# On Constrained Default Theories

**Torsten Schaub**[⋆]
*FG Intellektik, TH Darmstadt, Alexanderstraße 10, D–6100 Darmstadt*
torsten@intellektik.informatik.th-darmstadt.de

**Abstract.** We introduce a variant of default logic, called constrained default logic. With it, we make default logic commit to its assumptions without extending the first order language. In contrast to [1] wherein "formulas with constraints" were introduced, we introduce constrained extensions. Then, we employ constrained default logic in order to clarify the relationships among the constrained variants of default logic [4, 1, 3]. Finally, the proof–oriented notion of a lemma default rule is introduced that accounts for the practical impact of cumulativity: the capability of handling nonmonotonic lemmata.

## 1 Introduction

Default logic was defined by Reiter in [7] as a formal account of reasoning in the absence of complete information. On the one hand, default logic has gained much popularity due to its very natural way to encode default reasoning. On the other hand, general default theories lack several desired properties. Hence, default logic has been reevaluated and modified several times during the last decade. A first variant was proposed by Łukaszewicz in [4] which guaranteed semi–monotonicity and the existence of extensions. Semi–monotonicity stands for monotonicity wrt the defaults and allows for reasonable proof procedures (cf. [7]). Recently, some variants [1, 3] were proposed which made default logic commit to its assumptions [6] and restored cumulativity [5]. Intuitively, cumulativity stipulates that adding a theorem to the set of premises should not alter the theory under consideration. Commitment demands the consistency of a theory with all of its underlying assumptions.

Thus, in one respect the evolution of default logic was successful in that it brought up derivatives that share many desired properties. However, the process was diverging since the approaches differ basically in the way they achieve their results. Therefore, building on the work of [1] and [4], we introduce a variant of default logic called *constrained default logic*[1]. For one thing, the approach combines (almost) all advantages of its ancestors in an arguably simpler way. For another thing, constrained default logic serves as an instrument for comparing the derivatives of classical default logic.

In constrained default logic consistency assumptions are regarded as constraints on a given theory. Therefore, we provide the notion of a *constrained extension.* With it, we distinguish between our set of beliefs, ie. our extension, and its underlying constraints that form a *context* guiding our beliefs. Take the default theory $\left(\left\{\frac{A:B}{C}\right\}, \{A\}\right)$. Instead of a "flat" extension $Th(\{A, C\})$ as in classical default logic, we now obtain an extension that is embedded in a context, viz. the constrained extension $(Th(\{A, C\}), Th(\{A, B, C\}))$.

In [1], cumulativity was preserved by means of labelled formulas. Since cumulativity allows for nonmonotonic lemmata the question arises now "What is a nonmonotonic lemma and how should it be represented?" Therefore, we introduce the proof–oriented notion of a *lemma default rule* that applies to constrained as well as classical default logic.

The paper is organized as follows. In Section 2 we develop constrained default logic and use it in Section 3 to examine the relations between the derivatives of default logic. In Section 4 we introduce lemma default rules for constrained and classical default logic.

## 2 Towards constrained default logic

As defined by Reiter in [7] a *closed default theory* $(D, W)$ consists of a set of first order sentences $W$ and a set of *default rules* $D$. A default rule is of the form $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$ where $\alpha, \beta_1, \ldots, \beta_n$ and $\gamma$ are first order sentences. $\alpha$ is called the *prerequisite*, $\beta_1, \ldots, \beta_n$ the *justifications*, and $\gamma$ the *consequent.* An *extension* is defined as all formulas derivable from the facts using classical inference rules and all specified default rules. Informally, a classical extension $E$ of a default theory $(D, W)$ is the smallest deductively closed set of sentences containing $W$ such that for any $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D$, if $\alpha \in E$ and $\neg\beta_1, \ldots, \neg\beta_n \notin E$ then $\gamma \in E$. In the sequel, we shall consider only closed default rules of the form $\frac{\alpha:\beta}{\gamma}$. A default theory is said to be *normal* whenever the justification and the consequent of each default rule are logically equivalent.

The definition of a constrained extension relies on two sets of sentences: $E$ and $C$. For a default rule $\frac{\alpha:\beta}{\gamma}$ to apply in constrained default logic its prerequisite $\alpha$

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[1] Originally, described in [9].

must be in the extension $E$, whereas the consistency of the justification $\beta$ is checked wrt the set of constraints $C$. The constraints can be regarded as a *context* established by the premises, the nonmonotonic theorems as well as all underlying consistency assumptions. Formally, a constrained extension is defined as follows.

**Definition 2.1** *Let $(D, W)$ be a default theory. For any set of sentences $T$ let $\Upsilon(T)$ be the pair of smallest sets of sentences $(S', T')$ such that*

1. *$W \subseteq S' \subseteq T'$,* 2. *$S' = Th(S')$ and $T' = Th(T')$,*
3. *For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$.*

*A pair of sets of sentences $(E, C)$ is a constrained extension of $(D, W)$ iff $\Upsilon(C) = (E, C)$.*

When computing an extension, we have to preserve its consistency with all of the constraints. Thus, the fixed point condition itself relies merely on the constraints. Intuitively, this means that our context of reasoning has to coincide with our set of accumulated constraints.

A perhaps more intuitive characterization of constrained extensions is the following one.

**Theorem 2.1** *Let $(D, W)$ be a default theory and let $E, C$ be sets of sentences. Define $E_0 = W$ and $C_0 = W$ and for $i \geq 0$*

$$E_{i+1} = Th(E_i) \cup \{ \ \gamma \ \mid \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta, \gamma\} \not\vdash \bot\}$$

$$C_{i+1} = Th(C_i) \cup \{\beta \wedge \gamma \mid \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta, \gamma\} \not\vdash \bot\}$$

*$(E, C)$ is a constrained extension of $(D, W)$ iff $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.*

Observe that it is only referred to the previous partial extension $E_i$ whereas the consistency is checked wrt all constraints.

The approach taken by constrained default logic can be regarded as directly induced by the focused model semantics [8] which is based on the following order:

**Definition 2.2** *Let $\delta = \frac{\alpha:\beta}{\gamma}$ and $\Pi$ be a class of models. The order $\succeq_\delta$ on $2^\Pi \times 2^\Pi$ is defined as follows. For all $(\Pi_1, \breve{\Pi}_1), (\Pi_2, \breve{\Pi}_2) \in 2^\Pi \times 2^\Pi$ we have $(\Pi_1, \breve{\Pi}_1) \succeq_\delta (\Pi_2, \breve{\Pi}_2)$ iff*

1. *$\Pi_2 \models \alpha$*
2. *$\breve{\Pi}_2 \not\models \beta \wedge \gamma$*
3. *$\Pi_1 = \{\pi \in \Pi_2 \mid \pi \models \gamma\}$*
4. *$\breve{\Pi}_1 = \{\pi \in \breve{\Pi}_2 \mid \pi \models \beta \wedge \gamma\}$*

The induced order $\succeq_D$ is defined as the transitive closure of all orders $\succeq_\delta$ such that $\delta \in D$. Thus, given a $\succeq_D$–maximal pair of classes of models $(\Pi, \breve{\Pi})$, an extension is formed by all formulas that are valid in $\Pi$ whereas the focused models $\breve{\Pi}$ reflect themselves as constraints surrounding the extension.

**Theorem 2.2** *Let $(D, W)$ be a default theory. Let $(\Pi, \breve{\Pi})$ be a pair of classes of models and $E, C$ deductively closed sets of sentences such that $\Pi = \{\pi \mid \pi \models E\}$ and $\breve{\Pi} = \{\pi \mid \pi \models C\}$. Then, $(E, C)$ is a constrained extension of $(D, W)$ iff $(\Pi, \breve{\Pi})$ is a $\succeq_D$–maximal element above $(\Pi_W, \Pi_W)$.*

Constrained default logic has many desired properties: the existence of constrained extensions is guaranteed, constrained default logic is semi–monotonic, all constrained extensions of a given default theory are weakly orthogonal (ie. the constraints are contradictory) to each other, and constrained extensions commit to their assumptions. As an example, consider the default theory

$$\left( \left\{ \frac{:B}{C}, \frac{:\neg B}{D} \right\}, \emptyset \right). \quad (1)$$

In classical default logic we obtain only one extension: $Th(\{C, D\})$. Unintuitively, both default rules have been applied although they have contradicting justifications. Thus, there has been no commitment to the assumption $B$ nor $\neg B$. In contrary, constrained extensions commit to their assumptions and we obtain two of them: $(Th(\{C\}), Th(\{C, B\}))$ and $(Th(\{D\}), Th(\{D, \neg B\}))$. Once a default rule has been applied the constraints admit only compatible conclusions based on compatible consistency assumptions.

Whenever we have normal default theories, the constraints coincide with the extension and they are equivalent to the corresponding classical extension.

**Proposition 2.3** *Let $(D, W)$ be a normal default theory and $E$ a set of sentences. Then, $E$ is an extension of $(D, W)$ iff $(E, E)$ is a constrained extension of $(D, W)$.*

## 3 Relationships among default logics

Default logic has evolved during the last decade. Two prevailing approaches were Łukaszewicz' justified default logic [4] and Brewka's cumulative default logic [1]. Łukaszewicz attached sets of sentences to extensions whereas Brewka labelled formulas with sets of sentences. Thus, both employed constraints but differ basically in the location they put them. Constrained default logic turns out to be an amalgamation of both approaches. Hence, it is perfectly suited as an instrument for comparing the descendents of classical default logic. Moreover, extensions of J–default logic [3] turn out to be equivalent to constrained extensions in the case of semi–normal default theories.[2] Thus, all results given below carry over to their approach.

At first, we describe the relationship between classical and constrained default logic by taking advantage of the justifications of the generating default rules, ie. $C_E = \{\beta \mid \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E, \neg\beta \notin E\}$.

[2] To see this, compare Theorem 2.1 and [3, Def. 4.1].

**Theorem 3.1** *Let $E$ be a classical extension of $(D,W)$. If $E \cup C_E$ is consistent, then $(E, Th(E \cup C_E))$ is a constrained extension of $(D,W)$.*

Observe, that the converse of the above theorem does not hold since default logic does not guarantee the existence of extension.

**Theorem 3.2** *Let $(D,W)$ be a default theory and let $E$ and $C$ be sets of sentences. If $(E,C)$ is a constrained extension of $(D,W)$ and $E$ is a classical extension of $(D,W)$, then $C \subseteq Th(E \cup C_E)$.*

Łukaszewicz also attaches constraints to extensions in order to strengthen the applicability condition of default rules. Informally, a justified extension of $(D,W)$ is a pair $(E,J)$ of smallest sets of sentences such that $E$ is deductively closed and contains $W$, and for any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $\forall \eta \in J \cup \{\beta\}$. $E \cup \{\gamma, \eta\} \not\vdash \bot$ then $\gamma \in E$ and $\beta \in J$.

We observe that the set of constraints $J$ merely consists of the justifications of applied default rules. It has neither to be deductively closed nor consistent and, consequently prevents Łukaszewicz' variant from committing to its assumptions. The default theory (1) has only one justified extension: $Th(\{C,D\})$ wrt $\{B, \neg B\}$. Both default rules apply although they have contradicting justifications. Thus, the extension is justified by an inconsistent set of constraints. Since Łukaszewicz is primarily interested in avoiding inconsistencies between justifications and consequents of individual default rules he neglects inconsistencies among the constraints.

Nevertheless, we have the following relationships between the two globally constrained approaches.

**Theorem 3.3** *Let $(E,J)$ be a justified extension of $(D,W)$. If $E \cup J$ is consistent, then $(E, Th(E \cup J))$ is a constrained extension of $(D,W)$.*

**Theorem 3.4** *Let $(E,C)$ be a constrained extension of $(D,W)$. Then, there is a justified extension $(E',J')$ of $(D,W)$ such that $E \subseteq E'$ and $C \subseteq Th(E' \cup J')$.*

**Theorem 3.5** *Let $(D,W)$ be a default theory and let $E,C$, and $J$ be sets of sentences. If $(E,C)$ is a constrained extension of $(D,W)$ and $E$ is a justified extension of $(D,W)$ wrt $J$ then $C \subseteq Th(E \cup J)$.*

Brewka [1] restored commitment and cumulativity to default logic also by strengthening the applicability condition for default rules and making the reasons for believing something explicit. But in order to keep track of assumptions, he introduced assertions, ie. formulas labelled with the set of justifications and consequents of applied default rules (eg. $\langle \alpha, \{\alpha_1, \ldots, \alpha_n\}\rangle$). An assertional default theory is a pair $(D, \mathcal{W})$, where $D$ is a set of default rules and $\mathcal{W}$ is a set of assertions. Informally, an assertional extension of $(D,\mathcal{W})$ is the smallest set of assertions $\mathcal{E}$ being deductively closed under an extended[3] theory operator $\widehat{Th}$ and containing $\mathcal{W}$ such that for any $\frac{\alpha:\beta}{\gamma} \in D$, if $\langle \alpha, s(\alpha)\rangle \in \mathcal{E}$ and $f(\mathcal{E}) \cup s(\mathcal{E}) \cup \{\beta,\gamma\} \not\vdash \bot$ then $\langle \gamma, s(\alpha) \cup \{\beta,\gamma\}\rangle \in \mathcal{E}$.

Assertional extensions commit to their assumptions and we obtain two from the default theory (1): $\widehat{Th}(\{\langle C, \{B,C\}\rangle\})$ and $\widehat{Th}(\{\langle D, \{\neg B, D\}\rangle\})$. Once we have derived a proposition, we are aware of its underlying assumptions. Therefore, cumulative default logic prevents the derivation of conclusions that contradict previously derived conclusions or their underlying consistency assumption.

Complementary to constrained default logic that constrains extensions in a global fashion, the justifications and consequents of the applied default rules are recorded locally to the conclusions. Thus, assertional extensions share the notion of "joint consistency" with constrained extensions — but in a distributed way. In this sense, constrained default logic has moved from "formulas with constraints" towards constrained extensions. Since cumulative as well as constrained default logic are captured by the focused models semantics [8], they are very close to each other.

**Theorem 3.6** *Let $(D,W)$ be a default theory and $(D,\mathcal{W})$ the assertional default theory, where $\mathcal{W} = \{\langle \alpha, \emptyset\rangle \mid \alpha \in W\}$. Then, if $(E,C)$ is a constrained extension of $(D,W)$ then there is an assertional extension $\mathcal{E}$ of $(D,\mathcal{W})$ such that $E = f(\mathcal{E})$ and $C = Th(f(\mathcal{E}) \cup s(\mathcal{E}))$; and, conversely if $\mathcal{E}$ is an assertional extension of $(D,\mathcal{W})$ then $(f(\mathcal{E}), Th(f(\mathcal{E}) \cup s(\mathcal{E})))$ is a constrained extension of $(D,W)$.*

Observe, that we get a one–to–one correspondence between the "real" extensions, ie. $E = f(\mathcal{E})$, whereas the constraints of a constrained extension correspond to the deductive closure of the supports and the asserted formulas. Thus, we can map assertional extensions onto constrained extensions only modulo equivalent sets of supports.

However, since constrained default logic sticks to first order formulas it does not run into the "floating conclusions" problem [2] that arises whenever we reason skeptically by intersecting several extensions. Take the assertional default theory $\left(\left\{\frac{:\neg B}{A}, \frac{:\neg A}{B}\right\}, \{\langle A \to C, \emptyset\rangle, \langle B \to C, \emptyset\rangle\}\right)$ that has two extensions: $\widehat{Th}(\{\langle A, \{\neg B, A\}\rangle, \langle C, \{\neg B, A\}\rangle\})$ and $\widehat{Th}(\{\langle B, \{\neg A, B\}\rangle, \langle C, \{\neg A, B\}\rangle\})$. Reasoning skeptically, we cannot draw any conclusion about $C$. Although the asserted formula $C$ is in both extensions the corresponding supports differ and the assertions themselves do not belong to the intersection. The constrained extensions of the above default rules and the axioms $A \to$

[3]Let $f(\xi)$ be the (asserted) formula and $s(\xi)$ the label (support) of an assertion $\xi$: if $\xi_1, \ldots, \xi_n \in \widehat{Th}(\mathcal{S})$ and $f(\xi_1), \ldots, f(\xi_n) \vdash \alpha$ then $\langle \alpha, \cup_{i=1}^n s(\xi_i)\rangle \in \widehat{Th}(\mathcal{S})$.

$C, B \rightarrow C$ are: $(Th(\{A, C\}), Th(\{A, C, \neg B\}))$ and $(Th(\{B, C\}), Th(\{B, C, \neg A\}))$. Intersecting both yields $(Th(\{A \vee B, C\}), Th(\{C, \neg(A \leftrightarrow B)\}))$ that provides us with the skeptical theorem $C$.

Using assertions we cannot apply any deduction to the supports apart from considering them when checking consistency. But encoding the underlying consistency assumptions as a context guiding our beliefs, we have the whole deductive machinery at hand.

In view of the above results, we can make use of the central role of constrained default logic and obtain as corollaries the corresponding relationships between cumulative and J–default logic on one side and classical and justified default logic on the other.

To conclude, let us observe that constrained default logic is closer to cumulative default logic than to Łukaszewicz' variant. Although Łukaszewicz also attaches constraints to extension, he employs a weaker consistency check. Similar to classical default logic, justifications need only to be separately consistent with an extension at hand. In particular, this is mirrored by the notion of commitment since assertional and constrained extensions commit to their assumptions, whereas classical and justified extensions do not. Since additionally every classical extension is also a justified extension (cf. [4]), Łukaszewicz' variant seems to be closer to classical default logic than to its constrained descendents. However, constrained default logic differs from its constrained relatives in employing a deductively closed set of constraints. With it, it does neither discard inconsistencies among the constraints nor run into the "floating conclusions" problem.

## 4 Nonmonotonic Lemmata

Aside its theoretical quality, cumulativity is of great practical importance. This is, because a cumulative consequence operator allows for the use of lemmata needed for reducing computational efforts.

In [5], the failure of default logic for cumulativity was revealed by the default theory

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right)$$

that has one classical extension: $Th(\{A\})$. Adding the conclusion $A \vee B$ to the facts yields the default theory $\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right)$ which has now two classical extensions: $Th(\{A\})$ and $Th(\{\neg A, B\})$. Regardless of whether or not we employ a skeptical or a credulous notion of theory formation — in both cases we change the theory under consideration. In contrary, cumulative default logic allows to derive the assertions $\langle A, \{A\}\rangle$ and $\langle A \vee B, \{A\}\rangle$. Adding the assertion $\langle A \vee B, \{A\}\rangle$ to the premises yields the assertional default theory $\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{\langle A \vee B, \{A\}\rangle\}\right)$ that has still the same assertional extension.

As shown in [1, 8], it is necessary to be aware of a conclusion's underlying assumptions if we want to preserve cumulativity. But since constrained default logic sticks to first order formulas the question arises how to represent those assumptions. Inspired by default logic's natural distinction between facts and defaults, we view nonmonotonic lemmata as abbreviations for the corresponding default inferences. Thus, it is natural to add them as default rules. We take a nonmonotonic theorem, one of its minimal default proofs and construct the corresponding lemma default rule.[4]

**Definition 4.1** *Let $(E,C)$ be a constrained extension of $(D, W)$. A default proof $D_\ell$ of $\ell$ in $(E,C)$ is a sequence $\langle D_1, \ldots, D_k\rangle$ of sets of default rules where $D_i \subseteq GD_D^{(E,C)}$ $(1 \le i \le k)$ and $\cup_{i=1}^k D_i$ is a minimal set of default rules such that* [5]

1. $W \vdash p(D_1)$
2. $W \cup c(D_i) \vdash p(D_{i+1})$
3. $W \cup c(D_k) \vdash \ell$

Then, a conclusion's lemma default rule is defined as follows.[6]

**Definition 4.2** *Let $(E,C)$ be a constrained extension of $(D, W)$. Let $\ell \in E$ and $D_\ell$ be a default proof of $\ell$ in $(E,C)$. We define a lemma default rule $\delta_\ell$ for $\ell$ as*

$$\delta_\ell := \frac{: \bigwedge_{\delta \in D_\ell} j(\delta) \wedge \bigwedge_{\delta \in D_\ell} c(\delta)}{\ell}$$

With it, we guarantee that adding a conclusion's lemma default rule does neither alter any extension nor produce any new ones.

**Theorem 4.1** *Let $(E', C')$ be a constrained extension of $(D, W)$ and let $\delta_\ell$ be a lemma default rule for $\ell \in E'$. Then, $(E,C)$ is a constrained extension of $(D, W)$ iff $(E,C)$ is a constrained extension of $(D \cup \{\delta_\ell\}, W)$.*

Thus, the approach provides a simple solution for generating and using nonmonotonic lemmata. Also, it clarifies the notion of nonmonotonic lemmata by distinguishing between theirselves and their original theorems. Let us look again at the canonical cumulativity example. Adding the sentence $A \vee B$ as a nonmonotonic lemma amounts to the addition of the lemma default rule $\frac{:A}{A \vee B}$. We obtain the default theory $\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}, \frac{:A}{A \vee B}\right\}, \emptyset\right)$ that has still the same constrained extension: $(Th(A), Th(A))$.

The major difference between the addition of assertions to the premises and the addition of lemma default rules to the default rules is that once we have added an assertion to the premises it is not retractable any more whenever an inconsistency

[4] $GD_D^{(E,C)} = \{\frac{\alpha : \beta}{\gamma} \mid \alpha \in E,\ C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}$

[5] $j(\delta)$ is the justification and $c(\delta)$ the consequent of $\delta$.

[6] We define, $\delta \in \langle D_1, \ldots, D_k\rangle$ iff $\delta \in \cup_{i=1}^k D_i$.

arises. Just take the above assertional default theory $\left(\left\{\frac{:A}{A}, \frac{A\vee B:\neg A}{\neg A}\right\}, \{\langle A\vee B, \{A\}\rangle\}\right)$ obtained after lemmatizing the assertion $\langle A\vee B, \{A\}\rangle$. Now, adding $\langle\neg A, \emptyset\rangle$ yields a hard contradiction since $s(\langle A\vee B, \{A\}\rangle) \cup f(\langle\neg A, \emptyset\rangle) \vdash \bot$. Thus, the smooth default properties of the original default conclusion have been lost. However, adding $\neg A$ in the presence of the lemma default rule $\frac{:A}{A\vee B}$ just blocks the default rule and does not harm the reasoning process itself.

Hence, the addition of assertions [1] is stronger than that of lemma default rules. All extensions inconsistent with the asserted formula or even its support are eliminated after its addition. On the contrary, lemma default rules preserve all extensions and therefore their purpose is more an abbreviation of default proofs in order to improve the computational efforts. Also, they admit credulous as well as skeptical lemmata.

What has been achieved? One of the original postulates of nonmonotonic formalisms was to "jump to conclusions" in the absence of information. But since the computation of nonmonotonic conclusions does not only involve deduction but also expensive consistency checks, the need to incorporate lemmata is even greater in nonmonotonic theorem proving than in classical theorem proving. Hence, nonmonotonic lemmata can be seen as a step in this direction. This becomes obvious by means of Theorem 2.1: it is possible to jump to a conclusion $\ell$ normally derived in layer $E_k$ by skipping all previous layers $E_0$ to $E_{k-1}$ and solely applying the (prerequisite–free) lemma default rule.

Let us look at a simplified default proof of a nonmonotonic theorem $\ell$ consisting of a chain of default rules $\langle\{\frac{\alpha_0:\beta_0}{\gamma_0}\}, \ldots, \{\frac{\alpha_i:\beta_i}{\gamma_i}\}, \ldots, \{\frac{\alpha_n:\beta_n}{\ell}\}\rangle$ such that $W \vdash \alpha_0$, $W \cup \{\gamma_i\} \vdash \alpha_{i+1}$ $(0 \leq i < n)$ and $W \cup \{\gamma_{n-1}\} \vdash \ell$. Normally, proving $\ell$ from scratch requires $n$ proofs and $n$ consistency checks. Each consistency check involves the justification as well as the consequent of each default rule. On the contrary, applying the corresponding lemma default rule requires *no* proofs since lemma default rules are prerequisite–free. The effort of checking consistency reduces to *one* consistency check. But although the justification of the lemma default rule contains all justifications and consequents of previously applied default rules we have the advantage that their joint consistency has already been proven.

Notably, the approach taken by lemma default rules carries over to classical default logic. We only have to eliminate the requirement of joint consistency in Definition 4.2. Thus, given a default theory $(D, W)$ and one of its classical extensions $E$ we construct the lemma default rule $\zeta_\ell$ for a $\ell \in E$ as follows: Given one of its default proofs $D_\ell = \langle D_1, \ldots, D_k\rangle$, where $\cup_{i=1}^{k} D_i = \{\delta_1, \ldots, \delta_n\}$, but now in $(E, E)$ wrt $(D, W)$ we define

$$\zeta_\ell := \frac{: j(\delta_1), \ldots, j(\delta_n)}{\ell}.$$

Unfortunately, we obtain a non–singular (prerequisite–free) default rule. This is due to the fact that we have to preserve the consistency of each justification separately. Observe also that in the above definition no reference is made to the consequents of any default rules. However, it is now possible to enrich default logic such that it admits the generation of nonmonotonic lemmata without altering the logical formalism as such.

## 5 Conclusion

We have presented a constrained variant of default logic that commits to its assumptions and allows for nonmonotonic lemmata. Constrained default logic possesses a clear correspondence to the focused model semantics [8]. Since the approach sticks to first order formulas, we can use conventional theorem provers and do not run into the "floating conclusions" problem. We have exploited constrained default logic's central role in order to establish the relationships between classical [7], justified [4], assertional [1], constrained and extensions of J–default logic [3].

The approach taken by lemma default rules enables us to separate the notions of commitment and "practical cumulativity", so that it became adaptable to classical default logic. By using lemma default rules we have escaped from assertions and the propagation of their supports. We merely look at the default proofs and hence regard the assumptions underlying a conclusion only by need. Moreover, lemma default rules are retractable and later inconsistencies are avoided.

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