}) \cup Conseq(GD_D^{\mathcal{E}}))$$

and

$$Supp(\mathcal{E}) = Supp(\mathcal{W}) \cup Conseq(GD_D^{\mathcal{E}}) \cup Justif(GD_D^{\mathcal{E}}).$$

Proof A.2.2 Let $\mathcal{E}$ be an assertional extension of an assertional default theory $(D, \mathcal{W})$. For the sake of readability, we abbreviate

$$\widehat{Th}\Big(\mathcal{W} \cup \Big\{\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \;\Big|\; \tfrac{\alpha:\beta}{\gamma} \in GD_D^{\mathcal{E}},\ \langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}\Big\}\Big)$$

by $\mathcal{E}_{GD}$.

"$\subseteq$" By definition, $\mathcal{W} \subseteq \mathcal{E}$. Also, if $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \mathcal{E}_{GD}$ then there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}$, $\mathit{Supp}(\gamma) = \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}$ and $\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \nvdash \perp$. Then, by Definition 5.3.3, $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \mathcal{E}$. Consequently, $\mathcal{W} \cup \mathcal{E}_{GD} \subseteq \mathcal{E}$. By monotonicity, $\widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD}) \subseteq \widehat{Th}(\mathcal{E})$. Since $\mathcal{E} = \widehat{Th}(\mathcal{E})$, we obtain $\widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD}) \subseteq \mathcal{E}$.

"$\supseteq$" First, $\mathcal{W} \subseteq \widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})$.

Second, by idempotence, $\widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD}) = \widehat{Th}\big(\widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})\big)$.

Third, consider $\frac{\alpha:\beta}{\gamma} \in D$. If $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})$ then $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}$ according to what we have just proved. If additionally, $\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \nvdash \perp$ then $\frac{\alpha:\beta}{\gamma} \in GD_D^{\mathcal{E}}$, whence $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}_{GD}$. As a consequence, $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})$.

Accordingly, by the minimality of Ω, we have $\Omega(\mathcal{E}) \subseteq \widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})$. Since $\mathcal{E}$ is an assertional extension, we obtain $\mathcal{E} \subseteq \widehat{Th}(\mathcal{W} \cup \mathcal{E}_{GD})$.

■

Theorem A.2.4 (Groundedness) *Let $\mathcal{E}$ be an assertional extension of $(D, \mathcal{W})$. Then, there exists an enumeration $\langle\delta_i\rangle_{i\in I}$ of $GD_D^{\mathcal{E}}$ such that for $i \in I$*

$$\mathit{Form}(\mathcal{W}) \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i).$$

Proof A.2.4 Let $\mathcal{E}$ be an assertional extension of $(D, \mathcal{W})$.[1] According to Theorem 5.3.1 $(\mathit{Form}(\mathcal{E}), Th(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E})))$ is a constrained extension of $(D, \mathit{Form}(\mathcal{W}))$. Then, analogously to Theorem 4.3.6, we obtain the same sequence of partial extensions E_i and in particular the same sequence of sets of default rules Δ_i which results in exactly the same enumeration of the set of generating default rules. ■

[1] Without loss of generality, we assume $\mathcal{W}$ to be a non-supported set of assertions, ie. $\mathit{Supp}(\mathcal{W})$ is empty.

Appendix B

Proofs of Theorems in Chapter 3

This chapter presents the proofs of the theorems given in Chapter 3.

Prerequisite-free default theories

Theorem 3.4.1 *Let (D, W) be a prerequisite-free default theory and E a set of formulas. Then,*

$$E \text{ is a classical extension of } (D,W) \text{ iff } E = Th\Big(W \cup \Big\{ \gamma \;\Big|\; \tfrac{:\beta}{\gamma} \in D, \neg\beta \notin E \Big\}\Big).$$

Proof 3.4.1 According to Theorem 3.2.1, we have that E is a classical extension of (D, W) iff $E = \bigcup_{i=0}^{\infty} E_i$, where

$$E_0 = W$$

and for $i \geq 0$

$$E_{i+1} \;=\; Th(E_i) \;\cup\; \Big\{\gamma \;\Big|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E\Big\}.$$

According to this characterization, it is sufficient to show

$$\bigcup_{i=0}^{\infty} E_i = Th\Big(W \cup \Big\{ \gamma \;\Big|\; \tfrac{:\beta}{\gamma} \in D, \neg\beta \notin E \Big\}\Big)$$

in the case of prerequisite-free default theories. Therefore, we reformulate $\bigcup_{i=0}^{\infty} E_i$ by expanding its first four elements as follows.

$$\begin{aligned}
E_0 &= W \\
E_1 &= Th(E_0) \;\cup\; \Big\{ \gamma \;\Big|\; \tfrac{:\beta}{\gamma} \in D, \neg\beta \notin E \Big\} \\
E_2 &= Th(E_1) \\
E_3 &= Th(E_2) = Th(Th(E_1)) = Th(E_1)
\end{aligned}$$

Obviously, we have $\bigcup_{i=0}^{\infty} E_i = E_2$. That is, $\bigcup_{i=0}^{\infty} E_i = Th\Big(W \cup \Big\{ \gamma \;\Big|\; \tfrac{:\beta}{\gamma} \in D, \neg\beta \notin E \Big\}\Big)$. ■

Lemma default rules in classical default logic

Theorem 3.5.2 *Let (D, W) be a default theory and let E and E' be classical extensions of (D, W). Let $\langle D_1, \ldots, D_k \rangle$ be a default proof of ρ in E', and let ζ_ρ be the corresponding lemma default rule for ρ. Then,*

$$\zeta_\rho \in GD^E_{D\cup\{\zeta_\rho\}} \textit{ iff } \bigcup_{i=1}^k D_i \subseteq GD^E_D.$$

Proof 3.5.2 The case for E being inconsistent is easily dealt with, so that we prove below the theorem for E being consistent.

According to [Reiter, 1980], we have for non-singular default rules

$$GD^E_D = \left\{ \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D \;\middle|\; \alpha \in E,\ \neg\beta_1, \ldots, \neg\beta_n \notin E \right\}. \tag{B.1}$$

only-if part Assume $\zeta_\rho \in GD^E_{D\cup\{\delta_\rho\}}$. Let $\langle D_1, \ldots, D_k\rangle$ be the corresponding default proof of ρ in E' from (D, W). Let $D_\rho = \bigcup_{i=1}^k D_i$.

According to (B.1), we have for each $\delta \in D_\rho$

$$\neg \mathit{Justif}(\delta) \notin E. \tag{B.2}$$

We show by induction that $D_i \subseteq GD^E_D$.

Base Consider $\delta \in D_1$. By definition, $W \vdash \mathit{Prereq}(\delta)$. Then, $W \subseteq E$ and the fact that E is deductively closed implies $\mathit{Prereq}(\delta) \in E$. By (B.2), $\neg\mathit{Justif}(\delta) \notin E$. Then, by Definition 3.2.1 we obtain $\delta \subseteq GD^E_D$. Thus, $D_1 \subseteq GD^E_D$.

Step Assume, we have $D_i \subseteq GD^E_D$. Then, by Theorem 3.2.4, $\mathit{Conseq}(D_i) \subseteq E$.

Consider $\delta \in D_{i+1}$. By definition, $W \cup \mathit{Conseq}(D_i) \vdash \mathit{Prereq}(\delta)$. Then, since $W \cup \mathit{Conseq}(D_i) \subseteq E$ and E is deductively closed, we obtain $\mathit{Prereq}(\delta) \in E$. By (B.2), we have $\neg\mathit{Justif}(\delta) \notin E$. Then, by Definition 3.2.1 we obtain $\delta \subseteq GD^{(E,C)}_D$. Thus, $D_{i+1} \subseteq GD^E_D$.

Hence, we obtain $D_\rho \subseteq GD^E_D$.

if part Assume $D_\rho \subseteq GD^E_D$. Then, by Definition 3.2.1, $\neg\mathit{Justif}(\delta) \notin E$ for each $\delta \in D_\rho$. By (B.1), $\zeta_\rho \in GD^E_{D\cup\{\delta_\rho\}}$. ■

Theorem 3.5.3 *Let (D, W) be a default theory and let E' be a classical extension of (D, W). Let ζ_ρ be a lemma default rule for $\rho \in E'$. Then,*

E is a classical extension of (D, W) iff E is a classical extension of $(D \cup \{\zeta_\rho\}, W)$.

Proof 3.5.3

only-if part Let E be a classical extension of (D, W). By Definition 3.5.2,

$$\zeta_\rho = \frac{: \mathit{Justif}(\delta_1), \ldots, \mathit{Justif}(\delta_n)}{\rho}.$$

where $\{\delta_1, \ldots, \delta_n\} = \bigcup_i D_i$ for some default proof $\langle D_1, \ldots, D_k\rangle$ of ρ in E' from (D, W). We distinguish the following two cases. Let us abbreviate $\{\delta_1, \ldots, \delta_n\}$ by D_ρ.

1. Let $D_\rho \subseteq GD^E_D$. Then, $\mathit{Conseq}(D_\rho) \subseteq E$. That is, $\mathit{Conseq}(\zeta_\rho) \in E$.

2. Let $D_\rho \not\subseteq GD^E_D$. Then, there is a least k and a default rule $\delta \in D_k$ such that $\delta \notin GD^E_D$. By Definition 3.5.1, $W \cup \mathit{Conseq}(D_{k-1}) \vdash \mathit{Prereq}(D_k)$. By assumption, $D_{k-1} \subseteq GD^E_D$. Then, $W \cup \mathit{Conseq}(D_{k-1}) \subseteq E$ and the fact that E is deductively closed implies $\mathit{Prereq}(\delta) \in E$. According to Definition 3.2.1, this implies $\neg\mathit{Justif}(\delta) \notin E$.

Since E is a classical extension of (D, W), E is the smallest set satisfying the properties

1. $W \subseteq E$,
2. $E = Th(E)$,
3. For any $\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D$, if $\alpha \in E$ and $\beta_1, \ldots, \beta_n \notin E$ then $\gamma \in E$.

In both cases we have that E is also the smallest set satisfying the conditions 1. and 2. and, moreover the modified condition

3. For any $\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D \cup \{\zeta_\rho\}$, if $\alpha \in E$ and $\beta_1, \ldots, \beta_n \notin E$ then $\gamma \in E$.

This is because in the first case $Conseq(\zeta_\rho) \in E$, whereas in the second ζ_ρ is not applicable. Consequently, E is also a classical extension of $(D \cup \{\zeta_\rho\}, W)$.

if part Let E be a classical extension of $(D \cup \{\zeta_\rho\}, W)$. We regard the following two cases.

1. Let $\zeta_\rho \notin GD^E_{D \cup \{\zeta_\rho\}}$. Clearly, E is then also a classical extension of (D, W).
2. Let $\zeta_\rho \in GD^E_{D \cup \{\zeta_\rho\}}$. According to [Reiter, 1980], $E = \bigcup_{i=0}^{\infty} E_i$ such that $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D \cup \{\delta_\rho\}, \alpha \in E_i, \neg\beta_1, \ldots, \beta_n \notin E \right\}.$$

Clearly, we have $Conseq(\zeta_\rho) \subseteq E_1$ since ζ_ρ is prerequisite-free.

Analogously, E is a classical extension of (D, W) iff $E = \bigcup_{i=0}^{\infty} E'_i$ such that $E'_0 = W$ and for $i \geq 0$

$$E'_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D, \alpha \in E_i, \neg\beta_1, \ldots, \beta_n \notin E \right\}.$$

Since $\zeta_\rho \in GD^E_{D \cup \{\zeta_\rho\}}$, we have for each $\delta \in D_\rho$ that $\neg Justif(\delta) \notin E$. This and Definition 3.5.1 implies that there is a k such that $Conseq(D_\rho) \subseteq E'_k$. That is, $Conseq(\delta_\rho) \subseteq E'_k$. As a consequence, $\bigcup_{i=0}^{\infty} E'_i = \bigcup_{i=0}^{\infty} E_i$. Thus, $(E, C) = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$. ■

Appendix C

Proofs of Theorems in Chapter 4

This chapter presents the proofs of the theorems given in Chapter 4.

Coherence of the definition

Theorem 4.2.1 *The set of all pairs of sets of formulas* (S, T) *satisfying the conditions 1. to 3. of Definition 4.2.1 is closed under intersection.*

Proof 4.2.1 Clearly, it suffices to show that the intersection of two pairs of sets of sentences satisfying the conditions 1. to 3. satisfies these conditions, too.

Let T be an arbitrary set of formulas and let (S', T') and (S'', T'') be two pairs of sets of formulas satisfying the following three conditions (here written down for S' and T').

1. $W \subseteq S' \subseteq T'$,
2. $S' = Th(S')$ and $T' = Th(T')$,
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$.

By definition, $W \subseteq S' \subseteq T'$ and $W \subseteq S'' \subseteq T''$, whence $W \subseteq S' \cap S'' \subseteq T' \cap T''$.

By standard logic, the intersection of two deductively closed sets is also deductively closed. Hence, $S' \cap S'' = Th(S' \cap S'')$ and $T' \cap T'' = Th(T' \cap T'')$.

Consider a default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in S' \cap S''$ and $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$. Clearly, $\alpha \in S'$ and $\alpha \in S''$ and $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ implies $\gamma \in S'$ and $\beta \wedge \gamma \in T'$, and $\gamma \in S''$ and $\beta \wedge \gamma \in T''$, respectively. That is, $\gamma \in S' \cap S''$ and $\beta \wedge \gamma \in T' \cap T''$.

Therefore, we have shown that $(S' \cap S'', T' \cap T'')$ satisfies the above conditions as well. ■

Properties of constrained default logic

Theorem 4.3.1 *Let* (D, W) *be a default theory and let* E *and* C *be sets of formulas. Define*

$$E_0 = W \quad \text{and} \quad C_0 = W$$

and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp \right\} \\ C_{i+1} &= Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp \right\} \end{aligned}$$

(E, C) *is a constrained extension of* (D, W) *iff* $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Proof 4.3.1 First, observe that we have the following properties

1. $W \subseteq \bigcup_{i=0}^{\infty} E_i \subseteq \bigcup_{i=0}^{\infty} C_i$.
2. $\bigcup_{i=0}^{\infty} E_i = Th(\bigcup_{i=0}^{\infty} E_i)$ and $\bigcup_{i=0}^{\infty} C_i = Th(\bigcup_{i=0}^{\infty} C_i)$.
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in \bigcup_{i=0}^{\infty} E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in \bigcup_{i=0}^{\infty} E_i$ and $\beta \wedge \gamma \in \bigcup_{i=0}^{\infty} C_i$.

By the minimality of $\Upsilon(C)$, we have[1]

$$\Upsilon_1(C) \subseteq \bigcup_{i=0}^{\infty} E_i \qquad \text{and} \qquad \Upsilon_2(C) \subseteq \bigcup_{i=0}^{\infty} C_i, \tag{C.1}$$

regardless of whether (E,C) is a constrained extension or not.

only-if part Assume (E,C) is a constrained extension.

"$\supseteq$" We have to show that $E_i \subseteq E$ and $C_i \subseteq C$ for $i \geq 0$

Base Clearly, $E_0 = W \subseteq E$ and $C_0 = W \subseteq C$.

Step Assume $E_i \subseteq E$ and $C_i \subseteq C$ and consider $\eta \in E_{i+1} \cup C_{i+1}$.

1. $\eta \in Th(E_i)$. Since $E_i \subseteq E$ and $E = Th(E)$ we obtain $\eta \in Th(E_i) \subseteq Th(E) = E$.
2. $\eta \in Th(C_i)$. Analogous to 1.
3. $\eta \in \{\beta, \gamma\}$ for some $\frac{\alpha:\beta}{\gamma}$ such that $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
   Since $E_i \subseteq E$ we have $\alpha \in E$. Together, $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ imply $\gamma \in E$ and $\beta \wedge \gamma \subseteq C$, and both cases for η are covered.

Thus, we have $E_{i+1} \subseteq E$ and $C_{i+1} \subseteq C$, respectively.

"$\subseteq$" From (C.1) and the fact that $(E,C) = \Upsilon(C)$ we obtain $E \subseteq \bigcup_{i=0}^{\infty} E_i$ and $C \subseteq \bigcup_{i=0}^{\infty} C_i$, respectively.

We obtain $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

if part Assume $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

"$\supseteq$" Now, we have to show that $E_i \subseteq \Upsilon_1(C)$ and $C_i \subseteq \Upsilon_2(C)$ for $i \geq 0$.

Base Clearly, $E_0 = W \subseteq \Upsilon_1(C)$ and $C_0 = W \subseteq \Upsilon_2(C)$.

Step Assume $E_i \subseteq \Upsilon_1(C)$ and $C_i \subseteq \Upsilon_2(C)$ and consider $\eta \in E_{i+1} \cup C_{i+1}$.

1. $\eta \in Th(E_i)$. Since $E_i \subseteq \Upsilon_1(C)$ and $\Upsilon_1(C) = Th(\Upsilon_1(C))$ we obtain $\eta \in Th(E_i) \subseteq Th(\Upsilon_1(C)) = \Upsilon_1(C)$.
2. $\eta \in Th(C_i)$. Analogous to 1.
3. $\eta \in \{\beta, \gamma\}$ for some $\frac{\alpha:\beta}{\gamma}$ such that $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
   Since $E_i \subseteq \Upsilon_1(C)$ we have $\alpha \in \Upsilon_1(C)$. Together, $\alpha \in \Upsilon_1(C)$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ imply $\gamma \in \Upsilon_1(C)$ and $\beta \wedge \gamma \in \Upsilon_2(C)$ and both cases for η are covered.

Accordingly, we have $E_{i+1} \subseteq \Upsilon_1(C)$ and $C_{i+1} \subseteq \Upsilon_2(C)$, respectively.

"$\subseteq$" Follows from (C.1).

[1] We refer to the components of Υ as Υ_1 and Υ_2, respectively.

We have shown that $(\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i) = \Upsilon(C)$. Together with the assumption $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$, we obtain that (E,C) is a constrained extension of (D,W). ■

Theorem 4.3.4 (Pairwise maximality) *Let (D,W) be a default theory and let (E,C) and (E',C') be constrained extensions of (D,W). Then $E \subseteq E'$ and $C \subseteq C'$ implies $E = E'$ and $C = C'$.*

Proof 4.3.4 The case where W is unsatisfiable is trivial.

According to Theorem 4.3.1, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \ \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \ \middle| \ \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

Also $(E',C') = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$ where E'_i and C'_i are defined analogously.

We inductively prove $E'_i \subseteq E_i$ and $C'_i \subseteq C_i$ for all $i \geq 0$, whence $E' \subseteq E$ and $C' \subseteq C$, whence $E' = E$ and $C' = C$.

Base By definition, $E'_0 \subseteq E_0$ and $C'_0 \subseteq C_0$.

Step The induction hypothesis is: $E'_i \subseteq E_i$ and $C'_i \subseteq C_i$.

Consider $\eta \in E'_{i+1} \cup C'_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E'_i)$. By the induction hypothesis we have $\eta \in E_{i+1}$.
2. $\eta \in Th(C'_i)$. By the induction hypothesis we have $\eta \in C_{i+1}$.
3. $\eta \in \left\{ \beta \wedge \gamma \ \middle| \ \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E'_i, C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. That is, η is either γ or β such that there is a default rule $\frac{\alpha : \beta}{\gamma} \in D$ with $\alpha \in E'_i$ and $C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. By the induction hypothesis, $\alpha \in E_i$. By assumption, $C \subseteq C'$. Since C' is consistent, we have by monotonicity that $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ and we obtain together with $\alpha \in E_i$ that $\gamma \in E_{i+1}$ and $\beta \wedge \gamma \in C_{i+1}$ and both cases for η are covered.

From the three cases, we obtain $E'_{i+1} \subseteq E_{i+1}$ and $C'_{i+1} \subseteq C_{i+1}$.

■

Theorem 4.3.5 *Let (E,C) be a constrained extension of a default theory (D,W). We have*

$$\begin{aligned} E &= Th\Big(W \cup Conseq\Big(GD_D^{(E,C)}\Big)\Big), \\ C &= Th\Big(W \cup Conseq\Big(GD_D^{(E,C)}\Big) \cup Justif\Big(GD_D^{(E,C)}\Big)\Big). \end{aligned}$$

Proof 4.3.5 Let (E,C) be a constrained extension of a default theory (D,W). For the sake of readability, let us abbreviate $Conseq\Big(GD_D^{(E,C)}\Big)$ by E_{GD} and $Conseq\Big(GD_D^{(E,C)}\Big) \cup Justif\Big(GD_D^{(E,C)}\Big)$ by C_{GD}.

"$\subseteq$" By definition, $W \subseteq E \subseteq C$. Also, if $\gamma \in E_{GD}$ then there is a default rule $\frac{\alpha : \beta}{\gamma} \in D$ such that $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. Then by Definition 4.2.1, $\gamma \in E$ and $\beta \wedge \gamma \in C$. Consequently, $W \cup E_{GD} \subseteq E$ and $W \cup C_{GD} \subseteq C$. By monotonicity, $Th(W \cup E_{GD}) \subseteq Th(E)$ and $Th(W \cup C_{GD}) \subseteq Th(C)$. That is, since E and C are deductively closed, $Th(W \cup E_{GD}) \subseteq E$ and $Th(W \cup C_{GD}) \subseteq C$.

"$\supseteq$" First, $W \subseteq Th(W \cup E_{GD}) \subseteq Th(W \cup C_{GD})$.

Second, by idempotence, $Th(W \cup E_{GD}) = Th(Th(W \cup E_{GD}))$ and $Th(W \cup C_{GD}) = Th(Th(W \cup C_{GD}))$.

Third, consider $\frac{\alpha : \beta}{\gamma} \in D$. If $\alpha \in Th(W \cup E_{GD})$ then $\alpha \in E$ according to what we have just proved. If additionally, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ then $\frac{\alpha : \beta}{\gamma} \in GD_D^{(E,C)}$, whence $\gamma \in E_{GD}$ and $\beta \wedge \gamma \in C_{GD}$. Clearly, $\gamma \in Th(W \cup E_{GD})$ and $\beta \wedge \gamma \in Th(W \cup C_{GD})$.

Accordingly, by the minimality of Υ, we have $\Upsilon_1(C) \subseteq Th(W \cup E_{GD})$ and $\Upsilon_2(C) \subseteq Th(W \cup C_{GD})$. Since (E,C) is a constrained extension, we obtain $E \subseteq Th(W \cup E_{GD})$ and $C \subseteq Th(W \cup C_{GD})$.

■

Theorem 4.3.6 (Groundedness) *Let (E,C) be a constrained extension of (D,W). Then, there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,C)}$ such that for $i \in I$*

$$W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i).$$

Proof 4.3.6 Let (E,C) be a constrained extension of (D,W) and $GD_D^{(E,C)}$ the corresponding set of generating default rules. Since (E,C) is a constrained extension of (D,W), we have $E = \bigcup_{i=0}^{\infty} E_i$ such that $E_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp \right\}.$$

From this sequence, we can define a sequence of sets of default rules $\langle \Delta_i \rangle$ as follows. $\Delta_0 = \emptyset$, and for $i \geq 0$

$$\Delta_{i+1} = \left\{ \tfrac{\alpha : \beta}{\gamma} \in D \;\middle|\; \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp \right\}.$$

Clearly, $GD_D^{(E,C)} = \bigcup_{i=0}^{\infty} \Delta_i$. By compactness, there exists for each $\delta \in \Delta_i$ a finite subset $\{\delta_1, \ldots, \delta_n\} \subset \Delta_{i-1}$ such that $W \cup Conseq(\{\delta_1, \ldots, \delta_n\}) \vdash Prereq(\delta)$. Then, a standard method for diagonalization yields an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,C)}$ such that for $i \in I$ $W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i)$. ■

Theorem 4.3.7 (Semi-monotonicity) *Let (D,W) be a default theory and D' a set of default rules such that $D \subseteq D'$. If (E,C) is a constrained extension of (D,W), then there is a constrained extension (E',C') of (D',W) such that $E \subseteq E'$ and $C \subseteq C'$.*

Proof 4.3.7 The inconsistent case is easily dealt with, so that we prove below the theorem for E and C being consistent.

We define a sequence $\langle \Delta_\iota \rangle$ of subsets of D' as follows. For the sake of simplicity, let us abbreviate $Th(W \cup Conseq(\Delta_\iota))$ by E^κ and $Th(W \cup Conseq(\Delta_\iota) \cup Justif(\Delta_\iota))$ by C^κ.

$$\Delta_\iota = \begin{cases} GD_D^{(E,C)} & \text{if } \iota = 0 \\ \bigcup_{\kappa < \iota} \Delta_\kappa & \text{if } \iota \text{ is a limit ordinal} \\ \Delta_\kappa \cup \{\delta\} & \text{if } \iota = \kappa + 1 \text{ is a successor ordinal in the case there exists} \\ & \qquad \delta = \frac{\alpha : \beta}{\gamma} \in D' \setminus \Delta_\kappa \text{ such that } \alpha \in E^\kappa \text{ and } C^\kappa \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp \end{cases}$$

Since the sequence Δ is strictly increasing, the process eventually stops. Let χ be the greatest ordinal such that Δ_χ is defined. Define

$$E' = Th(E \cup Conseq(\Delta_\chi)) \text{ and } C' = Th(C \cup Conseq(\Delta_\chi) \cup Justif(\Delta_\chi)).$$

By definition, $E \subseteq E'$ and $C \subseteq C'$. Thus, it remains to be shown that (E', C') is a constrained extension of (D', W). First, observe the following properties.

1. By definition of E' and C', and since $W \subseteq E \subseteq C$, we have also $W \subseteq E' \subseteq C'$.
2. By definition, $E' = Th(E')$ and $C' = Th(C')$.
3. If $\frac{\alpha:\beta}{\gamma} \in D'$, and $\alpha \in E'$ and $C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$, we obtain $\gamma \in E'$ and $\beta \wedge \gamma \in C'$ because $\frac{\alpha:\beta}{\gamma} \in \Delta_\chi$ (otherwise $\Delta_{\chi+1}$ could be defined).

Then, by the minimality of $\Upsilon(C')$, we have[2] $\Upsilon_1(C') \subseteq E'$ and $\Upsilon_2(C') \subseteq C'$.

Now, assume $\Upsilon_1(C') \subset E'$ and $\Upsilon_2(C') \subset C'$, ie. none of the former inclusions are proper. Then (provided that $E \subseteq \Upsilon_1(C')$ and $C \subseteq \Upsilon_2(C')$), there exists a least ordinal κ such that $\Delta_\kappa = \Delta_{\kappa-1} \cup \{\delta\}$ where $\delta = \frac{\alpha:\beta}{\gamma} \in D'$, such that $\alpha \in E'$ and $C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$, and $\gamma \in E'$ and $\beta \wedge \gamma \in C'$ but either $\gamma \notin \Upsilon_1(C')$ or $\beta \wedge \gamma \notin \Upsilon_2(C')$. By definition of Δ, we have $\alpha \in E^{\kappa-1}$. Since κ is the least such ordinal, it follows that $\alpha \in \Upsilon_1(C')$. But by definition, $\alpha \in \Upsilon_1(C')$ and $C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$, implies $\gamma \in \Upsilon_1(C')$ and $\beta \wedge \gamma \in \Upsilon_2(C')$. Contradiction.

To show, that $E \subseteq \Upsilon_1(C')$ and $C \subseteq \Upsilon_2(C')$ recall that (E,C) is a constrained extension of (D, W). Thus, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \ \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \ \middle| \ \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

Proof by induction on i.

Base By definition.

Step The induction hypothesis is: $E_i \subseteq \Upsilon_1(C')$ and $C_i \subseteq \Upsilon_2(C')$.

Consider $\eta \in E_{i+1} \cup C_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis and the fact that $\Upsilon_1(C')$ is deductively closed, we have $\eta \in E_{i+1}$.
2. $\eta \in Th(C_i)$. By the induction hypothesis and the fact that $\Upsilon_2(C')$ is deductively closed, we have $\eta \in C_{i+1}$.
3. $\eta \in \left\{ \beta \wedge \gamma \ \middle| \ \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. That is, η is either γ or β such that there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ with $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. By the induction hypothesis, $\alpha \in \Upsilon_1(C')$. By definition, $\beta \wedge \gamma \in C \subseteq C'$. Since C' is consistent, we have $C' \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ and we obtain together with $\alpha \in \Upsilon_1(C')$ that $\gamma \in \Upsilon_1(C')$ and $\beta \wedge \gamma \in \Upsilon_2(C')$ and both cases for η are covered.

From the three cases, we obtain $E_{i+1} \subseteq \Upsilon_1(C')$ and $C_{i+1} \subseteq \Upsilon_2(C')$.

Hence, we have shown that $E_i \subseteq \Upsilon_1(C')$ and $C_i \subseteq \Upsilon_2(C')$ for $i \geq 0$. ■

Theorem 4.3.8 (Existence of extensions) *Every default theory has a constrained extension.*

[2]We refer to the components of Υ as Υ_1 and Υ_2, respectively.

Proof 4.3.8 Let (D, W) be a default theory. Then, there is a default theory $(\emptyset, W)$ which has a unique constrained extension $(Th(W), Th(W))$. From this and Theorem 4.3.7 the result follows immediately. ■

Theorem 4.3.9 (Weak orthogonality) *Let (D, W) be a default theory. If (E,C) and (E', C') are distinct constrained extensions*[3] *of (D, W), then $C \cup C'$ is inconsistent.*

Proof 4.3.9 The case where W is unsatisfiable is trivial.

According to Theorem 4.3.1, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

Also $(E', C') = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$ where E'_i and C'_i are defined analogously. Without loss of generality, we can assume that C and C' are distinct (cf. Corollary 4.3.2). Then, there exists a least k such that $C_{k+1} \neq C'_{k+1}$ in which case $C_k = C'_k$ (and $E_k = E'_k$). Then, there is a default rule $\frac{\alpha : \beta}{\gamma} \in D$ such that $\alpha \in E_k = E'_k$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ and $\beta \wedge \gamma \in C_{k+1}$ but $\beta \wedge \gamma \notin C'_{k+1}$. But $\alpha \in E'_k$ and $\beta \wedge \gamma \notin C'_{k+1}$ implies $C' \cup \{\beta\} \cup \{\gamma\} \vdash \bot$. Since $\beta \wedge \gamma \in C$ and C and C' are consistent, we have by monotonicity that $C \cup C' \vdash \bot$. That is, $C \cup C'$ is inconsistent. ■

Constrained versus classical default logic

Theorem 4.4.1 *Let (D, W) be a normal default theory and E a set of formulas. Then, E is a classical extension of (D, W) iff (E, E) is a constrained extension of (D, W).*

Proof 4.4.1 In order to prove the claim we reduce the characterization of constrained extensions given in Theorem 4.3.1 in the case of normal default theories. By definition, for any normal default rule $\frac{\alpha : \beta}{\gamma}$ we have $\beta \leftrightarrow \gamma$.

Now, according to Theorem 4.3.1 (E,C) is a constrained extension of the normal default theory (D, W) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ and $E_0 = W$ and $C_0 = W$ and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ C_{i+1} &= Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \end{aligned}$$

Clearly, since $\beta \leftrightarrow \gamma$ we have $Th(E_i) = Th(C_i)$. So, since $\bigcup_{i=0}^{\infty} E_i$ and $\bigcup_{i=0}^{\infty} C_i$ are deductively closed we also have $\bigcup_{i=0}^{\infty} E_i = \bigcup_{i=0}^{\infty} C_i$. Notice also, that due to the equivalence of β and γ the condition $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ reduces to $C \cup \{\beta\} \not\vdash \bot$, and furthermore, since C is deductively closed, we obtain $\neg\beta \notin C$.

Therefore, (E, E) is a constrained extension of a normal default theory (D, W) iff $(E, E) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} E_i)$ such that $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}$$

Obviously, this amounts to the same characterization of classical extensions given in Theorem 3.2.1 for classical extensions. ■

[3] According to Corollary 4.3.2, that is $C \neq C'$.

Theorem 4.4.2 *Let (D, W) be a default theory and let E be a classical extension of (D, W). If $E \cup C_E$ is consistent, then $(E, Th(E \cup C_E))$ is a constrained extension of (D, W).*

Proof 4.4.2
Let E be a classical extension of (D, W) and $C_E = \left\{\beta \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}$. Define $C = Th(E \cup C_E)$. We show that (E, C) is a constrained extension of (D, W).

First, observe the following properties.

1. By definition, $W \subseteq E \subseteq C$.
2. Also, by definition, $E = Th(E)$ and $C = Th(C)$.
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \wedge \gamma \in C$. Because, by monotonicity and the fact that E is deductively closed, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ implies $\neg\beta \notin E$ (since $E \subseteq C$).

Then, by the minimality of $\Upsilon(C)$, we have[4] $\Upsilon_1(C) \subseteq E$ and $\Upsilon_2(C) \subseteq C$. That is, $\Upsilon(C) \subseteq^2 (E,C)$.[5] Since E is a classical extension of (D, W) we have according to Theorem 3.2.1 that $E = \bigcup_{i=0}^{\infty} E_i$ such that $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Define $C_0 = \emptyset$, and for $i \geq 0$

$$C_{i+1} = \qquad \left\{ \beta \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Clearly, $C_E = \bigcup_{i=0}^{\infty} C_i$. We will show that $\bigcup_{i=0}^{\infty} E_i \subseteq \Upsilon_1(C)$ and $\bigcup_{i=0}^{\infty} C_i \subseteq \Upsilon_2(C)$, in order to show that $E \subseteq \Upsilon_1(C)$ and $C \subseteq \Upsilon_2(C)$.

Therefore, we show by induction $E_i \subseteq \Upsilon_1(C)$ and $C_i \subseteq \Upsilon_2(C)$ for $i \geq 0$.

Base Clearly, $E_0 = W \subseteq \Upsilon_1(C)$ and $C_0 = \emptyset \subseteq \Upsilon_2(C)$.

Step Assume $E_i \subseteq \Upsilon_1(C)$ and $C_i \subseteq \Upsilon_2(C)$ and consider $\eta \in E_{i+1} \cup C_{i+1}$.

1. If $\eta \in Th(E_i)$ then, by the induction hypothesis and the fact that $\Upsilon_1(C)$ is deductively closed, we obtain $\eta \in \Upsilon_1(C)$.
2. If $\eta \in C_i$ then, by the induction hypothesis, also $\eta \in \Upsilon_2(C)$.
3. Otherwise, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $\neg\beta \notin E$.
   By the induction hypothesis, $\alpha \in \Upsilon_1(C)$. By assumption, C is consistent. Since E is a classical extension of (D, W), $\alpha \in E_i$ and $\neg\beta \notin E$ implies $\gamma \in E$. Also, by definition of C_E, we have $\beta \in C_E$. Therefore, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ (since $C = Th(E \cup C_E)$). From $\alpha \in \Upsilon_1(C)$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ we conclude, by Definition 4.2.1, that $\gamma \in \Upsilon_1(C)$ and $\beta \wedge \gamma \in \Upsilon_2(C)$. Since $\Upsilon_2(C)$ is deductively closed the last membership implies $\beta \in \Upsilon_2(C)$. Clearly, both cases for v are covered.

Accordingly, $E_{i+1} \subseteq \Upsilon_1(C)$ and $C_{i+1} \subseteq \Upsilon_2(C)$.

With this, we have shown that $\bigcup_{i=0}^{\infty} E_i \subseteq \Upsilon_1(C)$ and $\bigcup_{i=0}^{\infty} C_i \subseteq \Upsilon_2(C)$. Since $\bigcup_{i=0}^{\infty} E_i = E$ and $\bigcup_{i=0}^{\infty} C_i = C_E$, that is $E \subseteq \Upsilon_1(C)$ and $C_E \subseteq \Upsilon_2(C)$. Since $\Upsilon_1(C) \subseteq \Upsilon_2(C)$ we have $E \cup C_E \subseteq \Upsilon_2(C)$. So, since $\Upsilon_2(C)$ is deductively closed, $C \subseteq \Upsilon_2(C)$. Hence, $(E,C) \subseteq^2 \Upsilon(C)$. ∎

[4] We refer to the components of Υ as Υ_1 and Υ_2, respectively.

[5] We sometimes abbreviate $\Upsilon_1(C) \subseteq E$ and $\Upsilon_2(C) \subseteq C$ by $\Upsilon_2(C) \subseteq^2 (E, C)$.

Theorem 4.4.3 *Let (D, W) be a default theory and let E and C be sets of formulas. If (E,C) is a constrained extension of (D, W) and E is a classical extension of (D, W), then $C \subseteq Th(E \cup C_E)$.*

Proof 4.4.3 Let (E,C) be a constrained extension of (D, W) and let E be a classical extension of (D, W) and $C_E = \left\{\beta \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}$. Then, according to Theorem 4.3.1 $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

We will show $\bigcup_{i=0}^{\infty} C_i \subseteq Th(E \cup C_E)$, in order to show $C \subseteq Th(E \cup C_E)$.
Therefore, we show by induction $C_i \subseteq Th(E \cup C_E)$ for $i \geq 0$.

Base Clearly, $C_0 = W \subseteq E \subseteq Th(E \cup C_E)$.

Step Assume $C_i \subseteq Th(E \cup C_E)$ and consider $\eta \in C_{i+1}$.

1. If $\eta \in Th(C_i)$ then, by the induction hypothesis, $\eta \in Th(E \cup C_E)$.
2. Otherwise, $\eta \in \{\beta, \gamma\}$ for some default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
   Clearly, $\gamma \in E$. According to the definition of C_E we have $\beta \in C_E$ only if $\alpha \in E$ and $\neg\beta \notin E$. Clearly, $\alpha \in E$ since $\alpha \in E_i$ and $E_i \subseteq E$. Since $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ by monotonicity, $E \cup \{\beta\} \not\vdash \bot$. That is, since E is deductively closed $\neg\beta \notin E$. Thus, both cases for η are covered.

From the two cases, we obtain $C_{i+1} \subseteq Th(E \cup C_E)$.

Therefore, we have shown that $\bigcup_{i=0}^{\infty} C_i \subseteq Th(E \cup C_E)$. That is, $C \subseteq Th(E \cup C_E)$. ■

Correctness and completeness for constrained default logic

Theorem 4.5.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $(\Pi, \breve{\Pi})$ be a pair of classes of first-order interpretations and E, C deductively closed sets of formulas such that $\Pi = \{\pi \mid \pi \models E\}$ and $\breve{\Pi} = \{\pi \mid \pi \models C\}$. Then, (E,C) is a constrained extension of (D, W) iff $(\Pi, \breve{\Pi})$ is a $\succcurlyeq_D$-maximal element above (Π_W, Π_W).*

Proof 4.5.1 First, we need the following definition.

Definition C.1.2 *Let (D, W) be a default theory. Given a possibly infinite sequence of default rules $\Delta = \langle \delta_0, \delta_1, \delta_2, \ldots \rangle$ in D, also denoted $\langle \delta_i \rangle_{i \in I}$ where I is the index set for Δ, we define a sequence of focused models structures $\langle (\Pi_i, \breve{\Pi}_i) \rangle_{i \in I}$ as follows:*

$$\begin{array}{lcl} (\Pi_0, \breve{\Pi}_0) & = & (\Pi_W, \Pi_W) \\ (\Pi_{i+1}, \breve{\Pi}_{i+1}) & = & (\{\pi \in \Pi_i \mid \pi \models \gamma_i\}, \{\pi \in \breve{\Pi}_i \mid \pi \models \beta_i \wedge \gamma_i\}), \quad \textit{where } \delta_i = \frac{\alpha_i:\beta_i}{\gamma_i}. \end{array}$$

The unsatisfiable case is easily dealt with, so that we prove below the theorem for E and C being satisfiable.

Proof 4.5.1 (Correctness) Assume (E,C) is a consistent constrained extension of (D,W). Then according to Theorem 4.3.6, there exists an enumeration $\langle\delta_i\rangle_{i\in I}$ of the set of generating default rules $GD_D^{(E,C)}$ such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0,\ldots,\delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{C.2}$$

Let $\langle(\Pi_i,\breve{\Pi}_i)\rangle_{i\in I}$ be a sequence of focused models structures obtained from the enumeration $\langle\delta_i\rangle_{i\in I}$ according to Definition C.1.2. We will show that $(\Pi,\breve{\Pi})$ coincides with $\bigcap_{i\in I}(\Pi_i,\breve{\Pi}_i)$ and is $\succcurlyeq_D$-maximal above (Π_W,Π_W).

Since (E,C) is a constrained extension, we have according to Theorem 4.3.5 that

$$\begin{aligned} E &= Th\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big),\\ C &= Th\Big(W \cup \mathit{Justif}\Big(GD_D^{(E,C)}\Big) \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big). \end{aligned}$$

Then, since $(\Pi,\breve{\Pi}) = (\mathrm{MOD}(E),\mathrm{MOD}(C))$ we have obviously that $(\Pi,\breve{\Pi}) = \bigcap_{i\in I}(\Pi_i,\breve{\Pi}_i)$.

First, let us show that $(\Pi_{i+1},\breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i,\breve{\Pi}_i)$ for $i \in I$.

- Since $\Pi_i \subseteq \Pi_W$ and $\Pi_W \models W$, then by definition of Π_i we have $\Pi_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\Pi_{i+1} \subseteq \Pi_i$ for $i \in I$ implies that $\Pi_i \models W \cup \mathit{Conseq}(\{\delta_0,\ldots,\delta_{i-1}\})$. By (C.2), it follows that $\Pi_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $(\Pi_{i+1},\breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i,\breve{\Pi}_i)$ fails for some $k \in I$. By definition of $\langle(\Pi_i,\breve{\Pi}_i)\rangle_{i\in I}$ and the fact that we have just proven that $\Pi_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\breve{\Pi}_k \models \neg(\gamma_k \wedge \beta_k)$ for $\delta_k = \frac{\alpha_k:\beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0,\ldots,\delta_{k-1}\}) \cup \mathit{Justif}(\{\delta_0,\ldots,\delta_{k-1}\})$ by C^k. By definition, $\breve{\Pi}_k = \mathrm{MOD}(C^k)$. Then, $C^k \models \neg(\gamma_k \wedge \beta_k)$. That is, $C^k \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$. By monotonicity, $C \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$, contradictory to the fact that $\delta_k \in GD_D^{(E,C)}$.

Therefore, $(\Pi_{i+1},\breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i,\breve{\Pi}_i)$ for $i \in I$. As a consequence, $\bigcap_{i\in I}(\Pi_i,\breve{\Pi}_i) \succcurlyeq_{GD_D^{(E,C)}} (\Pi_W,\Pi_W)$. That is, $(\Pi,\breve{\Pi}) \succcurlyeq_D (\Pi_W,\Pi_W)$.

Second, assume $(\Pi,\breve{\Pi})$ is not $\succcurlyeq_D$-maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^{(E,C)}$ such that $\Pi \models \alpha$ and $\breve{\Pi} \not\models \neg(\gamma\wedge\beta)$. [6] First, since $\Pi \models E$ we have $E \models \alpha$. Second, since $\breve{\Pi} = \mathrm{MOD}(C)$, we also have $C \not\models \neg(\gamma\wedge\beta)$. Of course, $E \models \alpha$ and $C \not\models \neg(\gamma\wedge\beta)$ implies $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$, a contradiction. ∎

Proof 4.5.1 (Completeness) Let $(\Pi,\breve{\Pi})$ be a $\succcurlyeq_D$-maximal element above (Π_W,Π_W) such that $\Pi = \{\pi \mid \pi \models E\}$ and $\breve{\Pi} = \{\pi \mid \pi \models C\}$.

According to Theorem 4.3.1, (E,C) is a constrained extension iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta\wedge\gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

We will show that $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$. Therefore, we consider the following two cases.

[6] For readablity, we abbreviate $\exists\pi \in \breve{\Pi}.\pi \models \beta\wedge\gamma$ by $\breve{\Pi} \not\models \beta\wedge\gamma$.

1. $\bigcup_{i=0}^{\infty} E_i \subseteq E$, $\bigcup_{i=0}^{\infty} C_i \subseteq C$.

   We show by induction that $E_i \subseteq E$ and $C_i \subseteq C$ for $i \geq 0$.

   Base By definition, $\Pi_W \models E_0$. Since $\Pi \subseteq \Pi_W$, we have $E \models E_0$. That is, $E_0 \subseteq E$. Analogously, we obtain $C_0 \subseteq C$.

   Step Let $E_i \subseteq E$ and $C_i \subseteq C$. Consider $\eta \in E_{i+1} \cup C_{i+1}$.

   (a) If $\eta \in Th(E_i)$ then, by the induction hypothesis and the fact that E is deductively closed, we obtain $\eta \in E$.

   (b) Similarly, if $\eta \in Th(C_i)$ we obtain $\eta \in C$.

   (c) Otherwise, $\eta \in \{\beta, \gamma\}$ such that there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ where $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
   By the induction hypothesis $\alpha \in E$. That is, $\Pi \models \alpha$. Also, by definition of C, we have $\breve{\Pi} \not\models \neg(\gamma \wedge \beta)$. Since $(\Pi, \breve{\Pi})$ is $\succcurlyeq_D$-maximal we also have $\Pi \models \gamma$ and $\breve{\Pi} \models \beta \wedge \gamma$ That is, $\gamma \in E$ and $\beta \wedge \gamma \in C$ and both cases for η are covered.

   From the three cases, we obtain $E_{i+1} \subseteq E$ and $C_{i+1} \subseteq C$.

2. $E \subseteq \bigcup_{i=0}^{\infty} E_i$, $C \subseteq \bigcup_{i=0}^{\infty} C_i$.

   Since $(\Pi, \breve{\Pi})$ is a $\succcurlyeq_D$-maximal element above (Π_W, Π_W) for (D, W), we have that $(\Pi, \breve{\Pi}) = (\bigcap_{i=0}^{\infty} \Pi_i, \bigcap_{i=0}^{\infty} \breve{\Pi}_i)$ where $\langle(\Pi_i, \breve{\Pi}_i)\rangle_{i \in I}$ is a sequence of focused models structures defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition C.1.2 such that $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ for $i \in I$. Then, we define $E_i^{\Pi} = \{v \mid \Pi_i \models v\}$ and $C_i^{\breve{\Pi}} = \{v \mid \breve{\Pi}_i \models v\}$ to be the sets of Π_i-valid and $\breve{\Pi}_i$-valid sentences, respectively. Clearly, $E = \bigcup_{i=0}^{\infty} E_i^{\Pi}$ and $C = \bigcup_{i=0}^{\infty} C_i^{\breve{\Pi}}$.

   Hence, we show inductively that $E_i^{\Pi} \subseteq \bigcup_{i=0}^{\infty} E_i$ and $C_i^{\breve{\Pi}} \subseteq \bigcup_{i=0}^{\infty} C_i$ for $i \geq 0$.

   Base Since $E_0^{\Pi} = C_0^{\breve{\Pi}} = \breve{\Pi}_W$ and $E_0 = C_0 = W$, the result is obvious.

   Step According to the induction hypothesis, $E_i^{\Pi} \subseteq \bigcup_{i=0}^{\infty} E_i$ and $C_i^{\breve{\Pi}} \subseteq \bigcup_{i=0}^{\infty} C_i$. Because $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ we have $\Pi_i \models \alpha_i$ and $\breve{\Pi}_i \not\models \neg(\beta_i \wedge \gamma_i)$, and $(\Pi_{i+1}, \breve{\Pi}_{i+1}) = (\{\pi \in \Pi_i \mid \pi \models \gamma_i\}, \{\pi \in \breve{\Pi}_i \mid \pi \models \beta_i \wedge \gamma_i\})$, where $\delta_i = \frac{\alpha_i:\beta_i}{\gamma_i}$.
   By the induction hypothesis and the fact that $\Pi_i \models \alpha_i$ we obtain $\alpha_i \in \bigcup_{i=0}^{\infty} E_i$. By compactness and monotonicity, there exists a k such that $\alpha_i \in E_k$. By definition, $\breve{\Pi}_{i+1} \models \beta_i \wedge \gamma_i$. Therefore, $\breve{\Pi} \models \beta_i \wedge \gamma_i$ since $\breve{\Pi} = \bigcap_{i=0}^{\infty} \breve{\Pi}_i$. Thus, $\gamma_i \wedge \beta_i \in C$. Since C is satisfiable, $C \cup \{\beta_i\} \cup \{\gamma_i\} \not\vdash \bot$. Then, $E_k \models \alpha_i$ and $C \cup \{\beta_i\} \cup \{\gamma_i\} \not\vdash \bot$, implies $\gamma_i \in E_{k+1}$ and $\gamma_i \wedge \beta_i \in C_{k+1}$. Hence, $\gamma_i \in \bigcup_{i=0}^{\infty} E_i$ and $\gamma_i \wedge \beta_i \in \bigcup_{i=0}^{\infty} C_i$.
   By the definition of Π_{i+1} and $\breve{\Pi}_{i+1}$, (or E_{i+1}^{Π} and $C_{i+1}^{\breve{\Pi}}$, respectively) and the fact that $\bigcup_{i=0}^{\infty} E_i$ and $\bigcup_{i=0}^{\infty} C_i$ are deductively closed, we obtain $E_{i+1}^{\Pi} \subseteq \bigcup_{i=0}^{\infty} E_i$ and $C_{i+1}^{\breve{\Pi}} \subseteq \bigcup_{i=0}^{\infty} C_i$.

■

■

Cumulativity for prerequisite-free default theories

Theorem 4.6.1 *Let (D, W) be a prerequisite-free default theory and let E, C be sets of formulas. Then, (E,C) is a constrained extension of (D, W) iff*

$E = Th(W \cup Conseq(D'))$

$$C = Th(W \cup Conseq(D') \cup Justif(D'))$$

for a maximal set of default rules $D' \subseteq D$ *such that* $W \cup Conseq(D') \cup Justif(D') \not\vdash \bot$.

Proof 4.6.1

only-if part Let (E,C) be a constrained extension of (D,W) containing α. Clearly, $GD_D^{(E,C)}$ is a maximal set of default rules that meets all requirements.

if part Let

$$E = Th(W \cup Conseq(D')) \tag{C.3}$$
$$C = Th(W \cup Conseq(D') \cup Justif(D')) \tag{C.4}$$

for a maximal set of default rules $D' \subseteq D$ such that $W \cup Conseq(D') \cup Justif(D') \not\vdash \bot$. According to Definition 4.3.1, (E,C) is a constrained extension of (D,W) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ where $E_0 = W$ and $C_0 = W$, and

$$E_1 = Th(W) \cup \left\{ \gamma \;\middle|\; \tfrac{:\beta}{\gamma} \in D, Th(W \cup Conseq(D') \cup Justif(D')) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_1 = Th(W) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{:\beta}{\gamma} \in D, Th(W \cup Conseq(D') \cup Justif(D')) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

and $E_{i+1} = Th(E_1)$ and $C_{i+1} = Th(C_1)$ for $i \geq 1$. As a consequence,

$$E = Th\Big(W \cup \left\{ \gamma \;\middle|\; \tfrac{:\beta}{\gamma} \in D, Th(W \cup Conseq(D') \cup Justif(D')) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}\Big)$$

$$C = Th\Big(W \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{:\beta}{\gamma} \in D, Th(W \cup Conseq(D') \cup Justif(D')) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}\Big).$$

By the maximality of D' and the fact that E and C are deductively closed, this amounts to the characterization of E and C given in (C.3) and (C.4), respectively. ■

Theorem 4.6.2 *Let* (D,W) *be a prerequisite-free default theory and let* $\alpha \in E'$ *for all constrained extension* (E',C') *of* (D,W). *Then,*

(E,C) *is a constrained extension of* (D,W) *iff* (E,C) *is a constrained extension of* $(D, W \cup \{\alpha\})$.

Proof 4.6.2

only-if part Let (E,C) be a constrained extension of (D,W) containing α. We have to show that (E,C) is a constrained extension of $(D, W \cup \{\alpha\})$.

This follows immediately from Theorem 4.6.3.

if part Let (E,C) be a constrained extension of $(D, W \cup \{\alpha\})$. Then, according to Theorem 4.6.1,

$$E = Th(W \cup \{\alpha\} \cup Conseq(D'))$$

$$C = Th(W \cup \{\alpha\} \cup Conseq(D') \cup Justif(D'))$$

for a maximal set of default rules D' such that $W \cup \{\alpha\} \cup Conseq(D') \cup Justif(D') \not\vdash \bot$.

It remains to be shown that (E,C) is a constrained extension of (D, W). By monotonicity, $W \cup Conseq(D') \cup Justif(D') \not\vdash \bot$. Consequently, there is a maximal set of default rules $D'' \supseteq D'$ such that $W \cup Conseq(D'') \cup Justif(D'') \not\vdash \bot$. Then, there is a constrained extension (E', C') of (D, W), where

$$E' = Th(W \cup Conseq(D''))$$

$$C' = Th(W \cup Conseq(D'') \cup Justif(D'')).$$

By definition, $\alpha \in E'$. As a consequence, (E', C') is also a constrained extension of $(D, W \cup \{\alpha\})$, by Theorem 4.6.3.

Since $D'' \supseteq D'$, we have $E \subseteq E'$ and $C \subseteq C'$. By Theorem 4.3.4, this implies $(E, C) = (E', C')$. Consequently, (E,C) is a constrained extension of (D, W). ■

Theorem 4.6.3 *Let (E,C) be a constrained extension of a default theory (D, W). If $F \subseteq E$ then (E,C) is also a constrained extension of the default theory $(D, W \cup F)$.*

Proof 4.6.3 Let (E,C) be a constrained extension of (D, W) and let $F \subseteq E$. According to Definition 4.2.1 (E,C) is the pair of smallest sets of formulas such that

1. $W \subseteq E \subseteq C$
2. $E = Th(E)$ and $C = Th(C)$
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \wedge \gamma \in C$.

Since $F \subseteq E$, E and C are also the smallest sets of formulas satisfying the stronger condition

1′. $W \cup F \subseteq E \subseteq C$,

2. and 3. Therefore, (E,C) is also a constrained extension of the default theory $(D, W \cup F)$. ■

Pre-constrained versus constrained default logic

Theorem 4.8.1 *Let (D, W, C_B) be a pre-constrained default theory and let*

$$D' = \left\{ \frac{\alpha:\beta\wedge\hat{C}_B}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D \right\} \cup \left\{ \frac{:\hat{C}_B}{\top} \right\},$$

where $\hat{C}_B$ is the conjunction of all formulas contained in the finite set of pre-constraints C_B. Let E and C be sets of formulas. Then, (E,C) is a pre-constrained extension of (D, W, C_B) iff (E,C) is a constrained extension of (D', W).

Proof 4.8.1 The inconsistent case is easily dealt with, so that we prove below the theorem for E and C being consistent.

According to Definition 4.8.1, we have that (E,C) is a pre-constrained extension of (D,W,C_B) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ and $E_0 = W$ and $C_0 = W \cup C_B$ and for $i \geq 0$

$$\begin{aligned} E_{i+1} &= Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \, \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ C_{i+1} &= Th(C_i) \cup \left\{ \beta \wedge \gamma \, \middle| \, \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \end{aligned}$$

Also, we have according to Theorem 4.3.1 that (E,C) is a pre-constrained extension of (D', W) where

$$D' = \left\{ \tfrac{\alpha:\beta \wedge \hat{C}_B}{\gamma} \, \middle| \, \tfrac{\alpha:\beta}{\gamma} \in D \right\} \cup \left\{ \tfrac{:\hat{C}_B}{\top} \right\}$$

(where $\hat{C}_B$ is the conjunction of all formulas contained in the set of pre-constraints C_B) iff $(E,C) = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$ and $E'_0 = W$ and $C'_0 = W$ and for $i \geq 0$

$$\begin{aligned} E'_{i+1} &= Th(E'_i) \cup \left\{ \qquad \gamma \qquad \middle| \, \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E'_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \\ C'_{i+1} &= Th(C'_i) \cup \left\{ (\beta \wedge \hat{C}_B) \wedge \gamma \, \middle| \, \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E'_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\} \cup \left\{ \hat{C}_B \right\} \end{aligned}$$

Notice, that the justification of the default rule $\frac{:\hat{C}_B}{\top}$ is added to C'_{i+1} for $i \geq 0$. This is because $W \cup C_B$ being inconsistent implies W being inconsistent.

Accordingly, it remains to be shown that $(\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i) = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$.

We show by induction that $(E_0 \cup E_1 \cup E_2, C_0 \cup C_1 \cup C_2) = (E'_0 \cup E'_1 \cup E_2, C'_0 \cup C'_1 \cup C'_2)$ and $(E_i, C_i) = (E'_i, C'_i)$ for $i \geq 3$.

Base Clearly, $E_0 = E'_0$. Consequently, also $E_1 = E'_1$ and $E_2 = E'_2$.

First, we have $C_0 = W \cup C_B$, and $C'_0 = W$. Second, we have

$$C_1 = Th(W \cup C_B) \cup \left\{ \beta \wedge \gamma \, \middle| \, \tfrac{\alpha:\beta}{\gamma} \in GD_D^{(E_1,C)} \right\}$$

and

$$C'_1 = Th(W) \cup \left\{ \hat{C}_B \right\} \cup \left\{ (\beta \wedge \hat{C}_B) \wedge \gamma \, \middle| \, \tfrac{\alpha:\beta}{\gamma} \in GD_D^{(E_1,C)} \right\}$$

This implies $Th(C_1) = Th(C'_1)$ and, Third, $C_2 = C'_2$.

As a result, $(E_0 \cup E_1 \cup E_2, C_0 \cup C_1 \cup C_2) = (E'_0 \cup E'_1 \cup E_2, C'_0 \cup C'_1 \cup C'_2)$.

Step Assume $E_i = E'_i$. Obviously, this implies $E_{i+1} = E'_{i+1}$ and $C_{i+1} = C'_{i+1}$.

With this, we have shown that $(\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i) = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$. ∎

Lemma default rules in constrained default logic

Theorem 4.10.1 *Let (D,W) be a default theory and let (E,C) and (E',C') be constrained extensions of (D,W). Let $\langle D_1, \ldots, D_k \rangle$ be a default proof of ρ in (E',C'), and let δ_ρ be the corresponding lemma default rule for ρ. Then,*

$$\delta_\rho \in GD_{D \cup \{\delta_\rho\}}^{(E,C)} \text{ iff } \bigcup_{i=1}^{k} D_i \subseteq GD_D^{(E,C)}.$$

Proof 4.10.1 The inconsistent case is easily dealt with, so that we prove below the theorem for E and C being consistent.

only-if part Assume $\delta_\rho \in GD^{(E,C)}_{D\cup\{\delta_\rho\}}$. Let $\langle D_1,\ldots,D_k\rangle$ be the corresponding default proof of ρ in (E',C') from (D,W). Let $D_\rho = \bigcup_{i=1}^k D_i$.

According to Definition 4.3.1, we have $C \cup \mathit{Justif}(\delta_\rho) \cup \mathit{Conseq}(\delta_\rho) \not\vdash \bot$. That is,

$$C \cup \left\{\bigwedge_{\delta\in D_\rho} \mathit{Justif}(\delta) \wedge \mathit{Conseq}(\delta)\right\} \cup \{\rho\} \not\vdash \bot \tag{C.5}$$

We show by induction that $D_i \subseteq GD^{(E,C)}_D$.

Base Consider $\delta \in D_1$. By definition, $W \vdash \mathit{Prereq}(\delta)$. Then, $W \subseteq E$ and the fact that E is deductively closed implies $\mathit{Prereq}(\delta) \in E$. By (C.5), we have $C \cup \mathit{Justif}(\delta) \cup \mathit{Conseq}(\delta) \not\vdash \bot$. Then, by Definition 4.3.1 we obtain $\delta \subseteq GD^{(E,C)}_D$. Thus, $D_1 \subseteq GD^{(E,C)}_D$.

Step Assume, we have $D_i \subseteq GD^{(E,C)}_D$. Then, by Theorem 4.3.5, $\mathit{Conseq}(D_i) \subseteq E$.

Consider $\delta \in D_{i+1}$. By definition, $W \cup \mathit{Conseq}(D_i) \vdash \mathit{Prereq}(\delta)$. Then, since $W \cup \mathit{Conseq}(D_i) \subseteq E$ and E is deductively closed, we obtain $\mathit{Prereq}(\delta) \in E$. By (C.5), we have $C \cup \mathit{Justif}(\delta) \cup \mathit{Conseq}(\delta) \not\vdash \bot$. Then, by Definition 4.3.1 we obtain $\delta \subseteq GD^{(E,C)}_D$. Thus, $D_{i+1} \subseteq GD^{(E,C)}_D$.

Hence, we obtain $D_\rho \subseteq GD^{(E,C)}_D$.

if part Assume $D_\rho \subseteq GD^{(E,C)}_D$. According to Theorem 4.3.5, we have that $\mathit{Justif}\left(GD^{(E,C)}_D\right) \cup \mathit{Conseq}\left(GD^{(E,C)}_D\right) \subseteq C$. Hence, $\mathit{Justif}(D_\rho) \cup \mathit{Conseq}(D_\rho) \subseteq C$. By Definition 4.10.1, $W \cup \mathit{Conseq}(D_k) \vdash \rho$. Then, since $W \cup \mathit{Conseq}(D_k) \subseteq E$ and E is deductively closed, we obtain $\rho \in E \subseteq C$. Clearly, $C \cup \mathit{Justif}(D_\rho) \cup \mathit{Conseq}(D_\rho) \cup \{\rho\}$ is consistent, since C is consistent. Thus, $C \cup \mathit{Justif}(\delta_\rho) \cup \mathit{Conseq}(\delta_\rho) \not\vdash \bot$. Since δ_ρ is prerequisite-free, this implies $\delta_\rho \in GD^{(E,C)}_{D\cup\{\delta_\rho\}}$. ∎

Theorem 4.10.2 *Let (D,W) be a default theory and let (E',C') be a constrained extension of (D,W). Let δ_ρ be a lemma default rule for $\rho \in E'$. Then,*

(E,C) is a constrained extension of (D,W) iff (E,C) is a constrained extension of $(D \cup \{\delta_\rho\}, W)$.

Proof 4.10.2

only-if part Let (E,C) be a constrained extension of (D,W). By Definition 4.10.2,

$$\delta_\rho = \frac{:\ \bigwedge_{\delta\in D_\rho} \mathit{Justif}(\delta) \wedge \bigwedge_{\delta\in D_\rho} \mathit{Conseq}(\delta)}{\rho}$$

where $D_\rho = \bigcup_i D_i$ for some default proof $\langle D_1,\ldots,D_k\rangle$ of ρ in (E',C') from (D,W). We distinguish the following two cases.

1. Let $D_\rho \subseteq GD^{(E,C)}_D$. Then, $\mathit{Conseq}(D_\rho) \subseteq E$ and $\mathit{Justif}(D_\rho) \cup \mathit{Conseq}(D_\rho) \subseteq C$. That is, $\mathit{Conseq}(\delta_\rho) \in E$ and $\{\mathit{Justif}(\delta_\rho)\} \cup \{\mathit{Conseq}(\delta_\rho)\} \subseteq C$.

2. Let $D_\rho \not\subseteq GD^{(E,C)}_D$. Then, there is a least k and a default rule $\delta \in D_k \subseteq D_\rho$ such that $\delta \not\in GD^{(E,C)}_D$. By Definition 4.10.1, $W \cup \mathit{Conseq}(D_{k-1}) \vdash \mathit{Prereq}(D_k)$. By assumption, $D_{k-1} \subseteq GD^{(E,C)}_D$. Then, $W \cup \mathit{Conseq}(D_{k-1}) \subseteq E$ and the fact that E is deductively closed implies $\mathit{Prereq}(\delta) \in E$. According to Definition 4.3.1, this implies $C \cup \{\mathit{Justif}(\delta)\} \cup \{\mathit{Conseq}(\delta)\} \vdash \bot$. By monotonicity, $C \cup \mathit{Justif}(D_\rho) \cup \mathit{Conseq}(D_\rho) \vdash \bot$. That is, $C \cup \{\mathit{Justif}(\delta_\rho)\} \cup \{\mathit{Conseq}(\delta_\rho)\} \vdash \bot$.

Since (E,C) is a constrained extension of (D,W), E and C are the smallest sets of sentences satisfying the properties

1. $W \subseteq E \subseteq C$,
2. $E = Th(E)$ and $C = Th(C)$,
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \wedge \gamma \in C$.

In both cases, we have that E and C are also the smallest sets satisfying the conditions 1. and 2. and, moreover the modified condition

3. For any $\frac{\alpha:\beta}{\gamma} \in D \cup \{\delta_\rho\}$, if $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \wedge \gamma \in C$.

This is because in the first case $Conseq(\delta_\rho) \in E$ and $\{Justif(\delta_\rho)\} \cup \{Conseq(\delta_\rho)\} \subseteq C$ whereas in the second δ_ρ is not applicable. Consequently, (E,C) is also a constrained extension of $(D \cup \{\delta_\rho\}, W)$.

if part Let (E,C) be a constrained extension of $(D \cup \{\delta_\rho\}, W)$. We regard the following two cases.

1. Let $\delta_\rho \notin GD^{(E,C)}_{D\cup\{\delta_\rho\}}$. Clearly, (E,C) is then also a constrained extension of (D,W).
2. Let $\delta_\rho \in GD^{(E,C)}_{D\cup\{\delta_\rho\}}$. According to Theorem 4.3.1, $(E,C) = (\bigcup_{i=0}^\infty E_i, \bigcup_{i=0}^\infty C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \; \tfrac{\alpha:\beta}{\gamma} \in D \cup \{\delta_\rho\}, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \; \middle| \; \tfrac{\alpha:\beta}{\gamma} \in D \cup \{\delta_\rho\}, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

   Clearly, we have $Conseq(\delta_\rho) \subseteq E_1$ and $\{Conseq(\delta_\rho)\} \cup \{Justif(\delta_\rho)\} \subseteq C_1$, since δ_ρ is prerequisite-free.

   Analogously, (E,C) is a constrained extension of (D,W) iff $(E,C) = (\bigcup_{i=0}^\infty E'_i, \bigcup_{i=0}^\infty C'_i)$ such that $E'_0 = W$ and $C'_0 = W$, and for $i \geq 0$

$$E'_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C'_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \; \middle| \; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

   Since $\delta_\rho \in GD^{(E,C)}_{D\cup\{\delta_\rho\}}$ we have $C \cup \left\{ \bigwedge_{\delta \in D_\rho} Justif(\delta) \wedge Conseq(\delta) \right\} \cup \{\rho\} \not\vdash \bot$. By monotonicity, $C \cup \{Justif(\delta)\} \wedge \{Conseq(\delta)\} \not\vdash \bot$ for each $\delta \in D_\rho$. This and Definition 4.10.1 implies that there is a k such that $Conseq(D_\rho) \subseteq E'_k$ and $Conseq(D_\rho) \cup Justif(D_\rho) \subseteq C'_k$. That is, $Conseq(\delta_\rho) \subseteq E'_k$ and $\{Conseq(\delta_\rho)\} \cup \{Justif(\delta_\rho)\} \subseteq C'_k$. As a consequence, $\bigcup_{i=0}^\infty E'_i = \bigcup_{i=0}^\infty E_i$ and $\bigcup_{i=0}^\infty C'_i = \bigcup_{i=0}^\infty C_i$. Thus, $(E,C) = (\bigcup_{i=0}^\infty E'_i, \bigcup_{i=0}^\infty C'_i)$.

■

Appendix D

Proofs of Theorems in Chapter 5

This chapter presents the proofs of the theorems given in Chapter 5.

Constrained versus justified default logic

Theorem 5.2.1 *Let (D, W) be a default theory and E a justified extension of (D, W) wrt J. If $E \cup J$ is consistent then $(E, Th(E \cup J))$ is a constrained extension of (D, W).*

Proof 5.2.1 Let E be a justified extension of (D, W) wrt J. Define $C = Th(E \cup J)$. We show that (E, C) is a constrained extension of (D, W).

First, let us observe the following properties.

1. By definition, $W \subseteq E \subseteq C$.
2. Also, by definition, $E = Th(E)$ and $C = Th(C)$.
3. For any $\frac{\alpha : \beta}{\gamma} \in D$, if $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \wedge \gamma \in C$ since, by monotonicity, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ implies $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ (since $C = Th(E \cup J)$).

Then, by the minimality of $\Upsilon(C)$, we have[1] $\Upsilon_1(C) \subseteq E$ and $\Upsilon_2(C) \subseteq C$. That is, $\Upsilon(C) \subseteq^2 (E, C)$.

It remains to be shown that $(E, C) \subseteq^2 \Upsilon(C)$. Since E is a justified extension of (D, W) wrt J we have according to Theorem A.1.1 that $(E, J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$ where $E_0 = W$ and $J_0 = \emptyset$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

$$J_{i+1} = \quad J_i \quad \cup \left\{ \beta \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

We will show that $\bigcup_{i=0}^{\infty} E_i \subseteq \Upsilon_1(C)$ and $\bigcup_{i=0}^{\infty} J_i \subseteq \Upsilon_2(C)$, in order to show that $E \subseteq \Upsilon_1(C)$ and $C \subseteq \Upsilon_2(C)$.

We show by induction $E_i \subseteq \Upsilon_1(C)$ and $J_i \subseteq \Upsilon_2(C)$ for $i \geq 0$.

Base Clearly, $E_0 = W \subseteq \Upsilon_1(C)$ and $J_0 = \emptyset \subseteq \Upsilon_2(C)$.

Step Assume $E_i \subseteq \Upsilon_1(C)$ and $J_i \subseteq \Upsilon_2(C)$ and consider $v \in E_{i+1} \cup J_{i+1}$.

[1] We refer to the components of Υ as Υ_1 and Υ_2, respectively.

1. If $v \in Th(E_i)$ then, by the induction hypothesis and the fact that $\Upsilon_1(C)$ is deductively closed, we obtain $v \in \Upsilon_1(C)$.
2. If $v \in J_i$ then, by the induction hypothesis, also $v \in \Upsilon_2(C)$.
3. Otherwise, $v \in \{\beta, \gamma\}$ for some default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$.
   By the induction hypothesis, $\alpha \in \Upsilon_1(C)$. By assumption, $E \cup J \not\vdash \bot$. Since E is a justified extension of (D, W) wrt J, $\alpha \in E_i$ and $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ imply $\gamma \in E$ and $\beta \in J$. Therefore, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ (since $C = Th(E \cup J)$). From $\alpha \in \Upsilon_1(C)$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ we conclude, by Definition 4.2.1, that $\gamma \in \Upsilon_1(C)$ and $\beta \wedge \gamma \in \Upsilon_2(C)$. Since $\Upsilon_2(C)$ is deductively closed the last membership implies $\beta \in \Upsilon_2(C)$. Clearly, both cases for v are covered.

Accordingly, $E_{i+1} \subseteq \Upsilon_1(C)$ and $J_{i+1} \subseteq \Upsilon_2(C)$.

Therefore, we have shown that $\bigcup_{i=0}^{\infty} E_i \subseteq \Upsilon_1(C)$ and $\bigcup_{i=0}^{\infty} J_i \subseteq \Upsilon_2(C)$. Since $\bigcup_{i=0}^{\infty} E_i = E$ and $\bigcup_{i=0}^{\infty} J_i = J$, that is $E \subseteq \Upsilon_1(C)$ and $J \subseteq \Upsilon_2(C)$. Since $\Upsilon_1(C) \subseteq \Upsilon_2(C)$ we have $E \cup J \subseteq \Upsilon_2(C)$. So, since $\Upsilon_2(C)$ is deductively closed, $C \subseteq \Upsilon_2(C)$. Hence, $(E,C) \subseteq^2 \Upsilon(C)$. ■

Proposition D.1.27 *Let (E,C) be a constrained extension of (D, W). Then,*

1. *$(E, Justif\left(GD_D^{(E,C)}\right))$ is a justified extension of $(GD_D^{(E,C)}, E)$ and*
2. *$GD_D^{(E,C)} = GD_{GD_D^{(E,C)}}^{(E, Justif(GD_D^{(E,C)}))}$.*

Proof D.1.27

1. Obvious.
2. By definition, $GD_{GD_D^{(E,C)}}^{(E, Justif(GD_D^{(E,C)}))} \subseteq GD_D^{(E,C)}$. Assume, $GD_D^{(E,C)} \not\subseteq GD_{GD_D^{(E,C)}}^{(E, Justif(GD_D^{(E,C)}))}$. Then, there is a default rule $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$ but $\frac{\alpha:\beta}{\gamma} \notin GD_{GD_D^{(E,C)}}^{(E, Justif(GD_D^{(E,C)}))}$. By Definition 4.3.1, $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$ implies $\alpha \in E$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. That is, according to Theorem 4.3.5, $Th\left(E \cup Justif\left(GD_D^{(E,C)}\right)\right) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ $\left(E = Th\left(W \cup Conseq\left(GD_D^{(E,C)}\right)\right)\right)$. By monotonicity, we obtain $\forall \eta \in Justif\left(GD_D^{(E,C)}\right) \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$. According to Theorem A.1.2, this and $\alpha \in E$ implies $\frac{\alpha:\beta}{\gamma} \in GD_{GD_D^{(E,C)}}^{(E, Justif(GD_D^{(E,C)}))}$, a contradiction.

■

Theorem 5.2.2 *Let (D, W) be a default theory and (E,C) be a constrained extension of (D, W). Then, there is a justified extension (E', J') of (D, W) such that $E \subseteq E'$ and $C \subseteq Th(E' \cup J')$.*

Proof 5.2.2 Let (E,C) be a constrained extension of (D, W). By Proposition D.1.27, $(E, Justif\left(GD_D^{(E,C)}\right))$ is a justified extension of $(GD_D^{(E,C)}, E)$. By semi-monotonicity, there is a justified extension (E', J') of (D, E) such that

$$E \subseteq E' \tag{D.1}$$

$$Justif\left(GD_D^{(E,C)}\right) \subseteq J' \tag{D.2}$$

As a consequence, $C \subseteq \mathit{Th}(E' \cup J')$ since $C = \mathit{Th}\Big(E \cup \mathit{Justif}\Big(GD_D^{(E,C)}\Big)\Big)$.

It remains to be shown that (E', J') is a justified extension of (D, W). Since (E', J') is a justified extension of (D, E), E' and J' are the smallest sets of sentences such that

1. $E \subseteq E'$,
2. $E' = \mathit{Th}(E')$,
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E'$ and $\forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ then $\gamma \in E'$ and $\beta \in J'$.

Also the following conditions hold.

1. Clearly, $W \subseteq E'$, since $W \subseteq E$ and $E \subseteq E'$.
2. By definition, $E' = \mathit{Th}(E')$.
3. By definition, for any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E'$ and $\forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ then $\gamma \in E'$ and $\beta \in J'$.

Then, by the minimality of $\Psi(E', J')$, we have[2] $\Psi_1(E', J') \subseteq E'$ and $\Psi_2(E', J') \subseteq J'$. That is, $\Psi(E', J') \subseteq^2 (E, C)$.

It remains to be shown that $(E', J') \subseteq^2 \Psi(E', J')$. Since E' is a justified extension of (D, E) wrt J' we have according to Theorem A.1.1 that $(E', J') = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$ where $E_0 = E$ and $J_0 = \emptyset$, and for $i \geq 0$

$$E_{i+1} = \mathit{Th}(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

$$J_{i+1} = \quad J_i \quad \cup \left\{ \beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

We will show that $\bigcup_{i=0}^{\infty} E_i \subseteq \Psi_1(E', J')$ and $\bigcup_{i=0}^{\infty} J_i \subseteq \Psi_2(E', J')$, in order to show that $E' \subseteq \Psi_1(E', J')$ and $J' \subseteq \Psi_2(E', J')$.

We show by induction $E_i \subseteq \Psi_1(E', J')$ and $J_i \subseteq \Psi_2(E', J')$ for $i \geq 0$.

Base Clearly, $J_0 = \emptyset \subseteq \Psi_2(E', J')$.

By definition, $E_0 = E$. That is, $E_0 = \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big)$. By definition, $W \subseteq \Psi_1(E', J')$. By Proposition D.1.27, $GD_D^{(E,C)} = GD_{GD_D^{(E,C)}}^{(E, \mathit{Justif}(GD_D^{(E,C)}))}$. Also $GD_{GD_D^{(E,C)}}^{(E, \mathit{Justif}(GD_D^{(E,C)}))} \subseteq GD_D^{(E',J')}$, by semi-monotonicity. Therefore, we have for each $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$

$$\forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot. \tag{D.3}$$

Since (E, C) is a constrained extension there exists according to Theorem 4.3.6 an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,C)}$ such that $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i)$ for $i \in I$. We show that $\mathit{Conseq}(\delta_i) \in \Psi_1(E', J')$ for $i \in I$.

Base By definition, $W \vdash \mathit{Prereq}(\delta_0)$. Since $W \subseteq \Psi_1(E', J')$ and the fact that $\Psi_1(E', J')$ is deductively closed, we have $\mathit{Prereq}(\delta_0) \in \Psi_1(E', J')$. This and (D.3) implies by Definition 5.2.1 that $\mathit{Conseq}(\delta_0) \in \Psi_1(E', J')$.

[2] We refer to the components of Ψ as Ψ_1 and Ψ_2, respectively.

Step Assume $Conseq(\{\delta_0, \ldots, \delta_i\}) \subseteq \Psi_1(E', J')$. By definition, $W \cup Conseq(\{\delta_0, \ldots, \delta_i\}) \vdash Prereq(\delta_{i+1})$. Since $W \cup Conseq(\{\delta_0, \ldots, \delta_i\}) \subseteq \Psi_1(E', J')$ by the induction hypothesis, and the fact that $\Psi_1(E', J')$ is deductively closed, we have $Prereq(\delta_{i+1}) \in \Psi_1(E', J')$. This and (D.3) implies by Definition 5.2.1 that $Conseq(\delta_{i+1}) \in \Psi_1(E', J')$.

We have shown that $Conseq(\delta_i) \in \Psi_1(E', J')$ for $i \in I$; hence, $Conseq\left(GD_D^{(E,C)}\right) \in \Psi_1(E', J')$.

From $W \subseteq \Psi_1(E', J')$ and $Conseq\left(GD_D^{(E,C)}\right) \subseteq \Psi_1(E', J')$ we conclude that $Th\left(W \cup Conseq\left(GD_D^{(E,C)}\right)\right) \subseteq \Psi_1(E', J')$, since $\Psi_1(E', J')$ is deductively closed. Consequently, $E_0 \subseteq \Psi_2(E', J')$.

Step Assume $E_i \subseteq \Psi_1(E', J')$ and $J_i \subseteq \Psi_2(E', J')$ and consider $v \in E_{i+1} \cup J_{i+1}$.

1. If $v \in Th(E_i)$ then, by the induction hypothesis and the fact that $\Psi_1(E', J')$ is deductively closed, we obtain $v \in \Psi_1(E', J')$.
2. If $v \in J_i$ then, by the induction hypothesis, also $v \in \Psi_2(E', J')$.
3. Otherwise, $v \in \{\beta, \gamma\}$ for some default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $\forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$.
   By the induction hypothesis, $\alpha \in \Psi_1(E', J')$. From $\alpha \in \Psi_1(E', J')$ and $\forall \eta \in J' \cup \{\beta\}.\ E' \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ we conclude, by Definition 5.2.1, that $\gamma \in \Psi_1(E', J')$ and $\beta \in \Psi_2(E', J')$. Clearly, both cases for v are covered.

Accordingly, $E_{i+1} \subseteq \Psi_1(E', J')$ and $J_{i+1} \subseteq \Psi_2(E', J')$.

Therefore, we have shown that $\bigcup_{i=0}^{\infty} E_i \subseteq \Psi_1(E', J')$ and $\bigcup_{i=0}^{\infty} J_i \subseteq \Psi_2(E', J')$. Since $\bigcup_{i=0}^{\infty} E_i = E'$ and $\bigcup_{i=0}^{\infty} J_i = J'$, that is $E' \subseteq \Psi_1(E', J')$ and $J' \subseteq \Psi_2(E', J')$. Hence, $(E', J') \subseteq^2 \Psi(E', J')$. ■

Theorem 5.2.3 *Let (D, W) be a default theory and let E, C, and J be sets of formulas. If (E,C) is a constrained extension of (D, W) and E is a justified extension of (D, W) wrt J then $C \subseteq Th(E \cup J)$.*

Proof 5.2.3 Let (E,C) be a constrained extension of (D, W) and let E be a justified extension of (D, W) wrt J. Then, according to Theorem 4.3.1 $E = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

We will show $\bigcup_{i=0}^{\infty} C_i \subseteq Th(E \cup J)$, in order to show $C \subseteq Th(E \cup J)$.
Therefore, we show by induction $C_i \subseteq Th(E \cup J)$ for $i \geq 0$.

Base Clearly, $C_0 = W \subseteq E \subseteq Th(E \cup J)$.

Step Assume $C_i \subseteq Th(E \cup J)$ and consider $v \in C_{i+1}$.

1. If $v \in Th(C_i)$ then, by the induction hypothesis, $v \in Th(E \cup J)$.

2. Otherwise, $v \in \{\beta, \gamma\}$ for some default rule $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$.
   Clearly, $\gamma \in E$. According to Definition 5.2.1 we have $\beta \in J$ only if $\alpha \in E$ and $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\eta\} \cup \{\gamma\} \not\vdash \perp$. Clearly, $\alpha \in E$ since $\alpha \in E_i$ and $E_i \subseteq E$. First, by monotonicity, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ implies $E \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ Second, since (E, J) is a justified extension we also have $\forall \eta \in J.\ E \cup \{\eta\} \cup \{\gamma\} \not\vdash \perp$. Hence, both cases for v are covered.

From the two cases, we obtain $C_{i+1} \subseteq Th(E \cup J)$.

Therefore, we have shown that $\bigcup_{i=0}^{\infty} C_i \subseteq Th(E \cup J)$. ■

Constrained versus cumulative default logic

Theorem 5.3.1 *Let (D, W) be a default theory and $(D, \mathcal{W})$ the assertional default theory, where $\mathcal{W} = \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\}$. Then, if (E,C) is a constrained extension of (D, W) then there is an assertional extension $\mathcal{E}$ of $(D, \mathcal{W})$ such that $E = Form(\mathcal{E})$ and $C = Th(Form(\mathcal{E}) \cup Supp(\mathcal{E}))$; and, conversely if $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ then $(Form(\mathcal{E}), Th(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ is a constrained extension of (D, W).*

Proof 5.3.1

only-if part Assume (E,C) is a constrained extension of (D, W). Let $\mathcal{F}$ be a set of assertions induced by $GD_D^{(E,C)}$, ie. $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ such that $\mathcal{F}_0 = \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\}$ and for each $i \geq 0$

$$\begin{aligned}\mathcal{F}_{i+1} \quad = \quad & \widehat{Th}(\mathcal{F}_i) \cup \{\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \mid \tfrac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}, \\ & \langle \alpha, Supp(\alpha) \rangle \in \mathcal{F}_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp\}\end{aligned}$$

Observe that due to our construction of $\mathcal{F}$ we have $E = Form(\mathcal{F})$ and also that $C = Th(Form(\mathcal{F}) \cup Supp(\mathcal{F}))$, and furthermore $\mathcal{F} = \widehat{Th}(\mathcal{F})$.

It remains to be shown that $\mathcal{F}$ is an assertional extension of $(D, \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\})$. According to [Brewka, 1991b, Proposition 1] we have that $\mathcal{F}$ is an assertional extension iff $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{E}_i$ such that $\mathcal{E}_0 = \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\}$ and for each $i \geq 0$

$$\begin{aligned}\mathcal{E}_{i+1} \quad = \quad & \widehat{Th}(\mathcal{E}_i) \cup \{\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \mid \tfrac{\alpha:\beta}{\gamma} \in D, \\ & \langle \alpha, Supp(\alpha) \rangle \in \mathcal{E}_i, Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp\}\end{aligned}$$

We have the following two cases.

1. $\bigcup_{i=0}^{\infty} \mathcal{E}_i \subseteq \mathcal{F}$. Therefore, we show by induction that $\mathcal{E}_i \subseteq \mathcal{F}$ for $i \geq 0$.

   Base Clearly, we have $\mathcal{E}_0 \subseteq \mathcal{F}$ since $\mathcal{E}_0 = \mathcal{F}_0$.

   Step Let $\mathcal{E}_i \subseteq \mathcal{F}$. Regard $\langle \gamma, Supp(\gamma) \rangle \in \mathcal{E}_{i+1}$.

   (a) If $\langle \gamma, Supp(\gamma) \rangle \in \widehat{Th}(\mathcal{E}_i)$ then by the induction hypothesis and the fact that $\mathcal{F} = \widehat{Th}(\mathcal{F})$ we also have $\langle \gamma, Supp(\gamma) \rangle \in \mathcal{F}$.

   (b) Otherwise, there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ where $\langle \alpha, Supp(\alpha) \rangle \in \mathcal{E}_i$ and $Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$.
   By the induction hypothesis $\langle \alpha, Supp(\alpha) \rangle \in \mathcal{F}$. By compactness there exists a k such that $\langle \alpha, Supp(\alpha) \rangle \in \mathcal{F}_k$. By definition, $C = Th(Form(\mathcal{F}) \cup Supp(\mathcal{F}))$.

Hence, $Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ implies $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. From $\langle\alpha, Supp(\alpha)\rangle \in \mathcal{F}_k$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ we conclude $\langle\gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{F}_{k+1}$. By monotonicity, $\langle\gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{F}$.

From the two cases, we obtain $\mathcal{E}_{i+1} \subseteq \mathcal{F}$.

2. $\mathcal{F} \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Therefore, we show by induction that $\mathcal{F}_i \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$ for $i \geq 0$.

Base Clearly, we have $\mathcal{F}_0 \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$ since $\mathcal{F}_0 = \mathcal{E}_0$.

Step Let $\mathcal{F}_i \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Regard $\langle\gamma, Supp(\gamma)\rangle \in \mathcal{F}_{i+1}$.

(a) If $\langle\gamma, Supp(\gamma)\rangle \in \widehat{Th}(\mathcal{F}_i)$ then by the induction hypothesis and the fact that $\bigcup_{i=0}^{\infty} \mathcal{E}_i = \widehat{Th}(\bigcup_{i=0}^{\infty} \mathcal{E}_i)$ we also have $\langle\gamma, Supp(\gamma)\rangle \in \bigcup_{i=0}^{\infty} \mathcal{E}_i$.

(b) Otherwise, there is a default rule $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$ where $\langle\alpha, Supp(\alpha)\rangle \in \mathcal{F}_i$ and $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
By the induction hypothesis $\langle\alpha, Supp(\alpha)\rangle \in \bigcup_{i=0}^{\infty} \mathcal{E}_i$. By compactness there exists a k such that $\langle\alpha, Supp(\alpha)\rangle \in \mathcal{E}_k$. By definition, $C = Th(Form(\mathcal{F}) \cup Supp(\mathcal{F}))$. Hence, $C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ implies $Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$. From $\langle\alpha, Supp(\alpha)\rangle \in \mathcal{E}_k$ and $Form(\mathcal{F}) \cup Supp(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ we conclude $\langle\gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}_{k+1}$. By monotonicity, $\langle\gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}$.

if part Assume $\mathcal{E}$ is an assertional extension of $(D, \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\})$. We show that $(Form(\mathcal{E}), Th(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ is a constrained extension of (D, W). Let us abbreviate $Form(\mathcal{E})$ by E^ε and $Th(Form(\mathcal{E}) \cup Supp(\mathcal{E}))$ by C^ε. Notice, that E^ε is also deductively closed, ie. $E^\varepsilon = Th(E^\varepsilon)$ since $\mathcal{E} = \widehat{Th}(\mathcal{E})$.

According to Theorem 4.3.1 we have that $(E^\varepsilon, C^\varepsilon)$ is an constrained extension iff $(E^\varepsilon, C^\varepsilon) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \ \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C^\varepsilon \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \ \middle| \ \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C^\varepsilon \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}.$$

According to [Brewka, 1991b, Proposition 1] we have that $\mathcal{E} = \bigcup_{i=0}^{\infty} \mathcal{E}_i$ such that $\mathcal{E}_0 = \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\}$ and for each $i \geq 0$

$$\begin{aligned} \mathcal{E}_{i+1} \ = \ & \widehat{Th}(\mathcal{E}_i) \cup \{\langle\gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \mid \tfrac{\alpha:\beta}{\gamma} \in D, \\ & \qquad \langle\alpha, Supp(\alpha)\rangle \in \mathcal{E}_i, Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\} \end{aligned}$$

We have to consider the following two cases.

1. $\bigcup_{i=0}^{\infty} E_i \subseteq E^\varepsilon$, $\bigcup_{i=0}^{\infty} C_i \subseteq C^\varepsilon$.

We show by induction that $E_i \subseteq E^\varepsilon$ and $C_i \subseteq C^\varepsilon$ for $i \geq 0$.

Base

(a) Clearly, $E_0 = W = Form(\{\langle\alpha, \emptyset\rangle \mid \alpha \in W\}) = E_0^\varepsilon \subseteq E^\varepsilon$.

(b) Also, $C_0 = W \subseteq E^\varepsilon \subseteq C^\varepsilon$.

Step Let $E_i \subseteq E^\varepsilon$ and $C_i \subseteq C^\varepsilon$. Regard $\eta \in E_{i+1} \cup C_{i+1}$.

(a) If $\eta \in Th(E_i)$ then, by the induction hypothesis and the fact that E^ε is deductively closed, we obtain $\eta \in E^\varepsilon$.

(b) Analogously, if $\eta \in Th(C_i)$ then $\eta \in C^e$.

(c) Otherwise, $\eta \in \{\beta, \beta \wedge \gamma\}$ such that there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ where $\alpha \in E_i$ and $C^e \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$.
By the induction hypothesis $\alpha \in E^e$. That is, $\alpha \in Form(\mathcal{E})$. Hence, $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}$. Also, $C^e \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ implies $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$. Since $\mathcal{E}$ is an assertional extension $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}$ and $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ imply $\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}$. Thus, $\gamma \in E^e$ and $\{\gamma, \beta\} \subseteq C^e$. Since C^e is deductively closed the latter implies $\gamma \wedge \beta \in C^e$.

From the three cases, we obtain $E_{i+1} \subseteq E^e$ and $C_{i+1} \subseteq C^e$.

2. $E^e \subseteq \bigcup_{i=0}^{\infty} E_i$, $C^e \subseteq \bigcup_{i=0}^{\infty} C_i$.

We show by induction that $Form(\mathcal{E}_i) \subseteq \bigcup_{i=0}^{\infty} E_i$ and $Form(\mathcal{E}_i) \cup Supp(\mathcal{E}_i) \subseteq \bigcup_{i=0}^{\infty} C_i$ for $i \geq 0$.

Base

(a) Clearly, $Form(\mathcal{E}_0) = W = E_0 \subseteq \bigcup_{i=0}^{\infty} E_i$.

(b) Similarly, $Th(Form(\mathcal{E}_0) \cup Supp(\mathcal{E}_0)) = Th(W) \subseteq C_1 \subseteq \bigcup_{i=0}^{\infty} C_i$.

Step Let $Form(\mathcal{E}_i) \subseteq \bigcup_{i=0}^{\infty} E_i$ and $Form(\mathcal{E}_i) \cup Supp(\mathcal{E}_i) \subseteq \bigcup_{i=0}^{\infty} C_i$.

Consider $\langle \eta, Supp(\eta)\rangle \in \mathcal{E}_{i+1}$.

(a) If $\langle \eta, Supp(\eta)\rangle \in \widehat{Th}(\mathcal{E}_i)$ then, by the induction hypothesis and the fact that $\bigcup_{i=0}^{\infty} E_i$ and $\bigcup_{i=0}^{\infty} C_i$ are deductively closed, we obtain $Form(\eta) \in \bigcup_{i=0}^{\infty} E_i$ and $Supp(\eta) \subseteq \bigcup_{i=0}^{\infty} C_i$.

(b) Otherwise, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D$ where $\langle \alpha, Supp(\alpha)\rangle \in \mathcal{E}_i$ and $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$. Then, $Supp(\eta) = Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}$.
By the induction hypothesis $\alpha \in \bigcup_{i=0}^{\infty} E_i$. Then, by compactness there exists a k such that $\alpha \in E_k$. Clearly, $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ implies $C^e \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$. From $\alpha \in E_k$ and $C^e \cup \{\beta\} \cup \{\gamma\} \not\vdash \perp$ we conclude $\gamma \in E_{k+1}$ and $\beta \wedge \gamma \in C_{k+1}$. Also, by the induction hypothesis, $Supp(\alpha) \subseteq \bigcup_{i=0}^{\infty} C_i$. Therefore, by monotonicity and the fact that $Supp(\eta) = Supp(\alpha) \cup \{\beta\} \cup \{\gamma\}$, we have $Form(\eta) \in \bigcup_{i=0}^{\infty} E_i$ and $Supp(\eta) \subseteq \bigcup_{i=0}^{\infty} C_i$.

■

Theorem 5.3.3 (Correctness & Completeness) *Let $(D, \mathcal{W})$ be an assertional default theory and let $(\Pi, \breve{\Pi})$ be a pair of classes of first-order interpretations.*

If $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$ then $(\mathrm{MOD}(Form(\mathcal{E})), \mathrm{MOD}(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ is a $\succcurlyeq_D$-maximal element above $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$.

If $(\Pi, \breve{\Pi})$ is a $\succcurlyeq_D$-maximal element above $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$ then there is an assertional extension $\mathcal{E}$ of $(D, \mathcal{W})$ such that $\Pi = \{\pi \mid \pi \models Form(\mathcal{E})\}$ and $\breve{\Pi} = \{\pi \mid \pi \models Form(\mathcal{E}) \cup Supp(\mathcal{E})\}$.

Proof 5.3.3 First, we need the following definition.

Definition D.1.3 *Let $(D, \mathcal{W})$ be an assertional default theory. Given a possibly infinite sequence of default rules $\Delta = \langle \delta_0, \delta_1, \delta_2, \ldots\rangle$ in D, also denoted $\langle \delta_i\rangle_{i\in I}$ where I is the index set for Δ, we define a sequence of focused models structures $\langle (\Pi_i, \breve{\Pi}_i)\rangle_{i\in I}$ as follows:*

$$\begin{array}{lcl}(\Pi_0, \breve{\Pi}_0) & = & (\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}}) \\ (\Pi_{i+1}, \breve{\Pi}_{i+1}) & = & (\{\pi \in \Pi_i \mid \pi \models \gamma_i\}, \{\pi \in \breve{\Pi}_i \mid \pi \models \beta_i \wedge \gamma_i\}), \quad \textit{where } \delta_i = \frac{\alpha_i:\beta_i}{\gamma_i}.\end{array}$$

The case where $(D, \mathcal{W})$ is not well based is easily dealt with, so that we prove below the theorem for $Form(\mathcal{W}) \cup Supp(\mathcal{W})$ being satisfiable.

Proof 5.3.3 (Correctness) Let $(D, \mathcal{W})$ be a well based assertional default theory. Assume $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$. Then, by assumption, also $Form(\mathcal{E}) \cup Supp(\mathcal{E})$ is consistent. Then according to Theorem A.2.4, there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of the set of generating default rules $GD_D^{\mathcal{E}}$ such that for $i \in I$

$$Form(\mathcal{W}) \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i). \tag{D.4}$$

Let $\langle (\Pi_i, \breve{\Pi}_i) \rangle_{i \in I}$ be a sequence of focused models structures obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition C.1.2. We will show that $(\Pi, \breve{\Pi})$ coincides with $\bigcap_{i \in I}(\Pi_i, \breve{\Pi}_i)$ and is $\succcurlyeq_D$-maximal above $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$.

Since $\mathcal{E}$ is a assertional extension, we have according to Theorem A.2.2 that

$$\begin{aligned} Form(\mathcal{E}) &= Th(Form(\mathcal{W}) \cup Conseq(GD_D^{\mathcal{E}})), \\ Supp(\mathcal{E}) &= Supp(\mathcal{W}) \cup Conseq(GD_D^{\mathcal{E}}) \cup Justif(GD_D^{\mathcal{E}}). \end{aligned}$$

Then, since $(\Pi, \breve{\Pi}) = (\mathrm{MOD}(Form(\mathcal{E})), \mathrm{MOD}(Form(\mathcal{E}) \cup Supp(\mathcal{E})))$ we have that $(\Pi, \breve{\Pi}) = \bigcap_{i \in I}(\Pi_i, \breve{\Pi}_i)$.

First, let us show that $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ for $i \in I$.

- Since $\Pi_i \subseteq \Pi_{\mathcal{W}}$ and $\Pi_{\mathcal{W}} \models Form(\mathcal{W})$, then by definition of Π_i we have $\Pi_i \models Form(\mathcal{W}) \cup Conseq(\delta_{i-1})$ for $i \in I$. Now, $\Pi_{i+1} \subseteq \Pi_i$ for $i \in I$ implies that $\Pi_i \models Form(\mathcal{W}) \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\})$. By (D.4), it follows that $\Pi_i \models Prereq(\delta_i)$ for $i \in I$.

- Let us assume that $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ fails for some $k \in I$. By definition of $\langle (\Pi_i, \breve{\Pi}_i) \rangle_{i \in I}$ and the fact that we have just proven that $\Pi_i \models Prereq(\delta_i)$ for $i \in I$, this means that $\breve{\Pi}_k \models \neg(\gamma_k \wedge \beta_k)$ for $\delta_k = \frac{\alpha_k : \beta_k}{\gamma_k}$. Let us abbreviate $Form(\mathcal{W}) \cup Conseq(\{\delta_0, \ldots, \delta_{k-1}\}) \cup Justif(\{\delta_0, \ldots, \delta_{k-1}\})$ by $Supp^k$. By definition, $\breve{\Pi}_k = \mathrm{MOD}(Supp^k)$. Then, $Supp^k \models \neg(\gamma_k \wedge \beta_k)$. That is, $Supp^k \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \perp$. By monotonicity, $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \perp$, contradictory to the fact that $\delta_k \in GD_D^{\mathcal{E}}$.

Therefore, $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ for $i \in I$. As a consequence, $\bigcap_{i \in I}(\Pi_i, \breve{\Pi}_i) \succcurlyeq_{GD_D^{\mathcal{E}}} (\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$. That is, $(\Pi, \breve{\Pi}) \succcurlyeq_D (\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$.

Second, assume $(\Pi, \breve{\Pi})$ is not $\succcurlyeq_D$-maximal. Then, there exists a default rule $\frac{\alpha : \beta}{\gamma} \in D \setminus GD_D^{\mathcal{E}}$ such that $\Pi \models \alpha$ and $\breve{\Pi} \not\models \neg(\gamma \wedge \beta)$. [3] First, since $\Pi \models Form(\mathcal{E})$ we have $Form(\mathcal{E}) \models \alpha$. Second, since $\breve{\Pi} = \mathrm{MOD}(Form(\mathcal{E}) \cup Supp(\mathcal{E}))$, we also have $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \not\models \neg(\gamma \wedge \beta)$. Of course, $Form(\mathcal{E}) \models \alpha$ and $Form(\mathcal{E}) \cup Supp(\mathcal{E}) \not\models \neg(\gamma \wedge \beta)$ implies $\frac{\alpha : \beta}{\gamma} \in GD_D^{\mathcal{E}}$, a contradiction. ■

Proof 5.3.3 (Completeness) Let $(\Pi, \breve{\Pi})$ be a $\succcurlyeq_D$-maximal element above $(\Pi_{\mathcal{W}}, \breve{\Pi}_{\mathcal{W}})$. for $(D, \mathcal{W})$. Then, we have that $(\Pi, \breve{\Pi}) = (\bigcap_{i=0}^{\infty} \Pi_i, \bigcap_{i=0}^{\infty} \breve{\Pi}_i)$ where $\langle (\Pi_i, \breve{\Pi}_i) \rangle_{i \in I}$ is a sequence of focused models structures defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition C.1.2 such that $(\Pi_{i+1}, \breve{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \breve{\Pi}_i)$ for $i \in I$.

Let $\mathcal{F}$ be a set of assertions induced by $\langle \delta_i \rangle_{i \in I}$, ie. $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$ such that $\mathcal{F}_0 = \mathcal{W}$ and for each $i \geq 0$

$$\mathcal{F}_{i+1} = \widehat{Th}(\mathcal{F}_i) \cup \left\{ \langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \;\middle|\; \delta_i = \frac{\alpha : \beta}{\gamma},\ \langle \alpha, Supp(\alpha) \rangle \in \mathcal{F}_i \right\}$$

Observe that due to our construction of $\mathcal{F}$ we have $\Pi = \{\pi \mid \pi \models Form(\mathcal{F})\}$ and $\breve{\Pi} = \{\pi \mid \pi \models Form(\mathcal{F}) \cup Supp(\mathcal{F})\}$. In particular, we have $\Pi_i = \{\pi \mid \pi \models Form(\mathcal{F}_i)\}$ and $\breve{\Pi}_i = \{\pi \mid \pi \models Form(\mathcal{F}_i) \cup Supp(\mathcal{F}_i)\}$.

[3] For readablity, we abbreviate $\exists \pi \in \breve{\Pi}.\pi \models \beta \wedge \gamma$ by $\breve{\Pi} \not\models \beta \wedge \gamma$.

It remains to be shown that $\mathcal{F}$ is an assertional extension of $(D, \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\})$. According to [Brewka, 1991b, Proposition 1] we have that $\mathcal{F}$ is an assertional extension iff $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{E}_i$ such that $\mathcal{E}_0 = \{\langle\alpha, \emptyset\rangle \mid \alpha \in W\}$ and for each $i \geq 0$

$$\begin{aligned}\mathcal{E}_{i+1} \;\; = \;\; & \widehat{Th}(\mathcal{E}_i) \cup \{\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \mid \tfrac{\alpha : \beta}{\gamma} \in D, \\ & \langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}_i, \mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}\end{aligned}$$

We have to regard the following two cases.

1. $\bigcup_{i=0}^{\infty} \mathcal{E}_i \subseteq \mathcal{F}$. Therefore, we show by induction that $\mathcal{E}_i \subseteq \mathcal{F}$ for $i \geq 0$.

   Base Clearly, we have $\mathcal{E}_0 \subseteq \mathcal{F}$ since $\mathcal{E}_0 = \mathcal{F}_0$.

   Step Let $\mathcal{E}_i \subseteq \mathcal{F}$. Regard $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \mathcal{E}_{i+1}$.

   (a) If $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \widehat{Th}(\mathcal{E}_i)$ then by the induction hypothesis and the fact that $\mathcal{F} = \widehat{Th}(\mathcal{F})$ we also have $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \mathcal{F}$.

   (b) Otherwise, there is a default rule $\frac{\alpha : \beta}{\gamma} \in D$ where $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}_i$ and $\mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$.
   By the induction hypothesis $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{F}$. That is, $\Pi \models \alpha$ Also, by compactness there exists a k such that $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{F}_k$.
   By definition, $\check{\Pi} = \mathrm{MOD}(\mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}))$. Thus, since $\check{\Pi}$ is non-empty, $\mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ implies $\check{\Pi} \not\models \neg(\beta \wedge \gamma)$. By the $\succcurlyeq_D$-maximality of $(\Pi, \check{\Pi})$ we conclude $\Pi \models \gamma$ and $\check{\Pi} \models \beta \wedge \gamma$. Hence, there is a $j \geq k$ such that $(\Pi_{j+1}, \check{\Pi}_{j+1}) \succcurlyeq_{\delta_j} (\Pi_j, \check{\Pi}_j)$. Therefore, by definition, $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{F}_{j+1}$. By monotonicity, $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{F}$.

   From the two cases, we obtain $\mathcal{E}_{i+1} \subseteq \mathcal{F}$.

2. $\mathcal{F} \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Therefore, we show by induction that $\mathcal{F}_i \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$ for $i \geq 0$.

   Base Clearly, we have $\mathcal{F}_0 \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$ since $\mathcal{F}_0 = \mathcal{E}_0$.

   Step Let $\mathcal{F}_i \subseteq \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Consider $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \mathcal{F}_{i+1}$.

   (a) If $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \widehat{Th}(\mathcal{F}_i)$ then by the induction hypothesis and the fact that $\bigcup_{i=0}^{\infty} \mathcal{E}_i = \widehat{Th}(\bigcup_{i=0}^{\infty} \mathcal{E}_i)$ we also have $\langle\gamma, \mathit{Supp}(\gamma)\rangle \in \bigcup_{i=0}^{\infty} \mathcal{E}_i$.

   (b) Otherwise, there is a default rule $\delta_i = \frac{\alpha : \beta}{\gamma}$ where $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{F}_i$. Then, by definition $(\Pi_{i+1}, \check{\Pi}_{i+1}) \succcurlyeq_{\delta_i} (\Pi_i, \check{\Pi}_i)$. As a consequence, $\check{\Pi}_{i+1} \models \beta \wedge \gamma$.
   By the induction hypothesis $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Then, by compactness there exists a k such that $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}_k$. By definition, $\check{\Pi}_{i+1} \models \beta \wedge \gamma$. Since $\check{\Pi} = \bigcap_{i=0}^{\infty} \check{\Pi}_i$, we have $\check{\Pi} \models \beta \wedge \gamma$. Also, by definition, $\check{\Pi} \models \mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F})$. Hence, $\mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ since $\check{\Pi}$ is non-empty. From $\langle\alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{E}_k$ and $\mathit{Form}(\mathcal{F}) \cup \mathit{Supp}(\mathcal{F}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ we conclude $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}_{k+1}$. By monotonicity, $\langle\gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{E}$.

■

■

Constrained default logic versus Theorist

Theorem 5.4.1 *Let W, E, C, C_B and Δ be sets of formulas and let*

$$D = \left\{ \frac{:\beta}{\beta} \;\middle|\; \beta \in \Delta \right\}.$$

Then, (E,C) is a pre-constrained extension of (D, W, C_B) iff E is a Theorist extension of (W, Δ, C_B).

Proof 5.4.1 According to [Poole, 1988], E is a Theorist extension of (W, Δ, C_B) iff

$$E = Th(W \cup \Delta')$$

for a maximal set of formulas $\Delta' \subseteq \Delta$ such that $W \cup \Delta' \cup C_B \not\vdash \bot$.

As a corollary to Theorem 4.6.1 and Definition 4.8.1, we have that (E,C) is a pre-constrained extension of a prerequisite-free normal default theory (D, W) iff

$$E = Th(W \cup Conseq(D'))$$

$$C = Th(W \cup C_B \cup Conseq(D'))$$

for a maximal set of default rules $D' \subseteq D$ such that $W \cup Conseq(D') \cup C_B \not\vdash \bot$.

Clearly, both characterizations coincide since

$$D = \left\{ \frac{:\beta}{\beta} \;\middle|\; \beta \in \Delta \right\}$$

and, therefore, $Conseq(D) = \Delta$ and $Conseq(D') = \Delta'$. ■

Appendix E

Proofs of Theorems in Chapter 6

This chapter presents the proofs of the theorems given in Chapter 6.

Coherence of the definition

Theorem 6.2.1 *The empty class of K-models is never preferred wrt (D, W) whenever W is consistent.*

Proof 6.2.1 Assume that $\mathfrak{M}_\emptyset \succ_D \mathfrak{M}_W$. By definition, there then exists a subset $D' = \{\delta_0, \delta_1, \ldots\}$ of D such that $\mathfrak{M}_\emptyset = \{\mathfrak{m} \mid \mathfrak{m} \models W \wedge \Box W \wedge \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$ for all $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}\}$. By compactness, there is a finite set $\{W \wedge \Box W \wedge \gamma_0 \wedge \Box(\gamma_0 \wedge \beta_0) \wedge \ldots \wedge \gamma_k \wedge \Box(\gamma_k \wedge \beta_k)\}$ which is inconsistent. By Corollary E.1.39, $\{W \wedge \gamma_0 \wedge \ldots \wedge \gamma_k\}$ is inconsistent. That is, $W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1} \models \neg\gamma_k$. By modal logic K, $\Box(W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1}) \models \Box\neg\gamma_k$ and $\Box(W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1}) \models \Box\neg(\gamma_k \wedge \beta_k)$. Then, it cannot be the case that $\mathfrak{M}_{k+1} \succ_{\delta_k} \mathfrak{M}_k$ because $\mathfrak{M}_j = \{\mathfrak{m} \mid \mathfrak{m} \models W \wedge \Box W \wedge \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$ for all $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$ such that $i < j\}$. Therefore, there is no such k and D' is empty. So, $\mathfrak{M}_\emptyset = \mathfrak{M}_W$ and, by Corollary E.1.39, W is inconsistent, a contradiction. ■

In the sequel, we frequently employ the following definition.

Definition E.1.4 *Let (D, W) be a default theory. Given a possibly infinite sequence of default rules $\Delta = \langle \delta_0, \delta_1, \delta_2, \ldots \rangle$ in D, also denoted $\langle \delta_i \rangle_{i \in I}$ where I is the index set for Δ, we define a sequence of classes of K-models $\langle \mathfrak{M}_i \rangle_{i \in I}$ as follows:*

$$\begin{aligned} \mathfrak{M}_0 &= \mathfrak{M}_W \\ \mathfrak{M}_{i+1} &= \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \odot\beta_i\}, \quad \textit{where } \delta_i = \tfrac{\alpha_i : \beta_i}{\gamma_i}. \end{aligned}$$

In constrained default logic, $\odot$ is $\Box$. In classical and justified default logic, $\odot$ is $\Diamond$.

We will be more liberal here about the orders $\succ_\delta, >_\delta, \rhd_\delta$ by relaxing the condition that $\mathfrak{M} \succ_\delta \mathfrak{M}'$ (similarly $\mathfrak{M} >_\delta \mathfrak{M}'$ and $\mathfrak{M} \rhd_\delta \mathfrak{M}'$) holds only if $\mathfrak{M}$ and $\mathfrak{M}'$ are distinct. That is, there will be cases where $\mathfrak{M} \succ_\delta \mathfrak{M}$ (similarly $\mathfrak{M} >_\delta \mathfrak{M}$ and $\mathfrak{M} \rhd_\delta \mathfrak{M}$) be true. Clearly, this does not affect the issues under consideration.

Correctness and completeness for constrained default logic

Theorem 6.2.2 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K-models and E, C deductively closed sets of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$. Then,*

(E,C) is a constrained extension of (D,W) iff $\mathfrak{M}$ is a $\succ_D$-maximal class above $\mathfrak{M}_W$.

Proof 6.2.2 The unsatisfiable case is easily dealt with, so that we prove below the theorem for E and C being satisfiable.

Proof 6.2.2 (Correctness) Assume (E,C) is a consistent constrained extension of (D,W). The set of generating default rules for (E,C) wrt D is defined as $GD_D^{(E,C)} = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. As has been shown in Theorem 4.3.6, then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,C)}$ such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{E.1}$$

Let $\langle \mathfrak{M}_i \rangle_{i \in I}$ be a sequence of classes of K-models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i \in I} \mathfrak{M}_i$ and is $\succ_D$-maximal above $\mathfrak{M}_W$.

Since (E,C) is a constrained extension, it has been proven in Theorem 4.3.5 that

$$\begin{aligned} E &= \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big), \\ C &= \mathit{Th}\Big(W \cup \mathit{Justif}\left(GD_D^{(E,C)}\right) \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big). \end{aligned}$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$ we have obviously that $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$.

First, let us show that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (E.1), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg(\gamma_k \wedge \beta_k)$ for $\delta_k = \frac{\alpha_k:\beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\}) \cup \mathit{Justif}(\{\delta_0, \ldots, \delta_{k-1}\})$ by C^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box C^k\}$. Since E is satisfiable, so is E^k. By applying Corollary E.1.43 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg(\gamma_k \wedge \beta_k)$ we obtain that $C^k \models \neg(\gamma_k \wedge \beta_k)$. That is, $C^k \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$. By monotonicity, $C \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$, contradictory to the fact that $\delta_k \in GD_D^{(E,C)}$.

Therefore, $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i \in I} \mathfrak{M}_i \succ_{GD_D^{(E,C)}} \mathfrak{M}_W$. That is, $\mathfrak{M} \succ_D \mathfrak{M}_W$.

Second, assume $\mathfrak{M}$ is not $\succ_D$-maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^{(E,C)}$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg(\gamma \wedge \beta)$. First, applying Corollary E.1.39 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, since $\mathfrak{M} \models E \wedge \Box C$, we get by monotonicity $\Box C \not\models \Box\neg(\gamma \wedge \beta)$, yielding $C \not\models \neg(\gamma \wedge \beta)$ by modal logic K. Of course, $E \models \alpha$ and $C \not\models \neg(\gamma \wedge \beta)$ implies $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$, a contradiction. ■

Proof 6.2.2 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$ is a $\succ_D$-maximal class of K-models above $\mathfrak{M}_W$.

Let us first establish a useful characterization of C, namely $\hat{C} = \{\eta \text{ non-modal} \mid \mathfrak{M} \models \Box\eta\}$. Obviously, $C \subseteq \hat{C}$. So, $\hat{C} \models C$. In order to prove the converse, notice that $\mathfrak{M} \models \Box\hat{C}$. Since E is satisfiable, $C \models \hat{C}$, by Corollary E.1.43. Since C and $\hat{C}$ are deductively closed, $C = \hat{C}$.

According to Theorem 4.3.1 (E,C) is a constrained extension iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = \mathit{Th}(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} C_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

First, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box C_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box C_0$. Since $\mathfrak{M} \succ_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box C_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box C_i$.

Consider $\eta \in E_{i+1} \cup C_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in Th(C_i)$. By the induction hypothesis, $\mathfrak{M} \models \Box\eta$.
3. $\eta \in \left\{ \beta, \gamma \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. That is, η is either γ or β such that there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ with $\alpha \in E_i$ and $\neg(\gamma \wedge \beta) \notin C$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Using the above characterization $\hat{C}$ of C, we have $\mathfrak{M} \not\models \Box\neg(\gamma \wedge \beta)$. Since $\mathfrak{M}$ is $\succ_D$-maximal, then $\mathfrak{M} \models \gamma \wedge \Box(\gamma \wedge \beta)$ must hold and both cases for η are covered.

From the three cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box C_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box C_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Second, since $\mathfrak{M}$ is a $\succ_D$-maximal class above $\mathfrak{M}_W$ for (D, W), then $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$ where $\langle \mathfrak{M}_i \rangle_{i \in I}$ is a sequence of classes of K-models defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4 such that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $E_0 = C_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$.

Since $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg(\gamma_i \wedge \beta_i)$ where $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. By Corollary E.1.39, $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Box(\gamma_i \wedge \beta_i)$, hence $\mathfrak{M} \models \Box(\gamma_i \wedge \beta_i)$ because $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. So, $\gamma_i \wedge \beta_i \in C$. Since C is satisfiable, $\neg(\gamma_i \wedge \beta_i) \notin C$. From $E_k \models \alpha_i$ and $\neg(\gamma_i \wedge \beta_i) \notin C$, we conclude that $\gamma_i \in E_{k+1}$ and $\gamma_i \wedge \beta_i \in C_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} C_i\}$. As a consequence, $\mathfrak{N} \models E$. By Corollary E.1.39, $\bigcup_{i=0}^{\infty} E_i \models E$. Clearly, the converse can be proved in a similar way. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Returning to $\mathfrak{M} = \mathfrak{N}$, we have $\mathfrak{N} \models \Box C$. Now, $\bigcup_{i=0}^{\infty} E_i$ is satisfiable since E is. Applying Corollary E.1.43, $\bigcup_{i=0}^{\infty} C_i \models C$. Again, the converse can be proved in a similar way. Then, $\bigcup_{i=0}^{\infty} C_i = C$ because $\bigcup_{i=0}^{\infty} C_i$ and C are both deductively closed sets of formulas.

Then, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ and according to Theorem 4.3.1 this means (E,C) is a constrained extension of (D, W). ■

■

Correctness and completeness for classical default logic

Theorem 6.3.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K-models and E be a deductively closed set of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$. Then,*

E is a consistent classical extension of (D, W) iff $\mathfrak{M}$ is a $>_D$-maximal non-empty class above $\mathfrak{M}_W$.

Proof 6.3.1

Proof 6.3.1 (Correctness) Assume E is a consistent classical extension of (D, W). The set of generating default rules for E wrt D is defined as $GD_D^E = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ \neg\beta \notin E \right\}$. As has been shown in [Schwind and Risch, 1991], then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of GD_D^E such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{E.2}$$

Let $\langle \mathfrak{M}_i \rangle_{i \in I}$ be a sequence of classes of K-models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i \in I} \mathfrak{M}_i$ and is $>_D$-maximal above $\mathfrak{M}_W$.

Since E is a classical extension, it has been proven in [Reiter, 1980] that

$$E \quad = \quad Th(W \cup \mathit{Conseq}(GD_D^E)).$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$ and $C_E = \mathit{Justif}(GD_D^E)$ we have obviously that $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Clearly, $E \wedge \beta$ is satisfiable for each $\beta \in C_E$.

First, let us show that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (E.2), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg\beta_k$ for $\delta_k = \frac{\alpha_k : \beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and $\mathit{Justif}(\{\delta_0, \ldots, \delta_{k-1}\})$ by C^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box E^k \wedge \Diamond C^k\}$. Since $E^k \subseteq E$ and $C^k \subseteq C_E$, we have that $E^k \wedge \eta$ is satisfiable for each $\eta \in C_E$ and we can apply Corollary E.1.42 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg\beta_k$. We obtain that $E^k \models \neg\beta_k$. By monotonicity, $E \models \neg\beta_k$. Since E is deductively closed we have $\neg\beta_k \in E$, contradictory to the fact that $\delta_k \in GD_D^E$.

Therefore, $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i \in I} \mathfrak{M}_i >_{GD_D^E} \mathfrak{M}_W$. That is, $\mathfrak{M} >_D \mathfrak{M}_W$.

Second, assume $\mathfrak{M}$ is not $>_D$-maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^E$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg\beta$. As noted above, $E \wedge \eta$ is satisfiable for each $\eta \in C_E$. First, applying Corollary E.1.38 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, since $\mathfrak{M} \models E \wedge \Box E \wedge \Diamond C_E$, we get by monotonicity $\Box E \not\models \Box\neg\beta$, yielding $E \not\models \neg\beta$ by modal logic K. Of course, $E \models \alpha$ and $E \not\models \neg\beta$ implies $\frac{\alpha:\beta}{\gamma} \in GD_D^E$, a contradiction.

Third, assume $\mathfrak{M}$ is empty. Then, $\mathfrak{M} \models \Box\bot$. From the definition of $\mathfrak{M}$ and the fact that $E \wedge \eta$ is satisfiable for each $\eta \in C_E$, Corollary E.1.42 yields $E \models \bot$. This contradicts the consistency of E. ■

Proof 6.3.1 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$ is a non-empty $>_D$-maximal class of K-models above $\mathfrak{M}_W$.

According to [Reiter, 1980] E is a classical extension iff $E = \bigcup_{i=0}^{\infty} E_i$ such that $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Define $C_0 = \emptyset$, and for $i \geq 0$

$$C_{i+1} = \qquad\qquad \left\{ \beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond \bigcup_{i=0}^{\infty} C_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $E = \bigcup_{i=0}^{\infty} E_i$.

First, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box E_0 \wedge \Diamond C_0$. Since $\mathfrak{M} >_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box E_0 \wedge \Diamond C_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box E_i \wedge \Diamond C_i$.

Consider $\eta \in E_{i+1} \cup C_{i+1}$. Then, one of the two following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in \{\beta, \gamma\}$ for some $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $\neg\beta \notin E$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Assume $\mathfrak{M} \models \Box\neg\beta$. Since E is deductively closed, we obtain, by definition of C_E, that $E \wedge \eta$ is satisfiable for each $\eta \in C_E$. So, Corollary E.1.42 applies to $\mathfrak{M}$ and $\mathfrak{M} \models \Box\neg\beta$. As a result, $E \models \neg\beta$. Then, it follows that $\neg\beta \in E$, a contradiction. So, $\mathfrak{M} \not\models \Box\neg\beta$. Since $\mathfrak{M}$ is $>_D$-maximal, then $\mathfrak{M} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta$ must hold and both cases for η are covered.

From the two cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box E_{i+1} \wedge \Diamond C_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Second, since $\mathfrak{M}$ is a $>_D$-maximal class above $\mathfrak{M}_W$ for (D, W), then $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$ where $\langle \mathfrak{M}_i \rangle_{i \in I}$ is a sequence of classes of K-models defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4 such that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $C_0 \subseteq E_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$.

Since $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg\beta_i$ where $\delta_i = \frac{\alpha_i:\beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. Suppose that $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is unsatisfiable for some $\eta \in \bigcup_{i=0}^{\infty} C_i$. Then, there is some k such that $\eta \in C_k$ and $E_k \models \neg\eta$. We have shown above that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. Then, $\mathfrak{M} \models \Box E_k \wedge \Diamond\eta$. From $E_k \models \neg\eta$, modal logic K yields $\Box E_k \models \Box\neg\eta$. Therefore, $\mathfrak{M} \models \Box\neg\eta \wedge \Diamond\eta$. Then, $\mathfrak{M}$ is empty, a contradiction. So, $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is satisfiable for each $\eta \in \bigcup_{i=0}^{\infty} C_i$. Since $\mathfrak{N} \models \alpha_i$, we can now apply Corollary E.1.38 to obtain that $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Diamond\beta_i$, hence $\mathfrak{M} \models \Diamond\beta_i$ because $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Since $\mathfrak{M}$ is non-empty, it follows from $\mathfrak{M} \models \Diamond\beta_i$ and $\mathfrak{M} \models \Box E$ by modal logic K that $E \not\models \neg\beta_i$. That is, $\neg\beta_i \notin E$. From $E_k \models \alpha_i$ and $\neg\beta_i \notin E$, we conclude that $\gamma_i \in E_{k+1}$ and $\beta_i \in C_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond \bigcup_{i=0}^{\infty} C_i\}$. Since $\mathfrak{M}$ hence $\mathfrak{N}$ is non-empty, $\Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond\beta$ is satisfiable for each $\beta \in \bigcup_{i=0}^{\infty} C_i$ (as $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$ and $\Diamond\bot \to \bot$ are valid in modal logic K). By Corollary E.1.38, $\bigcup_{i=0}^{\infty} E_i \models E$. The converse is proved in a similar way, it is just simpler. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Then, $E = \bigcup_{i=0}^{\infty} E_i$ and according to [Reiter, 1980] this means E is a consistent classical extension of (D, W) (if E were not consistent, $\mathfrak{M}$ would be empty). ■

■

Correctness and completeness for justified default logic

Theorem 6.4.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K-models, E a deductively closed set of formulas, and J a set of formulas such that $J = C_{(E,J)}$ and $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$. Then,*

E is a justified extension of (D, W) wrt J iff $\mathfrak{M}$ is a $\rhd_D$-maximal class above $\mathfrak{M}_W$.

Proof 6.4.1 The unsatisfiable case is easily dealt with, so that we prove below the theorem for $E \wedge \beta$ being satisfiable for each $\beta \in J$ (equivalently, $\mathfrak{M}$ is non-empty as can be seen from modal logic K).

Proof 6.4.1 (Correctness) Assume E is a consistent justified extension of (D, W) wrt J. The set of generating default rules for (E, J) wrt D is defined as $GD_D^{(E,J)} = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$. As has been shown in [Risch, 1992], then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,J)}$ such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{E.3}$$

Let $\langle \mathfrak{M}_i \rangle_{i \in I}$ be a sequence of classes of K-models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i \in I} \mathfrak{M}_i$ and is $\rhd_D$-maximal above $\mathfrak{M}_W$.

Since E is a justified extension wrt J, it has been proven in [Risch, 1992] that

$$\begin{aligned} E &= \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,J)}\Big)\Big), \\ J &= \mathit{Justif}\Big(GD_D^{(E,J)}\Big). \end{aligned}$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$ and $C_{(E,J)} = \mathit{Justif}\Big(GD_D^{(E,J)}\Big)$ we have obviously that $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Clearly, if $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$ then $E \wedge \gamma \wedge \eta$ is satisfiable for each $\eta \in \mathit{Justif}\Big(GD_D^{(E,J)}\Big)$.

First, let us show that $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (E.3), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg\beta_k \vee \Diamond\neg\gamma_k$ for $\delta_k = \frac{\alpha_k:\beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and

Justif$(\{\delta_0,\ldots,\delta_{k-1}\})$ by J^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box E^k \wedge \Diamond J^k\}$. Clearly, $E^k \subseteq E$ and $J^k \subseteq J$. So, E^k is satisfiable. Also, if $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$ then $E \wedge \gamma \wedge \eta$ is satisfiable for each $\eta \in J^k$. Thus, we can apply Corollary E.1.41 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg\beta_k \vee \Diamond\neg\gamma_k$ to obtain that $E^k \models \neg\beta_k \vee \neg\gamma_k$. That is, $E^k \cup \{\beta_k\} \cup \{\gamma_k\} \vdash \bot$. By monotonicity, $E \cup \{\beta_k\} \cup \{\gamma_k\} \vdash \bot$, contradictory to the fact that $\delta_k \in GD_D^{(E,J)}$.

Therefore, $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i\in I} \mathfrak{M}_i \rhd_{GD_D^{(E,J)}} \mathfrak{M}_W$. That is, $\mathfrak{M} \rhd_D \mathfrak{M}_W$.

Second, assume $\mathfrak{M}$ is not $\rhd_D$-maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^{(E,J)}$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. As noted above, $E \wedge \eta$ is satisfiable for each $\eta \in C_{(E,J)}$. First, applying Corollary E.1.38 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$ implies by the definition of $\mathfrak{M}$ and monotonicity that $\Box E \wedge \Diamond C_{(E,J)} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. Then, $\Box E \wedge \Diamond C_{(E,J)} \not\models \Diamond\neg\gamma$. By modal logic K, it follows that $E \wedge \eta \not\models \neg\gamma$ whenever $\eta \in C_{(E,J)}$. So, $E \cup \{\gamma\} \cup \{\eta\}$ is satisfiable for each $\eta \in J$ (because $J = C_{(E,J)}$). Returning to $\Box E \wedge \Diamond C_{(E,J)} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$, another consequence is $\Box E \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. That is, $\Box E \not\models \Box\gamma \rightarrow \Box\neg\beta$. By modal logic K, it follows that $E \not\models \gamma \rightarrow \neg\beta$. So, $E \cup \{\gamma\} \cup \{\beta\}$ is satisfiable. In all, $E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ whenever $\eta \in J \cup \{\beta\}$. Together with $E \models \alpha$, this implies $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$, a contradiction. ■

Proof 6.4.1 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$ is a non-empty $\rhd_D$-maximal class of K-models above $\mathfrak{M}_W$.

According to [Łukaszewicz, 1988] E is a justified extension wrt J iff $(E,J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$ such that $E_0 = W$ and $J_0 = \emptyset$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall\eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

$$J_{i+1} = \quad J_i \quad \cup \left\{ \beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall\eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\bigcup_{i=0}^{\infty} J_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $E = \bigcup_{i=0}^{\infty} E_i$ and $J = \bigcup_{i=0}^{\infty} J_i$.

First, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond J_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box E_0 \wedge \Diamond J_0$. Since $\mathfrak{M} \rhd_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box E_0 \wedge \Diamond J_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box E_i \wedge \Diamond J_i$.

Consider $\eta \in E_{i+1} \cup J_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in J_i$. By the induction hypothesis, $\mathfrak{M} \models \Diamond\eta$.
3. $\eta \in \{\beta,\gamma\}$ for some $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $E \cup \{\gamma\} \cup \{v\} \not\vdash \bot$ for all $v \in J \cup \{\beta\}$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Assume $\mathfrak{M} \models \Box\neg\beta \vee \Diamond\neg\gamma$. By definition of $C_{(E,J)}$, we obtain that $E \wedge v$ is satisfiable for each $v \in C_{(E,J)}$. Also E is satisfiable. So, Corollary E.1.41 applies to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \Box\neg\beta \vee \Diamond\neg\gamma$. As a result, $E \models \neg\beta \vee \neg\gamma$. This contradicts the fact that $E \cup \{\gamma\} \cup \{v\} \not\vdash \bot$ for all $v \in J \cup \{\beta\}$. So, $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. Since $\mathfrak{M}$ is $\rhd_D$-maximal, then $\mathfrak{M} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta$ must hold and both cases for η are covered.

From the three cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box E_{i+1} \wedge \Diamond J_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond J_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Second, since $\mathfrak{M}$ is a $\triangleright_D$-maximal class above $\mathfrak{M}_W$ for (D, W), then $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$ where $\langle \mathfrak{M}_i \rangle_{i \in I}$ is a sequence of classes of K-models defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition E.1.4 such that $\mathfrak{M}_{i+1} \triangleright_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $J_0 \subseteq E_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$.

Since $\mathfrak{M}_{i+1} \triangleright_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg\beta_i \vee \Diamond\neg\gamma_i$ where $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. Suppose that $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is unsatisfiable for some $\eta \in \bigcup_{i=0}^{\infty} J_i$. Then, there is some k such that $\eta \in J_k$ and $E_k \models \neg\eta$. We have shown above that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. Then, $\mathfrak{M} \models \Box E_k \wedge \Diamond\eta$. From $E_k \models \neg\eta$, modal logic K yields $\Box E_k \models \Box\neg\eta$. Therefore, $\mathfrak{M} \models \Box\neg\eta \wedge \Diamond\eta$. Then, $\mathfrak{M}$ is empty, a contradiction. So, $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is satisfiable for each $\eta \in \bigcup_{i=0}^{\infty} J_i$. Since $\mathfrak{N} \models \alpha_i$, we can now apply Corollary E.1.38 to obtain that $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Box\gamma_i \wedge \Diamond\beta_i$, hence $\mathfrak{M} \models \Box\gamma_i \wedge \Diamond\beta_i$ because $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Since $\mathfrak{M}$ is non-empty, it follows from $\mathfrak{M} \models \Box\gamma_i \wedge \Diamond\beta_i$ and $\mathfrak{M} \models \Box E$ by modal logic K that $E \wedge \gamma_i \not\models \neg\beta_i$. That is, $E \cup \{\gamma_i\} \cup \{\beta_i\} \not\vdash \bot$. Also, since $\mathfrak{M}$ is non-empty, it follows from $\mathfrak{M} \models \Box\gamma_i$ and $\mathfrak{M} \models \Box E \wedge \Diamond C_{(E,J)}$ by modal logic K that $\mathfrak{M} \models \Diamond(E \wedge \gamma_i \wedge \eta)$ for $\eta \in C_{(E,J)}$. That is, $E \cup \{\gamma_i\} \cup \{\eta\} \not\vdash \bot$ for $\eta \in J$ (because $J = C_{(E,J)}$). From $E_k \models \alpha_i$ and $E \cup \{\gamma_i\} \cup \{\eta\} \not\vdash \bot$ for $\eta \in J \cup \{\beta_i\}$, we conclude that $\gamma_i \in E_{k+1}$ and $\beta_i \in J_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond \bigcup_{i=0}^{\infty} J_i\}$. Since $\mathfrak{M}$ hence $\mathfrak{N}$ is non-empty, $\Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond\beta$ is satisfiable for each $\beta \in \bigcup_{i=0}^{\infty} J_i$ (as $\Box p \wedge \Diamond q \rightarrow \Diamond(p \wedge q)$ and $\Diamond\bot \rightarrow \bot$ are valid in modal logic K). By Corollary E.1.38, $\bigcup_{i=0}^{\infty} E_i \models E$. The converse is proved in a similar way. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Since $E = \bigcup_{i=0}^{\infty} E_i$, the definitions of $C_{(E,J)}$ and J_i make it easy to verify that $C_{(E,J)} = \bigcup_{i=0}^{\infty} J_i$ That is, $J = \bigcup_{i=0}^{\infty} J_i$.

Then, $E = \bigcup_{i=0}^{\infty} E_i$ and $J = \bigcup_{i=0}^{\infty} J_i$, and according to [Łukaszewicz, 1988] this means E is a justified extension of (D, W) wrt J. ■

■

Some modal propositions

Proposition E.1.37 *Let p, q, r, $s_1, \ldots, s_n$ be non-modal formulas such that $q \wedge s_i$ is satisfiable for $i = 1, \ldots, n$.*

If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow r$ then $\models p \rightarrow r$.

Proof E.1.37 Assume the contrary. Then, $p \wedge \neg r$ is satisfiable. It is thus possible to define the K-model $\mathfrak{m} = \langle \omega_0, \{\omega_i \mid i = 0, \ldots, n\}, \{(\omega_0, \omega_i) \mid i = 1, \ldots, n\}, \mathcal{I} \rangle$ such that $\omega_0 \models p \wedge \neg r$ and $\omega_i \models q \wedge s_i$ for $i = 1, \ldots, n$. Clearly, $\mathfrak{m}$ contradicts the validity of $p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow r$ even in the limiting case where $n = 0$. ■

Corollary E.1.38 *Let S, T, U and V be sets of non-modal formulas and $T \wedge u$ is satisfiable for each $u \in U$.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models V$ then $S \models V$.

Proof E.1.38 Consider $v \in V$. $\mathfrak{M} \models v$ means $S \wedge \Box T \wedge \Diamond U \models v$. By compactness, $S' \wedge \Box T' \wedge \Diamond U' \models v$ for some finite subsets S', T' and U' of S, T and U, respectively. Since the deduction theorem for material implication holds in modal logic K, we get $\models S' \wedge \Box T' \wedge \Diamond U' \rightarrow v$. Applying Proposition E.1.37, $\models S' \rightarrow v$. That is, $S' \models v$. By monotonicity, $S \models v$. So, $S \models V$. ∎

Corollary E.1.39 *Let S, T and V be sets of non-modal formulas.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T\}$ and $\mathfrak{M} \models V$ then $S \models V$.

Proof E.1.39 Apply Corollary E.1.38 in the limiting case where U is empty ($n = 0$ in Proposition E.1.37). ∎

Proposition E.1.40 *Let p, q, r, $s_1, \ldots, s_n$, t be non-modal formulas, with p and $q \wedge s_i \wedge \neg t$ satisfiable for $i = 1, \ldots, n$.*
If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Box r \vee \Diamond t$ then $\models q \rightarrow r \vee t$.

Proof E.1.40 Assume the contrary. Then, $q \wedge \neg r \wedge \neg t$ is satisfiable. Define the K-model $\mathfrak{m} = \langle \omega_0, \{\omega_i \mid i = 0, \ldots, n+1\}, \{(\omega_0, \omega_i) \mid i = 1, \ldots, n+1\}, \mathcal{I}\rangle$ with $\mathcal{I}$ as follows. Let $\omega_0 \models p$. Let $\omega_{n+1} \models q \wedge \neg r \wedge \neg t$. For $i = 1, \ldots, n$, let $\omega_i \models q \wedge s_i \wedge \neg t$. Then, $\mathfrak{m}$ contradicts the validity of $p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Box r \vee \Diamond t$ even in the limiting case where $n = 0$. ∎

Corollary E.1.41 *Let S, T and U be sets of non-modal formulas and let p and q be non-modal formulas such that S is satisfiable and $T \wedge u \wedge \neg q$ is satisfiable for each $u \in U$.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models \Box p \vee \Diamond q$ then $T \models p \vee q$.

Proof E.1.41 $\mathfrak{M} \models \Box p \vee \Diamond q$ means $S \wedge \Box T \wedge \Diamond U \models \Box p \vee \Diamond q$. By compactness, $S' \wedge \Box T' \wedge \Diamond U' \models \Box p \vee \Diamond q$ for some finite subsets S', T' and U' of S, T and U, respectively. Since the deduction theorem for material implication holds in modal logic K, we get $\models S' \wedge \Box T' \wedge \Diamond U' \rightarrow \Box p \vee \Diamond q$. Applying Proposition E.1.40, $\models T' \rightarrow p \vee q$. That is, $T' \models p \vee q$. Accordingly, $T \models p \vee q$. ∎

Corollary E.1.42 *Let S, T, U and V be sets of non-modal formulas such that S is satisfiable and $T \wedge u$ is satisfiable for each $u \in U$.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models \Box V$ then $T \models V$.

Proof E.1.42 Consider $v \in V$. Then, $\mathfrak{M} \models \Box v$. Since $\perp$ and $\Diamond \perp$ are equivalent in modal logic K, $\mathfrak{M} \models \Box v \vee \Diamond \perp$. Applying Corollary E.1.41, $T \models v \vee \perp$. That is, $T \models v$. Accordingly, $T \models V$. ∎

Corollary E.1.43 *Let S, T and U be sets of non-modal formulas such that S is satisfiable.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T\}$ and $\mathfrak{M} \models \Box V$ then $T \models V$.

Proof E.1.43 Apply Corollary E.1.42 in the limiting case where U is empty ($n = 0$ in Proposition E.1.40). ∎

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