A context-based framework for default logics

Philippe Besnard
IRISA, Campus de Beaulieu
F-35042 Rennes Cédex, France
besnard@irisa.fr

Torsten Schaub
TH Darmstadt, Alexanderstraße 10
D-6100 Darmstadt, Germany
torsten@intellektik.informatik.th-darmstadt.de

Abstract

We present a new context-based approach to default logic, called contextual default logic. The approach extends the notion of a default rule and supplies each extension with a context. Contextual default logic allows for embedding all existing variants of default logic along with more traditional approaches like the closed world assumption. A key advantage of contextual default logic is that it provides a syntactical instrument for comparing existing default logics in a unified setting. In particular, it reveals that existing default logics mainly differ in the way they deal with an explicit or implicit underlying context.

Introduction

Default logic has become the prime candidate for formalizing consistency-based default reasoning since its introduction in [Reiter,1980]. Since then, several variants of default logic have been proposed, eg. [Łukaszewicz,1988; Brewka,1991; Delgrande *et al.*,1992]. Each such variant rectified purportedly counterintuitive features of the original approach. However, the evolution of default logic is diverging. Although it has resulted in diverse variants sharing many interesting properties, it has altered the notion of a default rule. In particular, most of the aforementioned variants deal with a different notion of consistency. For instance, Reiter's default logic employs some sort of local consistency, whereas others employ some sort of global consistency.

Up to now, we are then compelled to choose among one of the respective variants whenever we want to represent default knowledge. At first sight, this seems to be a good solution, since we may select one of the variants depending on its properties. However, our choice fixes the notion of a default rule. More freedom would be desirable: We should not be forced to commit ourselves to just a single variant of default logic, because all facets of default logic are worth considering. Instead, an integrated approach is proposed below, which is based on a very general notion of a default rule.

Thus, the primary purpose of this work is to integrate the different variants of default logic in a more general but uniform system, which combines the expressiveness of the various default logics. The basic idea is twofold. First, we supply each default extension (ie. a set of default conclusions) with an underlying context. Second, we extend the notion of a default rule in order to allow for a variety of different application conditions which arise naturally from the distinction between the initial set of facts, the default extension at hand, and its context.

Notions of consistency in default logics

Classical default logic was defined by Reiter in [1980] as a formal account of reasoning in the absence of complete information. It is based on first-order logic, whose sentences are hereafter simply referred to as formulas (instead of closed formulas). In default logics, default knowledge is incorporated by means of so-called *default rules*. A default rule is any expression of the form $\frac{\alpha:\beta}{\gamma}$, where α, β and γ are formulas. α is called the *prerequisite*, β the *justification*, and γ the *consequent* of the default rule. Accordingly, a *default theory* (D, W) consists of a set of formulas W and a set of default rules D. Informally, an *extension* of the initial set of facts W is defined as the set of all formulas derivable from W by applying classical inference rules and all applicable default rules. Usually, a default rule $\frac{\alpha:\beta}{\gamma}$ is applicable, if its prerequisite α is derivable and its justification β is *consistent* in a certain way.

In all "conventional" default logics, the prerequisite α of a default rule $\frac{\alpha:\beta}{\gamma}$ is checked wrt an extension E by requiring $\alpha \in E$. However, all of the aforementioned variants differ in the way they account for the consistency of the justification β. For instance, in classical default logic [Reiter,1980] the consistency of the justification β is checked wrt the extension E by $\neg\beta \notin E$, whereas in constrained default logic [Delgrande *et al.*,1992] the same is done wrt a set of constraints C, containing the extension E, by checking $\neg\beta \notin C$.

In default logics, there are thus two extreme notions of consistency: Individual and joint consistency. The former one is employed in classical default logic, whereas the latter can be found in cumulative and constrained default logic. Individual consistency requires that no justification of an applying default rule is contradictory with a given extension, whereas joint consistency stipulates that all justifications of all applying default rules are jointly consistent with the extension at hand. As an example, consider the default theory

$$\left(\left\{\frac{:\mathsf{B}}{\mathsf{C}}, \frac{:\neg\mathsf{B}}{\mathsf{D}}\right\}, \emptyset\right). \quad (1)$$

In classical default logic, this default theory has one extension $Th(\{\mathsf{C},\mathsf{D}\})$. Both default rules apply, although they have contradictory justifications. This is because each justification is separately consistent with $Th(\{\mathsf{C},\mathsf{D}\})$. In this case, the extension is somehow embedded in a "context" which gathers two incompatible "subcontexts": One containing the extension and the justification of the first default rule, $Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\})$, and another one containing the justification of the second default rule, $Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})$.

This is different from the approach taken in constrained default logic. There, we obtain two constrained extensions. We obtain one extension $Th(\{\mathsf{C}\})$ which is supplied with a set of constraints $Th(\{\mathsf{C},\mathsf{B}\})$ consisting of the justification B and the consequent C of the first default rule. We also obtain another extension $Th(\{\mathsf{D}\})$ whose constraints $Th(\{\mathsf{D},\neg\mathsf{B}\})$ contain the justification $\neg\mathsf{B}$ and the consequent D of the second default rule. Each set of constraints contains the extension and additionally all justifications of all applying default rules. Thus, each extension is embedded in a "context" given by the set of constraints.

In order to combine the variants of default logic, we have to compromise the notions of individual and joint consistency. In particular, we have to deal with joint consistency requirements in the presence of inconsistent individual consistency requirements. Therefore, we allow for "contexts" containing contradictory formulas, like B and $\neg\mathsf{B}$ as in the previous example in classical default logic, without containing all possible formulas. Thus, we admit contexts which are not deductively closed. In the previous example, the extension $Th(\{\mathsf{C},\mathsf{D}\})$ will then have the context $Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\}) \cup Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})$, which is composed of two incompatible subcontexts. A useful notion is then that of *pointwise closure* $Th_S(T)$.

Definition 1 *Let T and S be sets of formulas. The pointwise closure of T under S is defined as* $Th_S(T) = \bigcup_{\phi\in T} Th(S \cup \{\phi\})$.

If S is a singleton set $\{\varphi\}$, we simply write $Th_\varphi(T)$ instead of $Th_{\{\varphi\}}(T)$. Given two sets of formulas T and S, we say that T is *pointwisely closed under* S iff $T = Th_S(T)$. We simply say that T is pointwisely closed whenever $T = Th_\top(T)$ for any tautology $\top$.

Observe that the aforementioned context can now be represented as the pointwise closure of $\{\mathsf{B},\neg\mathsf{B}\}$ under $\{\mathsf{C},\mathsf{D}\}$, namely $Th_{\{\mathsf{C},\mathsf{D}\}}(\{\mathsf{B},\neg\mathsf{B}\})$.

Contextual default logic

We introduce a new approach to default logic by extending the notions of default rules and extensions. The resulting system is called *contextual default logic*. We consider three sets of formulas: A set of facts W, an extension E, and a certain *context* C such that $W \subseteq E \subseteq C$. The set of formulas C is somehow established from the facts, the default conclusions (ie. the consequences of the applied default rules), as well as all underlying consistency assumptions (ie. the justifications of all applied default rules). That is, our approach trivially captures the above application conditions for "conventional" default rules, eg. $\alpha \in E$ and $\neg\beta \notin E$ in the case of classical default logic.

This approach allows for even more ways of forming application conditions of default rules. Consider a formula φ and three consistent, deductively closed sets of formulas W, E, and C such that $W \subseteq E \subseteq C$. Six more or less strong application conditions are obtained which can be ordered from left to right by decreasing strength; whereby $>$ is read as "implies":

$$\varphi \in W > \varphi \in E > \varphi \in C > \neg\varphi \notin C > \neg\varphi \notin E > \neg\varphi \notin W$$

We can think of W as a deductively closed set of facts, E as a default extension of W, and C as the above mentioned context for E. Then, the first condition $\varphi \in W$ stands for first-order derivability from the facts W. The second condition $\varphi \in E$ stands for derivability from W using first-order logic and certain default rules. This is used in conventional default logics as the test for the prerequisite of a default rule. The third condition, $\varphi \in C$, is the strangest one. It expresses "membership in a context of reasoning". The last three conditions are consistency conditions. The fourth condition $\neg\varphi \notin C$ corresponds to the consistency condition used in constrained default logic, the fifth one $\neg\varphi \notin E$ is used in classical default logic. Finally, the last condition $\neg\varphi \notin W$ is the one used for the closed world assumption [Reiter,1977], where it is restricted to ground negative literals.

This variety of application conditions motivates an extended notion of a default rule.

Definition 2 *A contextual default rule δ is an expression of the form*

$$\frac{\alpha_W \mid \alpha_E \mid \alpha_C : \beta_C \mid \beta_E \mid \beta_W}{\gamma}$$

where α_W, α_E, α_C, β_C, β_E, β_W, and γ are formulas. α_W, α_E, α_C are called the W-, E-, and C-prerequisites, also noted $Prereq_W(\delta)$, $Prereq_E(\delta)$, $Prereq_C(\delta)$, β_C, β_E, β_W are called the C-, E-, and W-justifications, also noted $Justif_C(\delta)$, $Justif_E(\delta)$, $Justif_W(\delta)$, and γ is called the consequent, also noted $Conseq(\delta)$. [1]

The six antecedents of a contextual default rule are to be treated along the above intuitions. Accordingly, a contextual default theory is a pair (D, W), where D is a set of contextual default rules and W is a deductively

[1]These projections extend to sets of default rules in the obvious way (eg. $Justif_E(\Delta) = \bigcup_{\delta\in\Delta}\{Justif_E(\delta)\}$).

closed[2] set of formulas.

Now, a *contextual extension* is to be a pair (E, C), where E is a deductively closed set of formulas and C is a pointwisely closed set of formulas, as follows.

Definition 3 *Let (D, W) be a contextual default theory. For any pair of sets of formulas (T, S) let $\nabla(T, S)$ be the pair of smallest sets of formulas (T', S') such that $W \subseteq T' \subseteq S'$ and the following condition holds:*

For any $\frac{\alpha_W \mid \alpha_E \mid \alpha_C : \beta_C \mid \beta_E \mid \beta_W}{\gamma} \in D$, *if*

1. $\alpha_W \in W$ 2. $\alpha_E \in T'$ 3. $\alpha_C \in S'$
4. $\neg\beta_C \notin S$ 5. $\neg\beta_E \notin T$ 6. $\neg\beta_W \notin W$

then 7. $Th_\gamma(T') \subseteq T'$
8. $Th_{\beta_E}(T') \subseteq S'$
9. $Th_{\beta_C}(S') \subseteq S'$

A pair of sets of formulas (E, C) is a contextual extension of (D, W) iff $\nabla(E, C) = (E, C)$.

Notice that the operator ∇ is in fact parameterized by (D, W). Furthermore, observe that Conditions *1-6* basically correspond to those given above.

Intuitively, we start from (W, W) (ie. we take the facts W as our initial version of E and C) and try to apply a contextual default rule by checking conditions *1-6* and, if we are successful, we enforce *7-9*, ie. we add γ to our current version of E and we add $\phi \wedge \beta_E$ and $\varphi \wedge \beta_C$ to our current version of C, for each ϕ in the final E and for each φ in the final C.

Consider the contextual default theory

$$\left(\left\{\frac{\mathsf{A} \,||\!:\!|\, \mathsf{B} \,|}{\mathsf{C}}, \frac{|\, \mathsf{C} \,|\!:\! \mathsf{E} \,|\, \neg\mathsf{B} \,|}{\mathsf{D}}\right\}, Th(\mathsf{A})\right)$$

along with its only contextual extension (E, C), where

$E \;=\; Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}\})$

$C \;=\; Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{B}\}) \cup Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \neg\mathsf{B}\})$.

E represents the extension and C provides its context. This contextual extension is generated from the facts by applying first the first contextual default rule and then the second one.

Now, $\frac{\mathsf{A}||:|\mathsf{B}|}{\mathsf{C}}$ applies if its prerequisite A is monotonically derivable (ie. if A is derivable without contextual default rules according to Condition 1 in Definition 3) and if its E-justification B is consistent with the extension E (according to Condition 5). In other words, B has to be individually consistent. This being the case, we derive C. That is, C is nonmonotonically derivable by means of the first contextual default rule (cf. Condition 2). Thus, C establishes the prerequisite of the second contextual default rule, $\frac{|\mathsf{C}|:\mathsf{E}|\neg\mathsf{B}|}{\mathsf{D}}$. In order to derive D, we have to verify the consistency of the two justifications E and ¬B, ie. E has to be jointly consistent (ie. according to Condition 4, it has to be consistent with the context C), whereas ¬B has to be individually consistent (ie. according to Condition 5, it has to be consistent with the extension E). Since this is fulfilled, we obtain the above contextual extension satisfying our consistency requirements.

[2] This is no real restriction, but it simplifies matters.

Observe that the context C is composed of two incompatible subcontexts, $Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{B}\})$ and $Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \neg\mathsf{B}\})$. All such subcontexts contain a common "kernel" given by the extension and all jointly consistent C-justifications, here $Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}\})$ and E. The E-justifications, B and ¬B, create different subcontexts. Why is the joint consistency of E not affected by these two incompatible formulas? This is because in our approach joint consistency only requires the consistency of a justification with each subcontext in turn, whereas individual consistency requires the consistency of a justification with at least one such subcontext.

Embedding default logics

We show that classical [Reiter,1980], justified [Łukaszewicz,1988] and constrained default logic [Delgrande *et al.*,1992] are embedded in contextual default logic. Since cumulative default logic [Brewka,1991] is closely connected to constrained default logic, neglecting representational issues, we obtain that variant too.

As mentioned in the introductory section, *classical default logic* employs a sort of local consistency (which we also called individual consistency), as can be seen from the following definition of *classical extensions.*

Definition 4 *Let (D, W) be a default theory. For any set of formulas T let $\Gamma(T)$ be the smallest set of formulas T' such that*

1. $W \subseteq T'$, 2. $Th(T') = T'$,
3. *For any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in T'$ *and* $\neg\beta \notin T$ *then* $\gamma \in T'$.

A set of formulas E is a classical extension of (D, W) iff $\Gamma(E) = E$.

In order to have a comprehensive example throughout the text, we extend default theory (1) by introducing an additional default rule:

$$\left(\left\{\frac{:\mathsf{B}}{\mathsf{C}}, \frac{:\neg\mathsf{B}}{\mathsf{D}}, \frac{:\neg\mathsf{C}\wedge\neg\mathsf{D}}{\mathsf{E}}\right\}, \emptyset\right) \qquad (2)$$

This default theory still has one classical extension $Th(\{\mathsf{C}, \mathsf{D}\})$. As shown above, the first two default rules apply, although they have contradictory justifications, and then block the third default rule.

In order to relate classical with contextual default logic, let us identify default theories in classical default logic with contextual default theories as follows.

Definition 5 *Let (D, W) be a default theory. Define*

$$\Phi_{\mathsf{DL}}(D, W) = \left(\left\{ \frac{|\alpha|:|\beta|}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D\right\}, Th(W)\right).$$

Then, classical default logic corresponds to this fragment of contextual default logic.

Theorem 1 *Let (D, W) be a default theory and let E be a set of formulas. Then, E is a classical extension of (D, W) iff (E, C) is a contextual extension of $\Phi_{\mathsf{DL}}(D, W)$ for some C.*

Given a classical extension E, the context C is the pointwise closure of the justifications of the generating[3] default rules under E.

[3] Informally, the generating default rules are those which apply in view of E.

Consider the contextual counterpart of default theory (2):

$$\left(\left\{\frac{||:|\,\mathsf{B}\,|}{\mathsf{C}}, \frac{||:|\,\neg\mathsf{B}\,|}{\mathsf{D}}, \frac{||:|\,\neg\mathsf{C}\wedge\neg\mathsf{D}\,|}{\mathsf{E}}\right\}, \mathit{Th}(\emptyset)\right)$$

We obtain one contextual extension $(\mathit{Th}(\{\mathsf{C},\mathsf{D}\}), \mathit{Th}(\{\mathsf{C},\mathsf{D},\mathsf{B}\}) \cup \mathit{Th}(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\}))$ whose extension corresponds to the classical extension of default theory (2). The common kernel of the two subcontexts of the context is given by the extension. In addition, the first subcontext, $\mathit{Th}(\{\mathsf{C},\mathsf{D},\mathsf{B}\}))$, contains the E-justification of the first contextual default rule, whereas the second one, $\mathit{Th}(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\}))$, contains additionally the E-justification of the second contextual default rule. As in classical default logic, the third contextual default rule is blocked by the other ones.

Further evidence for the generality of our approach is that it also captures *justified default logic* [Łukaszewicz,1988]. In this approach, the justifications of the applying default rules are attached to extensions in order to strengthen the applicability condition of default rules. A *justified extension* is defined as follows.

Definition 6 *Let (D, W) be a default theory. For any pair of sets of formulas (T, S) let $\Psi(T, S)$ be the pair of smallest sets of formulas T', S' such that*

1. $W \subseteq T'$, *2. $\mathit{Th}(T') = T'$,*

3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in T'$ and $\forall \eta \in S \cup \{\beta\}.\ T \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ then $\gamma \in T'$ and $\beta \in S'$.

A pair of sets of formulas (E, J) is a justified extension of (D, W) iff $\Psi(E, J) = (E, J)$.

Let us return to default theory (2). This default theory has two justified extensions: $(\mathit{Th}(\{\mathsf{C},\mathsf{D}\}), \{\mathsf{B},\neg\mathsf{B}\})$ and $(\mathit{Th}(\{\mathsf{E}\}), \{\neg\mathsf{C}\wedge\neg\mathsf{D}\})$. The first one corresponds to the extension obtained in classical default logic. However, it is supplied with a set of justifications, $\{\mathsf{B},\neg\mathsf{B}\}$ (which, incidentally, is inconsistent). The second extension stems from applying the third default rule whose justification blocks the two other default rules by contradicting their consequents.

Now, let us identify default theories in justified default logic with contextual default theories:

Definition 7 *Let (D, W) be a default theory. Define*

$$\Phi_{\mathsf{JDL}}(D, W) = \left(\left\{\frac{|\,\alpha\,|:\gamma\,|\,\beta\wedge\gamma\,|}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D\right\}, \mathit{Th}(W)\right).$$

This leads to the following correspondence.

Theorem 2 *Let (D, W) be a default theory and let E be a set of formulas. Then, (E, J) is a justified extension of (D, W) for some J iff (E, C) is a contextual extension of $\Phi_{\mathsf{JDL}}(D, W)$ for some C.*

J consists of the justifications of the generating[4] default rules for E, whereas C is given by the pointwise closure of the same set of justifications under E.

It is interesting to observe how the relatively complicated consistency check in justified default logic is accomplished in contextual default logic. For a justified extension (E, J) and a default rule $\frac{\alpha:\beta}{\gamma}$ the condition is $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$. In fact, it is two-fold: It consists of a joint and an individual consistency check, ie. $\forall \eta \in J.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ and $E \cup \{\gamma\} \cup \{\beta\} \not\vdash \bot$. Transposed to the case of a contextual extension (E, C) the two subconditions are $\neg\gamma \notin C$ and $\neg(\beta \wedge \gamma) \notin E$. The first check cares about the joint consistency of the consequent γ, whereas the second one checks whether the conjunction of the justification and consequent of the default rule is individually consistent.

Now, let us see what happens to default theory (2) if we apply translation Φ_{JDL}:

$$\left(\left\{\frac{||:\mathsf{C}\,|\,\mathsf{B}\wedge\mathsf{C}\,|}{\mathsf{C}}, \frac{||:\mathsf{D}\,|\,\neg\mathsf{B}\wedge\mathsf{D}\,|}{\mathsf{D}}, \frac{||:\mathsf{E}\,|\,\neg\mathsf{C}\wedge\neg\mathsf{D}\wedge\mathsf{E}\,|}{\mathsf{E}}\right\}, \mathit{Th}(\emptyset)\right)$$

As in justified default logic, we get two contextual extensions: $(\mathit{Th}(\{\mathsf{C},\mathsf{D}\}), \mathit{Th}(\{\mathsf{C},\mathsf{D},\mathsf{B}\}) \cup \mathit{Th}(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})\})$ and $(\mathit{Th}(\{\mathsf{E}\}), \mathit{Th}(\{\mathsf{E},\neg\mathsf{C},\neg\mathsf{D}\}))$, whose extensions correspond to the extensions obtained in justified default logic. Observe that the respective subcontexts differ exactly in the justifications attached to the extensions in justified default logic.

Finally, we turn to *constrained default logic* [Delgrande *et al.*,1992], which employs a sort of joint consistency. In constrained default logic, an extension comes with a set of constraints. A *constrained extension* is defined as follows.

Definition 8 *Let (D, W) be a default theory. For any set of formulas S let $\Upsilon(S)$ be the pair of smallest sets of formulas (T', S') such that*

1. $W \subseteq T' \subseteq S'$, *2. $T' = \mathit{Th}(T')$ and $S' = \mathit{Th}(S')$,*

3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in T'$ and $S \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in T'$ and $\beta, \gamma \in S'$.

A pair of sets of formulas (E, C) is a constrained extension of (D, W) iff $\Upsilon(C) = (E, C)$.

As we have seen above, constrained default logic detects inconsistencies among the justifications of default rules. Thus, we obtain three constrained extensions, $(\mathit{Th}(\{\mathsf{C}\}), \mathit{Th}(\{\mathsf{C},\mathsf{B}\}))$, $(\mathit{Th}(\{\mathsf{D}\}), \mathit{Th}(\{\mathsf{D},\neg\mathsf{B}\}))$, $(\mathit{Th}(\{\mathsf{E}\}), \mathit{Th}(\{\mathsf{E},\neg\mathsf{C},\neg\mathsf{D}\}))$, of default theory (2). They are formed as described above.

A default theory in constrained default logic is identified with a contextual default theory as follows.

Definition 9 *Let (D, W) be a default theory. Define*

$$\Phi_{\mathsf{CDL}}(D, W) = \left(\left\{\frac{|\,\alpha\,|:\beta\wedge\gamma\,||}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D\right\}, \mathit{Th}(W)\right).$$

This yields the following correspondence.

Theorem 3 *Let (D, W) be a default theory and let E and C be sets of formulas. Then, (E, C) is a constrained extension of (D, W) iff (E, C) is a contextual extension of $\Phi_{\mathsf{CDL}}(D, W)$.*

Notice that C is always deductively closed whenever (E, C) is an extension in either sense.

Consider the contextual counterpart of default theory (2) from the perspective of constrained default logic:

$$\left(\left\{\frac{||:\mathsf{B}\wedge\mathsf{C}\,||}{\mathsf{C}}, \frac{||:\neg\mathsf{B}\wedge\mathsf{D}\,||}{\mathsf{D}}, \frac{||:\neg\mathsf{C}\wedge\neg\mathsf{D}\wedge\mathsf{E}\,||}{\mathsf{E}}\right\}, \mathit{Th}(\emptyset)\right)$$

As a result, we obtain three contextual extensions: $(\mathit{Th}(\{\mathsf{C}\}), \mathit{Th}(\{\mathsf{C},\mathsf{B}\}))$, $(\mathit{Th}(\{\mathsf{D}\}), \mathit{Th}(\{\mathsf{D},\neg\mathsf{B}\}))$,

[4] In the sense of justified default logic.

and $(Th(\{\mathsf{E}\}), Th(\{\mathsf{E}, \neg\mathsf{C}, \neg\mathsf{D}\}))$. These are identical to the respective constrained extensions.

Contextual default logic: Expressiveness

This section is devoted to the novel application conditions of contextual default rules and how their interplay may influence the contents of extensions.

Let us first consider the difference between W- and E-prerequisites. In general, W-prerequisites should be preferred over E-prerequisites whenever a prerequisite has to be verified, ie. whenever it should not be derivable by default inferences. This cannot be modelled in conventional default logics, since they do not distinguish between monotonic and nonmonotonic conclusions.

As an example, consider the assertion "usually, we can transplant an organ provided that the person is proven to be dead". Of course, the antecedent of this rule should be more than merely concluded by default. For instance, a person whose body is fully covered with a medical blanket is usually dead, but it takes more evidence for doctors to remove organs. Now, the above rule can be formalized by means of the contextual default rule $\frac{\mathsf{D}\,||:\,\mathsf{O}\,||}{\mathsf{O}}$, saying that an organ, O, can be transplanted, if this is consistent with the current context, and provided that the death, D, of the person has been verified. Importantly, adding the contextual default rule $\frac{|\,\mathsf{C}\,|:|\,\mathsf{D}\,|}{\mathsf{D}}$ (saying that a person whose body is covered, C, with a blanket is usually dead, D) does not allow $\frac{\mathsf{D}\,||:\,\mathsf{O}\,||}{\mathsf{O}}$ to apply, even in the case where $W = Th(\{\mathsf{C}\})$.

C-prerequisites are a means for weakening antecedents of default rules. This is because a C-prerequisite allows us not only to refer to default conclusions but also to their underlying consistency assumptions: A C-prerequisite is satisfied iff it belongs to some subcontext. Accordingly, certain contextual default rules can only be applied if a certain context has been established. For instance, a contextual default rule $\frac{||:|\,\mathsf{A}\,|}{\mathsf{B}}$ may establish, without actually asserting, a consistency assumption A on which other contextual default rules, like $\frac{||\,\mathsf{A}\,:|\,\mathsf{D}\,|}{\mathsf{D}}$, rely.

Let us now turn to the difference between C- and E-justifications of contextual default rules. Notably, it can serve for imposing priorities between two implicit assumptions. This cannot be modelled easily in conventional default logics. For instance, in default theory (1) a precedence among the two implicit assumptions can be modelled in contextual default logic in a very straightforward way by weakening the implicit assumption B compared to its negation:

$$\left(\left\{\frac{||:|\,\mathsf{B}\,|}{\mathsf{C}}, \frac{||:\,\neg\mathsf{B}\,||}{\mathsf{D}}\right\}, Th(\emptyset)\right)$$

This yields one contextual extension, $(Th(\{\mathsf{C}\}), Th(\{\mathsf{B}, \mathsf{C}\}))$.

The use of W-justifications is closely related to CWA, the closed world assumption [Reiter,1977]. CWA has been introduced in order to complete a given set of facts W. In CWA, a ground negative literal is derivable iff the original atom is not derivable from W. Considering a database about taxpayers, for instance, an individual is not a dead person unless stated otherwise. Given no other knowledge about an individual, we derive that he is not dead. This can be modelled by means of the contextual default rule $\frac{||:||\,\neg\mathsf{D}}{\neg\mathsf{D}}$.

Contextual default logic: The formal theory

In the sequel, we give alternative characterizations of contextual extensions and describe their structure in more detail. First, we define the set of generating contextual default rules.

Definition 10 *Let (D, W) be a contextual default theory and T and S sets of formulas. The set of generating contextual default rules for (T, S) wrt (D, W) is defined as*

$$GD^{(T,S)}_{(D,W)} = \left\{ \frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C : \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D \;\middle|\; \begin{array}{lll} \alpha_W \in W, & \alpha_E \in T, & \alpha_C \in S, \\ \neg\beta_C \notin S, & \neg\beta_E \notin T, & \neg\beta_W \notin W \end{array} \right\}$$

Now, we can make precise the claim made before Definition 3: In a contextual extension (E, C), the set E is deductively closed and the set C is pointwisely closed.

Theorem 4 *Let (E, C) be a contextual extension of (D, W) and $\Delta = GD^{(E,C)}_{(D,W)}$. Then,*

$$Th(W \cup Conseq(\Delta)) = Th(E) = E$$

$$Th_{E \cup Justif_C(\Delta)}(Justif_E(\Delta)) = Th_{\top}(C) = C.$$

The first inclusion shows that extensions of contextual default theories are formed in the same way as in conventional default logics. That is, they consist of the initial facts along with the consequents of all applying contextual default rules. The second inclusion describes the respective contexts. A context is the pointwise closure of the E-justifications of the applying contextual default rules (corresponding to the individual consistency requirements) under the extension and the C-justifications of the applying contextual default rules (corresponding to the joint consistency requirements). It follows that whenever (E, C) is a contextual extension, C contains the deductive closure of E and all formulas involved in joint consistency requirements. In symbols, $Th\left(E \cup Justif_C\left(GD^{(E,C)}_{(D,W)}\right)\right) \subseteq C$. Since this set is shared by all subcontexts of a context, we call it the *kernel* of a context.

Theorem 5 *Let (D, W) be a contextual default theory and let E and C be sets of formulas. Define $E_0 = W$ and $C_0 = W$ and for $i \geq 0$*

$$\Delta_i = \left\{ \frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C : \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D \;\middle|\; \begin{array}{lll} \alpha_W \in W, & \alpha_E \in E_i, & \alpha_C \in C_i, \\ \neg\beta_C \notin C, & \neg\beta_E \notin E, & \neg\beta_W \notin W \end{array} \right\}$$

$$E_{i+1} = Th(W \cup Conseq(\Delta_i))$$

$$C_{i+1} = Th_{E_i \cup Justif_C(\Delta_i)}(Justif_E(\Delta_i))$$

Then, (E,C) *is a contextual extension of* (D,W) *iff* $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

The extension E is built by successively introducing the consequents of all applying contextual default rules. Also, the deductive closure is computed at each stage. For each partial context C_{i+1}, the previous partial extension E_i is unioned with the C-justifications of all applying contextual default rules. This set is unioned in turn with each E-justification of all applying contextual default rules. Again, the deductive closure is computed when appropriate. In this way, each partial context C_{i+1} is built upon the kernel of the previous partial context, $Th(E_i \cup Justif_C(\Delta_i))$.

A possible worlds semantics

In analogy to [Besnard and Schaub,1993], we employ Kripke structures for characterizing contextual extensions. A Kripke structure has a distinguished world, the "actual" world, and a set of worlds accessible from it.

The idea is roughly as follows. In a class of Kripke structures, the actual worlds characterize an extension, whereas the accessible worlds characterize its context consisting of a number of subcontexts. In concrete terms, given a contextual extension (E,C) and a Kripke structure $\mathfrak{m}$, we require that the actual world ω_0 of $\mathfrak{m}$ be a model of the extension, E, and demand that each world in $\mathfrak{m}$ accessible from ω_0 be a model of some subcontext of C. Thus, each world of $\mathfrak{m}$ accessible from the actual world ω_0 is to be a model of the kernel of C.

First, we define the class of K-models[5] associated with W as $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models \gamma \wedge \Box\gamma, \gamma \in W\}$. We will semantically characterize contextual extensions by maximal elements of a strict partial order on classes of K-models. Given a contextual default rule δ, its application conditions and the result of applying it are captured by an order $>_\delta$ as follows.

Definition 11 *Let* $\delta = \frac{\alpha_W \mid \alpha_E \mid \alpha_C \,:\, \beta_C \mid \beta_E \mid \beta_W}{\gamma}$. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be distinct classes of K-models. Define* $\mathfrak{M} >_\delta \mathfrak{M}'$ *iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box\gamma \wedge \Box\beta_C \wedge \Diamond\beta_E\}$$

and

1. $\mathfrak{M}_W \models \alpha_W$ 2. $\mathfrak{M}' \models \alpha_E$ 3. $\mathfrak{M}' \models \Diamond\alpha_C$
4. $\mathfrak{M}' \not\models \Diamond\neg\beta_C$ 5. $\mathfrak{M}' \not\models \Box\neg\beta_E$ 6. $\mathfrak{M}_W \not\models \neg\beta_W$

Given a set of contextual default rules D, the strict partial order $>_D$ is defined as the transitive closure of the union of all orders $>_\delta$ such that $\delta \in D$.

Then, we obtain soundness and completeness:[6]

Theorem 6 *Let* (D,W) *be a contextual default theory. Let* $\mathfrak{M}$ *be a class of K-models, E a deductively closed set of formulas, C a pointwisely closed set of formulas, and* $C_K = Th\left(E \cup Justif_C\left(GD_{(D,W)}^{(E,C)}\right)\right)$ *and* $C_J = Justif_E\left(GD_{(D,W)}^{(E,C)}\right)$ *such that*

$$\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C_K \wedge \Diamond C_J\}.$$

(E,C) *is a consistent contextual extension of* (D,W) *iff* $\mathfrak{M}$ *is a* $>_D$*-maximal non-empty class above* $\mathfrak{M}_W$.

Observe that the requirements on a maximal class of K-models correspond to the aforementioned intuitions. Clearly, E is the extension, C the context, C_K the kernel and C_J consists of E-justifications distinguishing the subcontexts from each other.

Conclusion

Contextual default logic provides a unified framework for default logics by extending the notion of a default rule and supplying each extension with a context. Such contexts are formed by pointwisely closing certain consistency assumptions under a given extension.

We isolated six different application conditions for default rules. We showed that only three of them are employed in conventional default logics, even though two of the three remaining ones correspond to well-known notions, namely first-order derivability and the closed world assumption. The remaining condition expresses "membership in a context" and needs further elaboration.

Among various advantages, contextual default logic explicates the context-dependency of default logics and reveals that existing default logics differ mainly in the way they deal with an explicit or implicit underlying context. As a result, we saw that justified default logic compromises individual and joint consistency, whereas other variants strictly employ either of them.

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[5] K-models stand for models of the modal logic K.

[6] Given a set of formulas T let $\Box T$ stand for $\wedge_{\alpha \in T}\Box\alpha$ and $\Diamond T$ stand for $\wedge_{\alpha \in T}\Diamond\alpha$.