Skeptical Query-Answering in Constrained Default Logic

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Abstract: An approach to skeptical query-answering in Constrained Default Logic based on the Connection Method is presented. We adapt a recently proposed general method to skeptical reasoning in Default Logics—a method which does neither strictly require the inspection of all extensions nor the computation of entire extensions to decide whether a formula is skeptically entailed. We combine this method with a credulous reasoner which uses the Connection Method as the underlying calculus for classical logic. Furthermore, we develop the notion of a skeptical default proof and show how such a proof can be extracted whenever our calculus proves skeptical entailment of a particular query.

1 Introduction

Nonmonotonic Logics in general, and approaches like Autoepistemic [10] or Default Logic [14] in particular, aim at extending an underlying classical logical system in order to provide conclusions that go beyond this system. For this, they induce one or several so-called *extensions* of a given world description, each of which represents a reasonable set of beliefs. This phenomenon of multiple extensions suggests two natural approaches to query-answering: A *credulous* one, in which a query is said to be derivable if it belongs to a *single* extension, and a *skeptical* one, in which one stipulates that a query lies in *all* extensions.

So far, computational approaches to nonmonotonic logics have mainly focused on the computation of entire extensions, like [4, 19, 7, 22, 11], or credulous query-answering, like [14, 18]. [8] compute intersections of extensions in Autoepistemic Logic. Skeptical query-answering has up to now been primarily studied in restricted nonmonotonic reasoning frameworks, like Theorist [12] (corresponding to so-called prerequisite-free default theories in Default Logic) [13, 20]. From the perspective of Default Logic, implementations of Circumscription, like [5], fall into the same category since they use roughly the same restricted fragment of Default Logic. Finally, a major category of implementations for fragments of Default Logic is given by the wide body of implementations of Logic Programming.

In what follows, we develop a method for skeptical query-answering in Default Logics. This work builds on [21], where a general framework to skeptical reasoning in (semi-monotonic) Default Logics was proposed. There, we have given a high-level description of skeptical query-answering that abstracts from an underlying credulous reasoner. In this paper, we make the aforementioned meta-algorithm precise and employ it to extend an existing approach to credulous query-answering [18] based on the Connection Method [1].

The reader may wonder why we have chosen Constrained Default Logic [16, 3] rather than Reiter's original approach. In fact, Constrained Default Logic serves us as an exemplary Default Logic enjoying the property of *semi-monotonicity*, which stipulates that

⋆ On leave from FG Intellektik, TH Darmstadt the addition of default rules to a theory does not invalidate the application of previously applied default rules. As pointed out in [14], semi-monotonicity is indispensable for feasible query-answering in Default Logics. This is so because it allows for local proof procedures focusing on default rules relevant for answering a query; otherwise the *whole* set of default rules has to be taken into account. Now, semi-monotonicity is only enjoyed by so-called *normal* default theories in Reiter's Default Logic. However, since both aforementioned Default Logics coincide on normal default theories, our exposition applies to this fragment of Reiter's Default Logic, too. As concerns other variants of Default Logic, we note that Constrained Default Logic yields the same conclusions as Cumulative Default Logic [2]; both variants differ in representational issues only. Moreover, Constrained Default Logic coincides with Rational Default Logic on the large fragment of so-called semi-normal default theories [9]. All these interrelations render our exemplar, Constrained Default Logic, a prime candidate for exposing our approach.

Of course, a similar question may arise concerning the choice of the Connection Method [1]. Unlike resolution-based methods that decompose formulas in order to derive a contradiction, the Connection Method analyses the structure of formulas for proving their unsatisfiability. In fact, we will see that skeptical query-answering requires numerous variants of similar subproofs. In such a case, it is advisable to reuse information gathered on similar structures. This approach is supported by the structure-sensitive nature of the Connection Method. As a result, we obtain a homogeneous characterization of skeptical default proofs at the level of the underlying deduction method.

The paper is organized as follows. After recapitulating the basic concepts of Constrained Default Logic in Section 2, we elaborate in Section 3 on the general framework to skeptical query answering proposed in [21]. The latter provides an abstract decision procedure for skeptical query-answering in (semi-monotonic) Default Logics; it has its roots in [13, 20], which address skeptical reasoning in Theorist [12]. In fact, we formalize the general ideas presented in [21] by appeal to sequences of default rules. As a result, we obtain in Section 3 an algorithm instantiating the general framework in [21]. The resulting formal underpinnings given in Theorem 3, 4, and 5 are obtained as corollaries to results in [21]. In addition, Section 3 offers a general definition of a *skeptical default proof*—a point left open in [21]. Such a skeptical default proof is returned by the algorithm developed in the same section. Section 4 describes our underlying method for credulous query-answering based on the Connection Method [18]. The major contribution of this paper is presented in Section 5: We develop an analytic calculus for skeptical query-answering by combing the approaches described in Section 3 and 4. We give a soundness and completeness result of this approach and illustrate how skeptical default proofs can be extracted whenever a query has been successfully proven. Our results are summarized in Section 6.

2 Constrained Default Logic

We consider a straightforward yet powerful extension of Constrained Default Logic [3], called *Pre-Constrained Default Logic* [17]. The idea is to supplement some initial consistency constraints that direct the subsequent reasoning process. This is a well-known technique, also used in Theorist [12], in which the "context of reasoning" is predetermined and subsequently dominated by some initial consistency requirements. These additional constraints play an important role in our approach to skeptical query-answering, as we will see below.

A *pre-constrained default theory* $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ (*default theory*, for short) consists of a set of formulas $\mathcal{W}$, a set of default rules $\mathcal{D}$, and a set of formulas $\mathcal{C}$ representing some initial constraints. A *default rule* is any expression of the form $\frac{\alpha : \beta}{\gamma}$ where α, β, γ are formulas. For convenience, we denote the prerequisite α of a default rule by $\mathit{Prereq}(\delta)$, its justification β by $\mathit{Justif}(\delta)$, and its consequent γ by $\mathit{Conseq}(\delta)$.[3] A *normal default theory* is restricted to *normal default rules*, whose justification is equivalent to the consequent. For simplicity, we deal with a propositional language over a finite alphabet and assume that $\mathcal{W} \cup \mathcal{C}$ is satisfiable. A *constrained extension* is defined as follows.

Definition 1. Let $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory and let E and C be sets of formulas. Define $E_0 = \mathcal{W}$ and $C_0 = \mathcal{W} \cup \mathcal{C}$ and for $i \geq 0$

$$E_{i+1} = \mathit{Th}(E_i) \cup \{ \quad \gamma \quad \mid \tfrac{\alpha : \beta}{\gamma} \in \mathcal{D}, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}$$
$$C_{i+1} = \mathit{Th}(C_i) \cup \{ \beta \wedge \gamma \mid \tfrac{\alpha : \beta}{\gamma} \in \mathcal{D}, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot\}$$

(E, C) is a constrained extension of $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ if $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Observe that the initial constraints, $\mathcal{C}$, enter merely the final constraints C (at C_0) but not the extension E. Thus, the initial constraints $\mathcal{C}$ direct the reasoning process without actually becoming a part of it. In particular, they usually decrease the number of applicable default rules.

Let us guide the formal development of our approach by means of the following example. Consider the default statements "quakers are doves if they are no anti-pacifists," "republicans are hawks if they are no pacifists," "doves as well as hawks are traditionalists," along with the strict knowledge telling us that we have a "republican quaker." This is formalized in the following default theory:

$$\left(\left\{ \frac{Q : P}{D}, \frac{R : \neg P}{H}, \frac{D : T}{T}, \frac{H : T}{T} \right\}, \{Q, R\}, \emptyset\right) \tag{1}$$

The first and second default cannot be combined in a single constrained extension due to their mutually exclusive justifications, P and $\neg P$. Hence, this theory has two constrained extensions, one containing D and T and another one containing H and T.

In the sequel, we follow [18] in dealing with default theories in atomic format in the following sense: For a default theory $\Delta = (\mathcal{D}, \mathcal{W}, \mathcal{C})$ in some language $\mathcal{L}_{\Sigma}$, let $\mathcal{L}_{\Sigma'}$ be the language obtained by adding, for each $\delta \in \mathcal{D}$, three new propositions, named $\alpha_\delta, \beta_\delta, \gamma_\delta$, which do not occur elsewhere. Then, Δ is mapped into a default theory $\Delta' = (\mathcal{D}', \mathcal{W}', \mathcal{C}')$ in $\mathcal{L}_{\Sigma'}$ where

$$\mathcal{D}' = \left\{ \frac{\alpha_\delta : \beta_\delta}{\gamma_\delta} \;\middle|\; \delta \in \mathcal{D} \right\}$$
$$\mathcal{W}' = \mathcal{W} \cup \{\mathit{Prereq}(\delta) \rightarrow \alpha_\delta, \beta_\delta \rightarrow \mathit{Justif}(\delta), \gamma_\delta \rightarrow \mathit{Conseq}(\delta) \mid \delta \in \mathcal{D}\}$$
$$\mathcal{C}' = \mathcal{C} \,.$$

The resulting default theory Δ' is called the *atomic format* of the original default theory, Δ. As shown in [15], this transformation is a conservative extension of the formalism.

[3] These projections extend to sets and sequences of default rules in the obvious way.

Hence, it does not affect the computation of queries to the original default theory. We can therefore restrict our attention to atomic default rules without losing generality. The advantages of atomic default rules over arbitrary ones are, first, that their constituents are not spread over several clauses while transforming them into clausal format and, second, that these constituents, e.g. the consequents, are uniquely referable to. The motivations for this format are detailed in [18] and they are somehow similar to the ones for clausal form in automated theorem proving.

3 Skeptical reasoning in constrained default logic

In classical logic, we can say that a formula φ is derivable from a set of facts $\mathcal{W}$ iff it belongs to the deductive closure of $\mathcal{W}$, that is if $\varphi \in \mathit{Th}(\mathcal{W})$. Due to the possible existence of multiple extensions, this notion of derivability is not directly applicable to Default Logic. Rather we obtain two different notions of derivability: A formula φ is *credulously derivable* from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff $\varphi \in E$ for some constrained extension (E, C) of $(\mathcal{D}, \mathcal{W}, \mathcal{C})$.[4] And a formula φ is *skeptically derivable* from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff φ belongs to all such extensions of $(\mathcal{D}, \mathcal{W}, \mathcal{C})$. In our example, (1), D and H are only credulously derivable while T is also skeptically derivable.

In order to furnish a corresponding proof-theory, we need the following concepts. A *default proof segment* in a default theory $\Delta = (\mathcal{D}, \mathcal{W}, \mathcal{C})$ (or Δ-*segment*, for short) is a (finite) sequence of default rules $\langle \delta_i \rangle_{i \in I}$ such that

$$\mathcal{W} \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i) \text{ for } i \in I \quad \text{and} \tag{2}$$

$$\mathcal{W} \cup \mathcal{C} \cup \{\mathit{Conseq}(\delta_i), \mathit{Justif}(\delta_i) \mid i \in I\} \text{ is satisfiable.} \tag{3}$$

A *credulous default proof*, or CDP for short, for a formula φ from Δ is a Δ-segment $\langle \delta_i \rangle_{i \in I}$ such that $\mathcal{W} \cup \{\mathit{Conseq}(\delta_i) \mid i \in I\} \vdash \varphi$. Furthermore, we say that a formula φ is *provable from a Δ-segment* $\langle \delta_i \rangle_{i \in I}$ iff there is a CDP $\langle \delta_j \rangle_{j \in J}$ for φ such that $I \subseteq J$, that is, if the segment is extendible to a CDP for φ. In our example, there are five Δ-segments, each of which is extendible to one of the two CDPs of T, namely

$$\left\langle \frac{Q:P}{D}, \frac{D:T}{T} \right\rangle \quad \text{and} \quad \left\langle \frac{R:\neg P}{H}, \frac{H:T}{T} \right\rangle . \tag{4}$$

Clearly, a formula is credulously derivable iff it is provable from some Δ-segment, since in this case it has a CDP. Accordingly, a formula is skeptically derivable iff it is provable from all Δ-segments.[5]

A basic question is that on the concept of a default proof in skeptical reasoning. Since in general there is not a single CDP valid in each extension, it is however natural to view a skeptical default proof as being compound of multiple CDPs. In fact, we take a skeptical default proof of a formula to be a set $\mathcal{P}$ of CDPs such that $\mathcal{P}$ is *complete*, that is, for each constrained extension (E, C), $\mathcal{P}$ includes a proof which is valid in (E, C):

Definition 2 Skeptical Default Proof. Let $\Delta = (\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory and φ a formula. A skeptical default proof of φ from Δ is a set $\mathcal{P}$ of CDPs for φ such that for each constrained extension (E, C) of Δ there is some $\langle \delta_i \rangle_{i \in I} \in \mathcal{P}$ such that $C \cup \{\mathit{Conseq}(\delta_i), \mathit{Justif}(\delta_i) \mid i \in I\}$ is satisfiable.

[4] For brevity, we sometimes simply say φ "belongs to" (or "is contained in") an extension (E, C), which always means $\varphi \in E$.

[5] This is formally shown in [21].

This view relies heavily on the notion of CDPs. In fact, we keep this fundamental idea and base our method for skeptical reasoning on credulous reasoning, too [21]: The idea is to start with an arbitrary CDP of a given query. Then, we determine in some way a representative selection of Δ-segments incompatible with our initial CDP. These Δ-segments indicate extensions in which our initial default proof is invalid. Intuitively, they can be thought of as putative counterarguments challenging our initial CDP. Next, we verify in turn whether our query is derivable from each such Δ-segment. If this is indeed the case, then our initial query is skeptically derivable.

In order to illustrate this approach, let us verify that T is skeptically derivable in our example. We start with an arbitrary CDP of T from Default Theory (1). Consider the second CDP in (4): $\left\langle \frac{R:\neg P}{H}, \frac{H:T}{T} \right\rangle$. This proof takes place in the extension of (1) containing H and T. Next, we regard all Δ-segments 'challenging' default rules in $\left\langle \frac{R:\neg P}{H}, \frac{H:T}{T} \right\rangle$. This notion is captured formally by the property of *orthogonality*:[6] Two default proof segments $\langle \delta_i \rangle_{i \in I}$ and $\langle \delta_j \rangle_{j \in J}$ in a default theory $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ are called $\mathcal{C}$*-orthogonal* iff $\mathcal{W} \cup \mathcal{C} \cup \{\mathit{Conseq}(\delta_k), \mathit{Justif}(\delta_k) \mid k \in I \cup J\}$ is unsatisfiable. That is, a Δ-segment is orthogonal to another one if its induced constraints, i.e., the set of justifications and consequents of its default rules, are incompatible with the same constraints of the other Δ-segment. Observe that each Δ-segment orthogonal to a given CDP indicates one or more extensions in which our CDP is not valid.

The formal basis for our approach is laid in the following theorem.

Theorem 3. *Let $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory and φ a formula. Then, φ is skeptically derivable from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff there is a* CDP *$\langle \delta_i \rangle_{i \in I}$ for φ and φ is provable from all default proof segments which are $\mathcal{C}$-orthogonal to $\langle \delta_i \rangle_{i \in I}$.*

In our example, there are two Δ-segments orthogonal to $\left\langle \frac{R:\neg P}{H}, \frac{H:T}{T} \right\rangle$, namely

$$\left\langle \frac{Q:P}{D} \right\rangle \text{ and } \left\langle \frac{Q:P}{D}, \frac{D:T}{T} \right\rangle .$$

This is so because the justification of the first default rule, $\frac{R:\neg P}{H}$, in our CDP is contradictory to the justification of default rule $\frac{Q:P}{D}$. There is no Δ-segment orthogonal to the second default rule in our default proof. As will be shown in Theorem 4, we can restrict our attention to minimal orthogonal Δ-segments. Accordingly, it is sufficient to consider the orthogonal Δ-segment $\left\langle \frac{Q:P}{D} \right\rangle$.

Intuitively, we then focus on all extensions of the initial default theory to which the Δ-segment $\left\langle \frac{Q:P}{D} \right\rangle$ contributes and check whether our initial query T belongs to these extensions too. Importantly, this is accomplished by using only default rules relevant for deriving T and hence without computing any extensions. We achieve this by checking whether T is skeptically derivable from the default theory obtained by 'applying' the default rules in our orthogonal Δ-segment. In this way, we try to prove our query under the restrictions imposed by the Δ-segment contesting our initial CDP. To this end, we add the consequence of the default rule $\frac{Q:P}{D}$ to the facts of Default Theory (1) while deleting the default rule itself. Furthermore, we have to add its justification to the set of

[6] Orthogonality usually refers to distinct extensions (c.f. [14]). In Constrained Default Logic, two constrained extensions (E, C) and (E', C') are orthogonal iff $C \cup C'$ is unsatisfiable [3].

initial constraints. This yields the following modified default theory:

$$\left(\left\{\frac{R:\neg P}{H}, \frac{D:T}{T}, \frac{H:T}{T}\right\}, \{Q, R, D\}, \{P\}\right) \tag{5}$$

Now, it remains to be shown that T be skeptically derivable from Default Theory (5). Proceeding recursively, we check first whether T is credulously derivable from this default theory. In fact, T can be proven by means of the CDP $\left\langle \frac{D:T}{T} \right\rangle$. Next, we have to proceed as above and in turn find all minimal Δ-segments orthogonal to $\frac{D:T}{T}$. However, there are no such segments since Default Theory (5) has a single extension containing D and T, in which, moreover, default rule $\frac{R:\neg P}{H}$ is blocked due to initial constraint P.

Importantly, the previous step supplies us with an alternative CDP of T from our original default theory in (1). This is obtained by appending Δ-segment $\left\langle \frac{Q:P}{D} \right\rangle$ and the CDP of T from default theory (5), viz. $\left\langle \frac{D:T}{T} \right\rangle$. This results in the first CDP given in (4). Since there are no other Δ-segments orthogonal to our initial default proof $\left\langle \frac{R:\neg P}{H}, \frac{H:T}{T} \right\rangle$, we are done. As a result, we have obtained a skeptical default proof consisting of two CDPs supporting the skeptical conclusion T. This approach is justified by the following theorem [21]:

Theorem 4. *Let $\langle\delta_i\rangle_{i\in I}$ be a default proof segment in a default theory $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ and φ a formula. Then, φ is provable from all default proof segments $\langle\delta_j\rangle_{j\in J}$ where $I \subseteq J$ iff φ is skeptically derivable from*

$$(\mathcal{D} \setminus \{\delta_i \mid i \in I\}, \mathcal{W} \cup \{\mathit{Conseq}(\delta_i) \mid i \in I\}, \mathcal{C} \cup \{\mathit{Justif}(\delta_i) \mid i \in I\}).$$

Let us summarize our approach to checking whether a query φ is skeptically derivable: We start with a CDP of φ. Then, we determine all minimal orthogonal Δ-segments contesting our initial CDP. Next, we check in turn whether φ is skeptically derivable under the restrictions imposed by each such Δ-segment. In this way, we check whether φ belongs to all extensions orthogonal to the one containing our initial CDP of φ. It is important to note that the choice of the initial CDP is a *"don't care"*-choice in so far that deciding whether φ is skeptically entailed is independent of which CDP is initially chosen. The design of the resulting skeptical default proof, on the other hand, depends on the latter choice.

The approach we have outlined so far can be put together to a concrete algorithm as follows. Let $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory and φ be a formula. We assume a function $cred(\mathcal{D}, \mathcal{W}, \mathcal{C}, \varphi)$ such that

$$cred(\mathcal{D}, \mathcal{W}, \mathcal{C}, \varphi) = \begin{cases} \langle\delta_i\rangle_{i\in I} & \text{if } \langle\delta_i\rangle_{i\in I} \text{ is a CDP of } \varphi \text{ from } (\mathcal{D}, \mathcal{W}, \mathcal{C}) \\ \bot & \text{otherwise} \end{cases}$$

Moreover, given a CDP $\langle\delta_i\rangle_{i\in I}$, let $orth(\mathcal{D}, \mathcal{W}, \mathcal{C}, \langle\delta_i\rangle_{i\in I})$ yield the set of all minimal Δ-segments $\mathcal{C}$-orthogonal to $\langle\delta_i\rangle_{i\in I}$.[7] These two functions yield the following algorithm for skeptical query-answering. Similar to its credulous source, it returns $\bot$ if φ is not skeptically derivable; otherwise it returns a set of CDPs forming a skeptical default proof. The function "$\circ$" concatenates two Δ-segments.

[7] Note that this set is finite in case of a finite propositional alphabet. The procedure $orth$ will be designed below.

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skep(D, W, C, φ) =
 if cred(D, W, C, φ) = ⊥
   then return ⊥
   else let ⟨δ_i⟩_{i∈I} = cred(D, W, C, φ) in
        let O = orth(D, W, C, ⟨δ_i⟩_{i∈I}) in
            P := ∅ ;
            while O ≠ ∅ do
                  select ⟨δ_j⟩_{j∈J} ∈ O ;
                  let D′ = D \ {δ_j | j ∈ J} in
                  let W′ = W ∪ {Conseq(δ_j) | j ∈ J} in
                  let C′ = C ∪ {Justif(δ_j) | j ∈ J} in
                     if skep(D′, W′, C′, φ) = ⊥
                       then return ⊥
                       else P := P ∪ {⟨δ_j⟩_{j∈J} ∘ ⟨δ_l⟩_{l∈L} | ⟨δ_l⟩_{l∈L} ∈ skep(D′, W′, C′, φ)}
                     fi ;
                  O := O \ {⟨δ_j⟩_{j∈J}}
            od ;
            return {⟨δ_i⟩_{i∈I}} ∪ P
 fi
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The variable $\mathcal{P}$ accumulates the set of resulting CDPs. Whenever the procedure ends up with success, the CDPs in $\mathcal{P}$ form a skeptical default proof of φ from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$.

Finally, we have to address the determination of minimal Δ-segments orthogonal to a CDP at hand. Again, this is accomplishable by appeal to credulous reasoning. Note that the notion of orthogonality refers to the consistency constraints induced by a CDP. For this, we have to consider additionally a default rule's justification any time the default rule applies: For a set of atomic default rules $\mathcal{D}$, we define the set of *normalization rules* as $N(\mathcal{D}) = \{\gamma_\delta \rightarrow \beta_\delta \mid \frac{\alpha_\delta : \beta_\delta}{\gamma_\delta} \in \mathcal{D}\}$. Intuitively, the addition of normalization rules to the facts of a default theory in atomic format turns each atomic default rule $\frac{\alpha_\delta : \beta_\delta}{\gamma_\delta}$ into a default rule of the form $\frac{\alpha_\delta : \beta_\delta}{\beta_\delta \wedge \gamma_\delta}$. With this, the following theorem tells us how to determine Δ-segments orthogonal to a CDP at hand:[8]

Theorem 5. *Let $\langle\delta_i\rangle_{i\in I}$ and $\langle\delta_j\rangle_{j\in J}$ be default proof segments in a default theory $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ in atomic format. Then, $\langle\delta_i\rangle_{i\in I}$ and $\langle\delta_j\rangle_{j\in J}$ are $\mathcal{C}$-orthogonal iff there is some $i \in I$ such that $\langle\delta_j\rangle_{j\in J}$ is a* CDP *for $\neg Conseq(\delta_i) \vee \neg Justif(\delta_i)$ in this default theory:*

$$(\mathcal{D}\setminus\{\delta_k \mid k < i\}, \mathcal{W} \cup N(\mathcal{D}\setminus\{\delta_k \mid k < i\}) \cup \{Conseq(\delta_k), Justif(\delta_k) \mid k < i\}, \mathcal{C}) \quad (6)$$

That is, in order to find Δ-segments orthogonal to our CDP $\left\langle \delta_1 = \frac{R : \neg P}{H}, \delta_2 = \frac{H : T}{T} \right\rangle$, we consider the default theories[9]

$$\left(\left\{ \frac{Q : P}{D}, \frac{R : \neg P}{H}, \frac{D : T}{T}, \frac{H : T}{T} \right\}, \{Q, R\} \cup \{D \rightarrow P, H \rightarrow \neg P\}, \emptyset \right) \quad (7)$$

[8] The following is a variant of Theorem 5.7 in [21], where a more complex normalization procedure is employed. By using $N(\mathcal{D})$ instead, we exploit the fact that we deal with default theories in atomic format only (c.f. Section 2).

[9] For sake of readability, we refrain from presenting Theory (7) and (8) in atomic format. Moreover, we discard normalization rules for normal default rules since they are tautological.

$$\left(\left\{\frac{Q:P}{D},\frac{D:T}{T},\frac{H:T}{T}\right\},\{Q,R\}\cup\{D\rightarrow P\}\cup\{H,\neg P\},\emptyset\right) \tag{8}$$

In turn, we must determine all minimal CDPs of the negated consequences or the negated justifications of $\frac{R:\neg P}{H}$ and $\frac{H:T}{T}$, respectively. That is, first we search for a proof for $\neg H \vee P$ from Theory (7) and then for a proof of $\neg T$ from Theory (8). While $\neg T$ is not provable from (8), $\neg H \vee P$ is provable from (7) yielding a single orthogonal Δ-segment, $\left\langle \frac{Q:P}{D} \right\rangle$.

Theorem 5 leads to the following algorithm for the function $orth$:

$orth(\mathcal{D},\mathcal{W},\mathcal{C},\langle\delta_i\rangle_{i\in I})$ =
for $i \in I$ **do**
  let $\mathcal{D}' = \mathcal{D} \setminus \{\delta_k \mid k < i\}$ **in**
  let $\mathcal{W}' = \mathcal{W} \cup N(\mathcal{D}') \cup \{Conseq(\delta_k), Justif(\delta_k) \mid k < i\}$ **in**
  let $\mathcal{O} = \emptyset$ **in**
  let $\varphi = \neg Conseq(\delta_i) \vee \neg Justif(\delta_i)$ **in**
    $\mathcal{O}$ **:=** $\mathcal{O} \cup \{\langle\delta_j\rangle_{j\in J} \mid \langle\delta_j\rangle_{j\in J} = cred(\mathcal{D}',\mathcal{W}',\mathcal{C},\varphi)$ *and there is no* $J' \subset J$
      such that $\langle\delta_{j'}\rangle_{j'\in J'} = cred(\mathcal{D}',\mathcal{W}',\mathcal{C},\varphi)\ \}$
od ;
return $\mathcal{O}$

This procedure yields a set containing all minimal Δ-segments orthogonal to a given CDP $\langle\delta_i\rangle_{i\in I}$.

In all, our approach has the following advantages. First, it avoids the computation of entire extensions. Second, it is goal-directed and thus restricted to default rules relevant to proving a given query. Third, it takes advantage of basic techniques developed for credulous reasoning. Clearly, it is necessary to consider all mutually orthogonal CDPs belonging to distinct extensions. In this way, we cannot get around the exponential factor present in worst-case, where there is an exponential number of extensions each of which comprises a CDP orthogonal to all CDPs in all other extensions. In fact, skeptical reasoning is Π_2^P-complete [6]. While this theoretical threshold is inevitable in the worst-case, our local proof theory avoids investigating all or even entire extensions whenever a large domain is 'locally structured,' that is, if the query under consideration is deductively connected with merely a small fraction of the entire theory. This advantage becomes obvious by looking at some more examples. Suppose that we extend Default Theory (1) with a default rule like $\frac{R:W}{W}$ saying that "republicans are Western fans." Proving that W is skeptically derivable is doable by a single CDP, $\left\langle\frac{R:W}{W}\right\rangle$, since this CDP is not contested by any orthogonal Δ-segments. For another example, suppose that we extend Default Theory (1) with default rules like $\frac{Q:V}{V}$ and $\frac{R:\neg V}{\neg V}$ saying that "quakers are vegetarians" and "republicans aren't vegetarians." This leads to four distinct extensions. Proving that T is skeptically derivable however is doable with the same steps as described above. That is, the two additional extensions do not increase computational efforts. This is so because the new default rules are irrelevant to proving T.

4 Credulous query-answering

In what follows, we extend the approach for query-answering in Default Logics developed in [18] to Pre-Constrained Default Logic. This approach is based on the Connection Method [1], which allows for testing unsatisfiability of formulas in conjunctive normal form (CNF). Unlike resolution-based methods that decompose formulas in order to derive a contradiction, the Connection Method analyses the structure of formulas for proving their unsatisfiability.

In the Connection Method, formulas in CNF are displayed two-dimensionally in the form of *matrices* (see (9) for an exemplar). A matrix is a set of sets of literals (literal occurrences, to be precise).[10] Each column of a matrix represents a *clause* of the CNF of a formula. In order to show that a sentence φ is entailed by a sentence $\mathcal{W}$, we prove $\mathcal{W} \wedge \neg\varphi$ be unsatisfiable. In the Connection Method this is accomplished by path checking: A *path* through a matrix is a set of literals, one from each clause. A *connection* is an unordered pair of literals which are identical except for the negation sign (and possible indices). A *mating* is a set of connections. A mating *spans* a matrix if each path through the matrix contains a connection from the mating. Finally, a formula, like $\mathcal{W} \wedge \neg\varphi$, is unsatisfiable iff there is a spanning mating for its matrix.

The approach of [18] relies on the idea that a default rule can be decomposed into a classical implication along with two qualifying conditions, one accounting for the character of an inference rule and another one enforcing the respective consistency conditions. The computational counterparts of these qualifying conditions are given by the proof-oriented concepts of *admissibility* and *compatibility*, which we introduce in the sequel.

In order to find out whether a formula φ is in some extension of a default theory $(\mathcal{D}, \mathcal{W}, \mathcal{C})$, we first transform the default rules in $\mathcal{D}$ into their sentential counterparts. This yields a set of indexed implications:

$$\mathcal{W}_{\mathcal{D}} = \left\{ \alpha_\delta \rightarrow \gamma_\delta \;\middle|\; \tfrac{\alpha_\delta : \beta_\delta}{\gamma_\delta} \in \mathcal{D} \right\}$$

Second, we transform both $\mathcal{W}$ and $\mathcal{W}_{\mathcal{D}}$ into their clausal forms, $C_{\mathcal{W}}$ and $C_{\mathcal{D}}$. The clauses in $C_{\mathcal{D}}$, like $\{\neg\alpha_\delta, \gamma_\delta\}$, are called δ*-clauses*; all other clauses like those in $C_{\mathcal{W}}$ are referred to as ω*-clauses*. Finally, a query φ is derivable from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff there is a spanning mating for the matrix $C_{\mathcal{W}} \cup C_{\mathcal{D}} \cup \{\neg\varphi\}$ agreeing with the concepts of admissibility and compatibility.[11]

A useful concept is that of a *core* of a matrix M wrt a mating Π, which allows for isolating the clauses relevant to the underlying proof. [18] defines the core of M wrt Π as[12]

$$\kappa(M, \Pi) = \{c \in M \mid \exists \pi \in \Pi \;.\; c \cap \pi \neq \emptyset\}\,.$$

For instance, the core of Matrix (9) below wrt the mating drawn is given by all clauses connected by arcs. Then, the proof-theoretic counterpart of condition (2), also called groundedness, can be captured as follows [18]:

Definition 6 Admissibility. Let $C_{\mathcal{W}}$ be a set of ω-clauses and $C_{\mathcal{D}}$ be a set of δ-clauses and let Π be a mating for $C_{\mathcal{W}} \cup C_{\mathcal{D}}$. Then, $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ is <u>admissible</u> iff there is an enumeration $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i \in I}$ of $\kappa(C_{\mathcal{D}}, \Pi)$ such that for each $i \in I$, Π is a spanning mating for $C_{\mathcal{W}} \cup \left(\bigcup_{j<i} \{\{\neg\alpha_{\delta_j}, \gamma_{\delta_j}\}\}\right) \cup \{\{\neg\alpha_{\delta_i}\}\}$.

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[10] In the sequel, we simply say literal instead of literal occurrences; the latter allow for distinguishing between identical literals in different clauses.

[11] Without loss of generality, we deal with atomic queries only, since any query can be transformed into ‘atomic format.’

[12] Recall that we deal with literal occurrences.

Note that sometimes not all connections in Π are needed for showing the unsatisfiability of the previous submatrices.

As regards compatibility, we have to extend the corresponding notion found in [18] in order to deal with a set of pre-constraints $\mathcal{C}$.

Definition 7 $\mathcal{C}$-compatibility. Let $C_{\mathcal{W}}$ and $C_{\mathcal{C}}$ be sets of ω-clauses and let $C_{\mathcal{D}}$ be a set of δ-clauses. Let Π be a mating for $C_{\mathcal{W}} \cup C_{\mathcal{D}}$ and let $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i\in I}$ be an enumeration of $\kappa(C_{\mathcal{D}}, \Pi)$. Then, $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ is $\underline{\mathcal{C}\text{-compatible wrt } I}$ iff there is no spanning mating for $C_{\mathcal{W}} \cup C_{\mathcal{C}} \cup \left(\bigcup_{i\in I}\{\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}, \{\beta_{\delta_i}\}\}\right)$.

The following theorem shows that our extended method is sound and complete:

Theorem 8. *Let $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory in atomic format and φ an atomic formula. Then, $\varphi \in E$ for some constrained extension (E, C) of $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff there is a spanning mating Π for the matrix $M = C_{\mathcal{W}} \cup C_{\mathcal{D}} \cup \{\{\neg\varphi\}\}$ and an enumeration $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i\in I}$ of $\kappa(C_{\mathcal{D}}, \Pi)$ which verifies $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ be admissible and $\mathcal{C}$-compatible (wrt I).*

Finally, (M, Π) represents the CDP $\langle\delta_i\rangle_{i\in I}$ for φ from $(\mathcal{D}, \mathcal{W}, \mathcal{C})$.

For illustration, let us verify that T is credulously derivable according to the recipe given above. The encoding of the set of default rules yields the set $\mathcal{W}_{\mathcal{D}}$ of implications: $\{Q_{\delta_1} \to D_{\delta_1}, R_{\delta_2} \to H_{\delta_2}, D_{\delta_3} \to T_{\delta_3}, H_{\delta_4} \to T_{\delta_4}\}$. The indexes denote the respective default rules in (1) from left to right. In order to verify that a republican quaker is traditionalist, T, we first transform the facts in Default Theory (1) and the implications in $\mathcal{W}_{\mathcal{D}}$ into their clausal form. The resulting clauses are given two-dimensionally as the first six columns of the matrix in (9). The full matrix is obtained by adding the clause containing the negated query, $\neg T$. In fact, the matrix has a spanning mating, viz $\{\{R, \neg R_{\delta_2}\}, \{H_{\delta_2}, \neg H_{\delta_4}\}, \{T_{\delta_4}, \neg T\}\}$. We have indicated these connections in (9) as arcs linking the respective literals.

$$\left[\begin{array}{ccccccc} & & \neg Q_{\delta_1} & \neg R_{\delta_2} & \neg D_{\delta_3} & \neg H_{\delta_4} & \neg T \\ Q & R & D_{\delta_1} & H_{\delta_2} & T_{\delta_3} & T_{\delta_4} & \end{array}\right] \tag{9}$$

This proof corresponds to the second one in (4) and yields the following enumeration:

$$\langle\{\neg R_{\delta_2}, H_{\delta_2}\}, \{\neg H_{\delta_4}, T_{\delta_4}\}\rangle \tag{10}$$

For admissibility, we must therefore consider the following two submatrices of Matrix (9):

$$\left[\begin{array}{ccc} & & \neg R_{\delta_2} \\ Q & R & \end{array}\right]\left[\begin{array}{cccc} & & \neg R_{\delta_2} & \neg H_{\delta_4} \\ Q & R & H_{\delta_2} & \end{array}\right] \tag{11}$$

Observe that each of these submatrices has a spanning mating, so the original matrix and its mating, given in (9), constitute an admissible proof.

For compatibility (or $\emptyset$-compatibility, to be precise), we have to verify that the following matrix has no spanning mating:[13]

$$\left[\begin{array}{cccccc} & & \neg R_{\delta_2} & & \neg H_{\delta_4} & \\ Q & R & H_{\delta_2} & \neg P_{\delta_2} & T_{\delta_4} & T_{\delta_4} \end{array}\right] \tag{12}$$

[13] The clauses $\{\neg P_{\delta_2}\}$ and $\{T_{\delta_4}\}$ represent the justifications of δ_2 and δ_4, respectively.

This is indeed the case since the matrix contains a non-complementary path, viz. $\{Q, R, H_{\delta_2}, T_{\delta_4}, \neg P_{\delta_2}\}$. We thus obtain an admissible and compatible proof for the original query, T, asking whether a republican quaker is traditionalist.

5 Skeptical query-answering

Let us now return to skeptical query-answering. The basic idea is to extend the given method for credulous query-answering by adding another specific condition on proofs ensuring that a query is skeptically derivable. This extra condition is motivated by the general idea described in Section 3.

At first, we account for the proof-theoretic counterpart of Δ-segments orthogonal to a given CDP. Let $C_{N(\mathcal{D})}$ be the clausal representation of $N(\mathcal{D})$, i.e., $C_{N(\mathcal{D})} = \{\{\neg\gamma_\delta, \beta_\delta\} \mid \delta \in \mathcal{D}\}$. These clauses are needed for adding the justifications of default rules while determining Δ-segments orthogonal to a CDP at hand. Clearly, this is obsolete for normal default theories.

Definition 9 Challenge. Let $C_\mathcal{W}$ and $C_\mathcal{C}$ be sets of ω-clauses and let $C_\mathcal{D}$ be a set of δ-clauses. Let Π be a mating for $C_\mathcal{W} \cup C_\mathcal{D} \cup \{\{\neg\varphi\}\}$ (for some φ) and let $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i\in I}$ be an enumeration of $\kappa(C_\mathcal{D}, \Pi)$. Then, a challenge Λ at $i \in I$ is a minimal (wrt set inclusion) set of default rules $\Lambda = \{\delta \mid \{\neg\alpha_\delta, \gamma_\delta\} \in \kappa(C_\mathcal{D}, \Pi_i)\}$ for some spanning mating Π_i for the matrix

$$M_i = C_\mathcal{W} \cup C_{N(\mathcal{D})} \cup \{\{\gamma_{\delta_k}\}, \{\beta_{\delta_k}\} \mid k < i\} \cup C_\mathcal{D} \cup \{\{\gamma_{\delta_i}\}, \{\beta_{\delta_i}\}\} \qquad (13)$$

such that (M_i, Π_i) is admissible and $\mathcal{C}$-compatible (wrt some index set I_i).

Observe that the matrix representation allows us to simplify (13) by replacing $C_\mathcal{D}$ and $C_{N(\mathcal{D})}$ by $(C_\mathcal{D} \setminus \{\{\neg\alpha_{\delta_k}, \gamma_{\delta_k}\} \mid k \le i\})$ and $(C_{N(\mathcal{D})} \setminus \{\{\neg\gamma_{\delta_k}, \beta_{\delta_k}\} \mid k \le i\})$, respectively, since the subtracted clauses are subsumed by $\{\{\gamma_{\delta_k}\}, \{\beta_{\delta_k}\} \mid k < i\}$ and the query clauses $\{\{\gamma_{\delta_i}\}, \{\beta_{\delta_i}\}\}$. Now, the overall idea is that $(C_\mathcal{W} \cup C_\mathcal{D}, \Pi)$ represents a CDP $\langle\delta_i\rangle_{i\in I}$ for the considered query. Then, each default proof (M_i, Π_i) induces a challenge Λ that corresponds to a minimal Δ-segment $\mathcal{C}$-orthogonal to $\langle\delta_i\rangle_{i\in I}$. This is so because (M_i, Π_i) represents a CDP of $\neg\mathit{Conseq}(\delta_i) \vee \neg\mathit{Justif}(\delta_i)$ from Default Theory (6) (c.f. Theorem 5). Of course, any matrix M_i may have several spanning matings Π_i and hence may induce different challenges.

Now, we are ready to formulate our additional condition on default proofs for skeptical reasoning:

Definition 10 Protection & Stability. Let $C_\mathcal{W}$ and $C_\mathcal{C}$ be sets of ω-clauses, $C_\mathcal{D}$ be a set of δ-clauses and φ an atomic formula. Let Π be a mating for $C_\mathcal{W} \cup C_\mathcal{D}$ and let $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i\in I}$ be an enumeration of $\kappa(C_\mathcal{D}, \Pi)$. Let $\langle\Lambda_{ij}\rangle_{j\in J_i}$ be the family of all challenges at $i \in I$.

We say that $(C_\mathcal{W} \cup C_\mathcal{D}, \Pi)$ is protected under $\mathcal{C}$ against Λ_{ij} by (M_{ij}, Π_{ij}) iff Π_{ij} is a spanning mating for the matrix

$$M_{ij} = C_\mathcal{W} \cup \{\{\gamma_\delta\} \mid \delta \in \Lambda_{ij}\} \cup C_\mathcal{D} \cup \{\{\neg\varphi\}\}$$

such that (M_{ij}, Π_{ij}) is admissible, $\mathcal{C} \cup \mathit{Justif}(\Lambda_{ij})$-compatible, and stable for φ under $\mathcal{C} \cup \mathit{Justif}(\Lambda_{ij})$. We say that $(C_\mathcal{W} \cup C_\mathcal{D}, \Pi)$ is stable for φ under $\mathcal{C}$ iff $(C_\mathcal{W} \cup C_\mathcal{D}, \Pi)$ is protected under $\mathcal{C}$ against all challenges Λ_{ij}.

The idea is that $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ represents a CDP for query φ. In order to verify whether φ is skeptically derivable, we need to show that $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ in addition satisfies the stability criterion. For this, we proceed as follows. First, we isolate all challenges Λ against our CDP $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$. In turn we verify whether φ is also skeptically derivable from the matrices obtained by adding the consequents of the default rules in Λ to $C_{\mathcal{W}}$ and taking the justifications of the default rules in Λ as additional constraints on the compatibility check. This amounts to verifying whether φ is in all extensions to which the default rules in Λ contribute. Accordingly, a skeptical default proof is given by a stable credulous default proof $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ along with all its protecting default proofs (M_{ij}, Π_{ij}). Of course, there may be several such skeptical proofs depending on the initial choice.

Now, let us examine whether our default proof in (9) is stable and thus renders T a skeptical conclusion of Default Theory (1). For this, we consider the obtained enumeration $\langle \{\neg R_{\delta_2}, H_{\delta_2}\}, \{\neg H_{\delta_4}, T_{\delta_4}\} \rangle$. In turn, we determine all emerging challenges. That is, we consider all minimal default proofs of $\neg H_{\delta_2} \vee P_{\delta_2}$ and $\neg T_{\delta_4}$. These formulas represent the negated consequents (and justifications) of the used default rules, δ_2 and δ_4, respectively.

In the first case, we consider the matrix obtained from our original matrix, (9), by replacing query clause $\{\neg T\}$ by clauses $\{H_{\delta_2}\}$ and $\{\neg P_{\delta_2}\}$. This allows us moreover to eliminate the δ-clause $\{\neg R_{\delta_2}, H_{\delta_2}\}$ from (9) since it is subsumed by $\{H_{\delta_2}\}$. Analogously, we can omit the normalization clause $\{\neg H_{\delta_2}, \neg P_{\delta_2}\}$ due to the presence of $\{\neg P_{\delta_2}\}$. Hence, we have to add only the normalization clause $\{\neg D_{\delta_1}, P_{\delta_1}\}$ since δ_3 and δ_4 are normal default rules. The modifications to our initial matrix in (9) are indicated as dashed boxes.[14] This results in the following derivative of Matrix (9):[15]

$$M_2 = \left[\begin{array}{ccccccccc} & & \neg Q_{\delta_1} & & \neg D_{\delta_3} & \neg H_{\delta_4} & \neg D_{\delta_1} & H_{\delta_2} & \neg P_{\delta_2} \\ Q & R & D_{\delta_1} & & T_{\delta_3} & T_{\delta_4} & P_{\delta_1} & & \end{array} \right] \qquad (14)$$

By discarding the two query clauses, Matrix[16] M_2 can be seen as the proof-theoretic counterpart of Default Theory (7). In fact, M_2 admits the spanning mating $\Pi_2 = \{\{Q, \neg Q_{\delta_1}\}, \{D_{\delta_1}, \neg D_{\delta_1}\}, \{P_{\delta_1}, \neg P_{\delta_2}\}\}$. This CDP involves a single default rule, viz. δ_1. We thus obtain the singleton enumeration $\langle \{\neg Q_{\delta_1}, D_{\delta_1}\} \rangle$ inducing the following two matrices for verifying admissibility and compatibility, respectively:

$$\left[\begin{array}{ccccc} & & \neg Q_{\delta_1} & \neg D_{\delta_1} & \neg H_{\delta_2} \\ Q & R & & P_{\delta_1} & \neg P_{\delta_2} \end{array} \right] \left[\begin{array}{ccccc} & & \neg Q_{\delta_1} & \neg D_{\delta_1} & \neg H_{\delta_2} \\ Q & R & D_{\delta_1} & P_{\delta_1} & \neg P_{\delta_2} \end{array} \right] \qquad (15)$$

Both matrices contain the normalization clauses $\{\neg D_{\delta_1}, P_{\delta_1}\}$ and $\{\neg H_{\delta_2}, \neg P_{\delta_2}\}$. The left matrix is complementary and thus confirms admissibility, while the right matrix has open path $\{Q, R, D_{\delta_1}, P_{\delta_1}, \neg H_{\delta_2}\}$ establishing compatibility. Consequently, (M_2, Π_2)

[14] This is done to underline the utility of structure-oriented theorem proving.

[15] For simplicity, we have refrained from turning the *two* query clauses $\{H_{\delta_2}\}$ and $\{\neg P_{\delta_2}\}$ into $\{H_{\delta_2}, \varphi\}, \{\neg P_{\delta_2}, \varphi\}$ along with the *single atomic* query clause $\{\neg\varphi\}$ (as stipulated in Theorem 8).

[16] The index 2 of M_2 and Π_2 reflects the index of the query $\neg H_{\delta_2} \vee P_{\delta_2}$.

provides us with a CDP of $\neg H_{\delta_2} \vee P_{\delta_2}$. This CDP is orthogonal to our initial proof in (9). As a result, (M_2, Π_2) induces the challenge $\Lambda_{21} = \{\delta_1\}$. There is no other challenge induced by M_2.

Now, let us first verify whether our CDP in (9) is protected against Λ_{21} by some other CDP before we determine more challenges: For establishing stability, intuitively, we verify whether our initial query, T, belongs to the extensions formed (among others) by the default rules in $\Lambda_{21} = \{\delta_1\}$. For this, we consider the matrix M_{21} obtained from our initial matrix in (9) by adding the consequents of all default rules in Λ_{21}. This amounts to replacing δ-clause $\{\neg Q_{\delta_1}, D_{\delta_1}\}$ in (9) by ω-clause $\{D_{\delta_1}\}$:

$$M_{21} = \left[\begin{array}{ccccccc} & & & \neg R_{\delta_2} & \neg D_{\delta_3} & \neg H_{\delta_4} & \neg T \\ Q & R & D_{\delta_1} & H_{\delta_2} & T_{\delta_3} & T_{\delta_4} & \end{array}\right] \tag{16}$$

According to Definition 10 we then verify whether there is a spanning mating Π_{21} for M_{21} such that (M_{21}, Π_{21}) is admissible, $\{P_{\delta_1}\}$-compatible (since *Justif*$(\Lambda_{21}) = \{P_{\delta_1}\}$), and stable. Admissibility and compatibility of (M_{21}, Π_{21}) are easily verified by checking the two following matrices induced by the only δ-clause used in (16), i.e., $\{\neg D_{\delta_3}, T_{\delta_3}\}$:

$$\left[\begin{array}{cccc} & & & \neg D_{\delta_3} \\ Q & R & D_{\delta_1} & \end{array}\right] \left[\begin{array}{ccccc} & & & \neg D_{\delta_3} & \\ Q & R & D_{\delta_1} & T_{\delta_3} & P_{\delta_1} \end{array}\right]$$

Complementarity of the left matrix establishes admissibility while non-complementarity (initiated by the open path $\{Q, R, D_{\delta_1}, T_{\delta_3}, P_{\delta_1}\}$) of the right matrix confirms $\{P_{\delta_1}\}$-compatibility. Now, it remains to be shown that (M_{21}, Π_{21}) along with its induced enumeration $\langle\{\neg D_{\delta_3}, T_{\delta_3}\}\rangle$ is stable for T under $\{P_{\delta_1}\}$. However, there is no challenge to $\{\neg D_{\delta_3}, T_{\delta_3}\}$ since $\neg T_{\delta_3}$ is not provable from the matrix obtained by replacing query clause $\{\neg T\}$ in (16) by $\{T_{\delta_3}\}$.[17] As a result, we obtain that our CDP in (9) is protected against Λ_{21} by (M_{21}, Π_{21}).

Next, we must consider all challenges of the second δ-clause in (10). For this, we determine all CDPs of $\neg T_{\delta_4}$ (the consequent of δ_4) from the matrix obtained in the following way. First, we add the clauses $\{H_{\delta_2}\}$ and $\{\neg P_{\delta_2}\}$ representing the 'consequent' and the 'justification' of the first δ-clause in (10) to our initial matrix, (9). The first addition is accomplished by replacing δ-clause $\{\neg R_{\delta_2}, H_{\delta_2}\}$ by $\{H_{\delta_2}\}$. Second, we add the normalization clause $\{\neg D_{\delta_1}, P_{\delta_1}\}$. As with Matrix (14), the second normalization clause $\{\neg H_{\delta_2}, \neg P_{\delta_2}\}$ can be omitted since it is subsumed by $\{\neg P_{\delta_2}\}$. Again, no normalization clauses are added for the normal default rules δ_3 and δ_4. Finally, we replace the original query clause $\{\neg T\}$ of (9) by $\{T_{\delta_4}\}$. This allows us to eliminate δ-clause $\{\neg H_{\delta_4}, T_{\delta_4}\}$ since it is subsumed by $\{T_{\delta_4}\}$:

$$\left[\begin{array}{ccccccccc} & & \neg Q_{\delta_1} & & \neg D_{\delta_3} & & \neg P_{\delta_2} & \neg D_{\delta_1} & T_{\delta_4} \\ Q & R & D_{\delta_1} & H_{\delta_2} & T_{\delta_3} & & & P_{\delta_1} & \end{array}\right]$$

[17] In fact, there is no compatible default proof.

Observe that this matrix can be regarded as the proof-theoretic counterpart of Default Theory (8) if we discard the query clause. The above matrix has a spanning mating inducing the enumeration $\langle\{\neg Q_{\delta_1}, D_{\delta_1}\}\rangle$. Admissibility of the previous proof can be verified in a straightforward way, and to test compatibility, we have to consider the following matrix:

$$\begin{bmatrix} & & \neg Q_{\delta_1} & H_{\delta_2} & \neg P_{\delta_2} & \neg D_{\delta_1} \\ Q & R & D_{\delta_1} & & & P_{\delta_1} \end{bmatrix}$$

Obviously, this matrix has no open path so that our proof is not compatible. Accordingly, $\neg T_{\delta_4}$ is not derivable (c.f. Default Theory (8)) and therefore there are no more challenges, apart from Λ_{21}. In this way, we have shown that our initial CDP in (9) is stable for T (in addition to its admissibility and compatibility verified in Section 4). This is so because it is protected against its only challenge Λ_{21} by (M_{21}, Π_{21}). This tells us that T is skeptically derivable from Default Theory (9).

In general, we have the following result stating the adequacy of our proof method:

Theorem 11. *Let $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ be a default theory in atomic format and φ an atomic formula. Then, $\varphi \in E$ for all constrained extension (E, C) of $(\mathcal{D}, \mathcal{W}, \mathcal{C})$ iff there is a spanning mating Π for the matrix $M = C_{\mathcal{W}} \cup C_{\mathcal{D}} \cup \{\{\neg\varphi\}\}$ and an enumeration $\langle\{\neg\alpha_{\delta_i}, \gamma_{\delta_i}\}\rangle_{i\in I}$ of $\kappa(C_{\mathcal{D}}, \Pi)$ such that $(C_{\mathcal{W}} \cup C_{\mathcal{D}}, \Pi)$ is admissible wrt I, $\mathcal{C}$-compatible wrt I, and stable for φ under $\mathcal{C}$.*

6 Conclusion

We have developed an approach to skeptical query-answering in Constrained Default Logic based on the Connection Method. This has been accomplished by elaborating on a recently proposed, general idea for skeptical reasoning in (semi-monotonic) Default Logics [21]. As a result, we have obtained a precise algorithm that returns a skeptical default proof if the query is contained in all extensions of the underlying default theory. The approach has then been combined with a method for credulous query-answering based on the Connection Method. This was accomplished by employing a further restriction on credulous default proofs, expressed by the *stability* criterion. This has led to a homogeneous characterization of skeptical default proofs at the level of the underlying deduction method. This approach was supported by the structure-sensitive nature of the Connection Method. The value of this for structure-sharing among the diverse subproofs involved is detailed for credulous query-answering in [18]. Even though we have not discussed it here, it should be obvious that the utility of structure-sharing applies to skeptical query-answering, too. We have tried to indicate this by stressing the common structures involved in the skeptical default proof carried out in the previous section.

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