Theory Solving made easy with *Clingo* 5
(Extended Version*)

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Abstract

Answer Set Programming (ASP) is a model, ground, and solve paradigm. The integration of application- or theory-specific reasoning into ASP systems thus impacts on many if not all elements of its workflow, viz. input language, grounding, intermediate language, solving, and output format. We address this challenge with the fifth generation of the ASP system *clingo* and its grounding and solving components by equipping them with well-defined generic interfaces facilitating the manifold integration efforts. On the grounder’s side, we introduce a generic way of specifying language extensions and propose an intermediate format accommodating their ground representation. At the solver end, this is accompanied by high-level interfaces easing the integration of theory propagators dealing with these extensions.

1 Introduction

The *clingo* system, along with its grounding and solving components *gringo* and *clasp*, is nowadays among the most widely used tools for Answer Set Programming (ASP; [22]). This does not only apply to end-users, but more and more to system developers who build upon *clingo*’s infrastructure for developing their own systems. Among them, we find (alphabetically) *clasp-nk* [13], *clingcon* [25], *dflat* [1], *dingo* [21], *dlvhex* [14], *inca* [12], and *mingo* [23]. None of these systems can use *clingo* or its components without workarounds or even involved modifications to realize the desired functionality. Moreover, since ASP is a model, ground, and solve paradigm, such modifications are rarely limited to a single component but often spread throughout the whole workflow. This begins with the addition of new language constructs to the input language, requiring in turn amendments to the grounder as well as syntactic means for passing the ground constructs to a downstream system. In case they are to be dealt with by an ASP solver, it must be enabled to treat the specific input and incorporate corresponding solving capacities. Finally, each such extension is application-specific and requires different means at all ends.

We address this challenge with the new *clingo* series 5 and its components. This is accomplished by introducing generic interfaces that allow for accommodating extensions to ASP at the salient stages of its workflow. To begin with, we extend *clingo*’s grounder component *gringo* with means for specifying simple theory grammars in which new theories can be represented. As theories are expressed using constructs close to ASP’s basic modeling language, the existing grounding machinery takes care of instantiating them. This also involves a new intermediate ASP format that allows for passing the enriched information from grounders to solvers in a transparent way. (Since this format is mainly for settings with stand-alone grounders and solvers, and thus outside the scope of *clingo*, we delegate details to the Appendix in A.) For a complement, *clingo* 5 provides several interfaces for reasoning with theory expressions. On the one hand, the existing Lua and Python APIs are extended by high-level interfaces for augmenting propagation in *clasp* with so-called *theory propagators*. Several such propagators can be registered with *clingo*, each implementing an interface of four basic methods. Our design is driven by the objective to provide means for rapid prototyping of dedicated reasoning procedures while enabling effective implementations. To this

* Abridged version appeared in [17].
#theory difference {
const { - : 0, unary };  
diff_term { - : 0, binary, left };  
linear_term { + : 2, unary; - : 2, unary;  
* : 1, binary, left;  
+ : 0, binary, left; - : 0, binary, left };  
domain_term { .. : 1, binary, left };  
show_term { / : 1, binary, left };  
&dom/0 : domain_term, {=} , linear_term, any;  
&sum/0 : linear_term, { <=,=, >=,<,>, !=} , linear_term, any;  
&diff/0 : diff_term, { <=}, constant, any;  
&show/0 : show_term, directive  
}.

#const n=2. #const m=1000.
task (1.. n). duration ( T,200 *T) :- task (T).
&dom { 1.. m } = start (T) :- task (T).
&dom { 1.. m } = end (T) :- task (T).
&diff { end (T)- start (T) } <= D :- duration ( T,D).
&sum { end (T) : task (T); -start (T) : task (T) } <= m.
&show { start /1; end /1 }.

Listing 1: Logic program enhanced with difference and linear constraints (diff.lp)

end, the interface supports, for instance, stateful theory propagators as well as multi-threading in the underlying solver. On the other hand, the functionality of the aforementioned extended APIs is now also offered via a C interface. This is motivated by the wide availability of foreign function interfaces for C, which enable the import of clingo in programming languages like Java or Haskell. A first application of this is the integration of clingo 5 into SWI-Prolog.¹

## 2 Input Language

This section introduces the novel features of clingo 5’s input language. All of them are situated in the underlying grounder gringo 5 and can thus also be used independently of clingo. We start with a detailed description of gringo 5’s generic means for defining theories and afterwards summarize further new features.

Our generic approach to theory specification rests upon two languages: the one defining theory languages and the theory language itself. Both borrow elements from the underlying ASP language, foremost an aggregate-like syntax for formulating variable length expressions. To illustrate this, consider Listing 1, where a logic program is extended by constructs for handling difference and linear constraints. While the former are binary constraints of the form \(x_1 - x_2 \leq k\), the latter have a variable size and are of form \(a_1 x_1 + \cdots + a_n x_n \circ k\), where \(x_i\) are integer variables, \(a_i\) and \(k\) are integers, and \(\circ \in \{\leq, \geq, <, >, =\}\) for \(1 \leq i \leq n\).² Note that solving difference constraints is polynomial, while solving linear equations (over integers) is NP-hard. The theory language for expressing both types of constraints is defined in Lines 1–13 and preceded by the directive #theory. The elements of the resulting theory language are preceded by & and used as regular atoms in the logic program in Lines 15–21.

To be more precise, a theory definition has the form

```
#theory T {D_1;...;D_n}.  
```

where \(T\) is the theory name and each \(D_i\) is a definition for a theory term or a theory atom for \(1 \leq i \leq n\). The language induced by a theory definition is the set of all theory atoms constructible from its theory atom definitions.

A theory atom definition has form

```
&p/k : t,o or &p/k : t,{o_1,...,o_m},t',o  
```

¹https://github.com/JanWielemaker/clingo
²For simplicity, we consider normalized difference constraints rather than general ones of form \(x_1 - x_2 \circ k\).
where $p$ is a predicate name and $k$ its arity, $t, t'$ are names of theory term definitions, each $\diamond$, is a theory operator for $m \geq 1$, and $o \in \{\text{head, body, any directive}\}$ determines where theory atoms may occur in a rule. Examples of theory atom definitions are given in Lines 9–12 of Listing 1. The language of a theory atom definition as above contains all theory atoms of form

$$\text{theory atom}: (D_1; \ldots; D_n)$$

where $k$ is an atom over predicate $p$ of arity $k$, each $D_i$ is a tuple of theory terms in the language for $t, c$ is a theory term in the language for $t', o$ is a theory operator among $\{o_1, \ldots, o_m\}$, and each $L_i$ is a regular condition (i.e., a tuple of regular literals) for $1 \leq i \leq n$. Whether the last part $'o c'$ is included depends on the form of a theory atom definition. Five occurrences of theory atoms can be found in Lines 17–21 of Listing 1.

A theory term definition has form

$$t \langle D_1; \ldots; D_n \rangle$$

where $t$ is a name for the defined terms and each $D_i$ is a theory operator definition for $1 \leq i \leq n$. A respective definition specifies the language of all theory terms that can be constructed via its operators. Examples of theory term definitions are given in Lines 2–8 of Listing 1. Each resulting theory term is one of the following:

- a constant term: $c$
- a variable term: $v$
- a binary theory term: $t_1 \diamond t_2$
- a unary theory term: $\diamond t$

where each $t_i$ is a theory term, $\diamond$ is a theory operator defined by some $D_i$, $c$ and $f$ are symbolic constants, $v$ is a first-order variable, $k \geq 1$, and $l \geq 0$. (The trailing comma in tuple theory terms is optional if $l \neq 1$.) Parentheses can be used to specify operator precedence.

A theory operator definition has form

$$\diamond : p, \text{unary} \quad \text{or} \quad \diamond : p, \text{binary}, a$$

where $\diamond$ is a unary or binary theory operator with precedence $p \geq 0$ (determining implicit parentheses). Binary theory operators are additionally characterized by an associativity $a \in \{\text{right, left}\}$. As an example, consider Line 5 of Listing 1, where the binary operator $\ast$ is defined with precedence 1 and left associativity. In total, Lines 2–8 of Listing 1 include nine theory operator definitions. Particular theory operators can be assembled (written consecutively without spaces) from the symbols ‘$+$’, ‘$-$’, ‘$\ast$’, ‘$\div$’, ‘$\backslash$’, ‘$\setminus$’, ‘$\cap$’, ‘$\cup$’..., ‘$\emptyset$’, ‘$\neg$’, ‘$\lt$’, ‘$\gt$’, ‘$\leq$’, ‘$\geq$’, ‘$\lt=$’, ‘$\gt=$’, ‘$\neq$’, ‘$\equiv$’, ‘$\dashv$’, ‘$\mapsto$’, ‘$\mid$’, ‘$\not|\not\mid$’, ‘$\leftarrow$’ and ‘$\rightarrow$’. As an aggregate-like construction for forming variable-length theory expressions. In this way, standard grounding techniques can be used for gathering theory terms. (However, the actual atom within a theory atom comprises regular terms only.) The treatment of theory terms still differs from their regular counterparts in that the grounder skips simplifications like, e.g., arithmetic evaluation. This can be nicely seen on the different results in Listing 2 of grounding terms formed with the regular and theory-specific variants of operator ‘$\ast$’. Observe that the fact task(1..n) in Line 16 of Listing 1 results in n ground facts, viz. task(1) and task(2) because of n=2. Unlike this, the theory expression 1..n stays structurally intact and is only transformed into 1..1000 in view of m=1000. That is, the grounder does not evaluate the theory term 1..1000 and leaves its interpretation to a downstream theory solver. A similar situation is encountered when comparing the treatment of the regular term ‘200*T’ in Line 16 of Listing 1 to the theory term ‘end(T)-start(T)’ in Line 19. While each instance of ‘200*T’ is evaluated during grounding, instances of the theory term are left in Line 6 of Listing 2. In fact, if ‘200*T’ had been a theory term as well, it would have resulted in the unevulated instances ‘200*1’ and ‘200*2’.

The remainder of this section is dedicated to other language extensions of gringo 5 aiming at a disentanglement of the various uses of #show directives (and their induced symbol table). Such directives were beforehand used for controlling the output of stable models, delineating the scope of reasoning modes (e.g., intersection, union, projection, etc.), and for passing special-purpose information to downstream
systems. For instance, theory and heuristic information was passed to clasp via dedicated predicates like _edge and _heuristic. This entanglement brought about several shortcomings. In fact, passing information via a symbol table did not only scramble the output, but also provoked overhead in grounding and filtering “artificial” symbolic information.

Now, in gringo 5, the sole purpose of #show is to furnish an output directive. There are three different kinds of such statements:

```
#show. #show p/n. #show t : l1,...,ln.
```

The first form hides all atoms, and the second all except those over predicates p/n indicated by #show statements. The third form does not hide any atoms and can be used to output arbitrary terms t, whenever the literals l1,...,ln in the condition after ‘:’ hold. This is particularly useful in meta-programming, e.g., ‘#show A : holds(A)’ can be used to map back reified atoms. Atoms used in reasoning modes are indicated by #project directives, having two forms:

```
#project p/n. #project a : l1,...,ln.
```

Here, p is a predicate name with arity n, a is an atom, and l1,...,ln are literals. While the first form declares all atoms over predicate p/n as subject to projection, the second includes instances of a obtained via grounding, as detailed in [18] for #external directives.

The last two new directives of interest abolish the need for the special-purpose predicates _edge and _heuristic, previously used in conjunction with the ASP solver clasp:

```
#edge (a,v) : l1,...,ln. #heuristic a : l1,...,ln. [k@p,m]
```

As above, a is an atom, and l1,...,ln are literals. Moreover, u, v, k, p, m are terms, where ‘(u,v)’ stands for an edge from u to v in an acyclicity extension [8]. Integer values for k and p along with init, factor, level, sign, true, or false for m determine a heuristic modifier [19]. Finally, note that zero is taken as default priority when the optional ‘@p’ part in ‘[k@p,m]’, resembling the syntax of ranks for weak constraints [10], is omitted.

3 Logical Characterization

The semantics of logic programs modulo theories rests upon ground programs P over two disjoint alphabets, A and T, consisting of regular and theory atoms. Accordingly, P is a set of rules r of the form h ← a1,...,am,¬a_{m+1},...,¬an, where the head h is constant ⊥, a0 or \{a0\} for an atom a0 ∈ A ∪ T, and \{a1,...,an\} ⊆ A ∪ T. If h = ⊥, r is called an integrity constraint, a normal rule if h = a0, or a choice rule if h = \{a0\}; as usual, we skip ⊥ when writing integrity constraints. We let h(r) = \{a0\} for an integrity constraint r, h(r) = \{a0\} for a normal or choice rule r, and define h(P) = \bigcup_{r∈P} h(r) as the head atoms of P. In analogy to inputs atoms from #external directives [18], we partition T into defined theory atoms T ∩ h(P) and external theory atoms T \ h(P).

Given a collection T of theories, we associate each T ∈ T with a scope T^T of atoms relevant to T, and let T = \bigcup_{T∈T} T^T be the corresponding set of theory atoms. Reconsidering the input language in Section 2, a natural choice for T^T consists of all (ground) atoms declared within a #theory directive for T. However, as we see in Section 5, a scope may in general include atoms written in regular as well as extended syntax (the latter preceded by ‘?’) in the input language.

In order to reflect different forms of theory propagation, we further consider a partition of the scope T^T of a theory T into strict theory atoms T^T_s and non-strict theory atoms T^T_ns such that T^T_s ∩ T^T_ns = ∅.
and $\mathcal{T}_T^e \cup \mathcal{T}_T^r = \mathcal{T}_T^r$. The strict theory atoms in $\mathcal{T}_T^e$ resemble equivalences as expressed by the constraint atoms of $\text{clingcon}$ [25], which must be assigned to true iff their associated constraints hold. This is complemented by viewing the non-strict theory atoms in $\mathcal{T}_T^r$ as implications similar to the constraint statements of $\text{ezcsp}$ [2], where only statements assigned to true impose requirements, while constraints associated with false ones are free to hold or not. Given the distinction of respective kinds of theory atoms, a combined theory $T$ may integrate constraints according to the semantics of $\text{clingcon}$ and $\text{ezcsp}$, e.g., indicated by dedicated predicates or arguments thereof in $T$’s theory language.

We now turn to mapping the semantics of logic programs modulo theories back to regular stable models. In the abstract sense, we call any $\mathcal{S}^T \subseteq \mathcal{T}_T^r$ a $T$-solution if $T$ is consistent with the conditions expressed by elements of $\mathcal{S}^T$ as well as the complements of conditions associated with the false strict theory atoms in $\mathcal{T}_T^e \setminus \mathcal{S}_T^T$.

Generalizing this concept to a collection $\mathcal{T}$ of theories, we say that $\mathcal{S} \subseteq \mathcal{T}$ is a $\mathcal{T}$-solution if $\mathcal{S} \cap \mathcal{T}_T^r$ is a $T$-solution for each $T \in \mathcal{T}$. Then, we define a set $X \subseteq \mathcal{A} \cup \mathcal{T}$ of (regular and theory) atoms as a $\mathcal{T}$-stable model of a ground program $P$ if there is some $T$-solution $\mathcal{S}$ such that $X$ is a (regular) stable model of the program

$$P \cup \{a \leftarrow \big| T \in \mathcal{T}, a \in (\mathcal{T}_T^e \setminus h(P)) \cap \mathcal{S}\} \cup \{\sim a \mid T \in \mathcal{T}, a \in (\mathcal{T}_T^e \setminus h(P)) \cap \mathcal{S}\}$$

(1)

$$\cup \{[a] \leftarrow \big| T \in \mathcal{T}, a \in (\mathcal{T}_T^r \setminus h(P)) \cap \mathcal{S}\} \cup \{\sim a \mid T \in \mathcal{T}, a \in (\mathcal{T}_T^r \setminus h(P)) \setminus \mathcal{S}\}.$$  

(2)

That is, the rules added to $P$ in (1) and (2) express conditions aligning $X \cap \mathcal{T}$ with an underlying $T$-solution $\mathcal{S}$. First, the facts in (1) make sure that external theory atoms that are strict, i.e., included in $\mathcal{T}_T^e \setminus h(P)$ for some $T \in \mathcal{T}$, and hold in $\mathcal{S}$ belong to $X$ as well. Unlike this, the corresponding set of choice rules in (2) merely says that non-strict external theory atoms from $\mathcal{S}$ may be included in $X$, thus not insisting on a perfect match between non-strict theory atoms and elements of $\mathcal{S}$. Moreover, the integrity constraints in (1) and (2) take care of defined theory atoms belonging to $h(P)$. The respective set in (1) again focuses on strict theory atoms and stipulates the ones from $\mathcal{S}$ to be included in $X$ as well. In addition, for both strict and non-strict defined theory atoms, the integrity constraints in (2) assert the falsity of atoms that do not hold in $\mathcal{S}$.

For example, consider a program $P = \{a \leftarrow b, \sim c\}$ subject to some theory $T$ with the strict and non-strict theory atoms $\mathcal{T}_T^e = \{a, b\}$ and $\mathcal{T}_T^r = \{c\}$, and let $\mathcal{S} = \{a, b, c\}$ be a $T$-solution. Then, the extended program for $\mathcal{S}$ is $P \cup \{b \leftarrow \{c\} \leftarrow \sim a\}$, whose (only) regular stable model $X = \{a, b\}$ is a $\{T\}$-stable model of $P$. Note that $\mathcal{S}$ assigns the non-strict theory atom $c$ to true, while $X$ excludes it to keep $a \leftarrow b, \sim c$ applicable for the (strict) defined theory atom $a$.

To summarize the main principles of the $T$-stable model concept, strict theory atoms (for some $T \in \mathcal{T}$) must exactly match their interpretation in a $T$-solution $\mathcal{S}$, while non-strict ones (not strict for any $T \in \mathcal{T}$) in $X$ are only required not to exceed $\mathcal{S}$. Second, external theory atoms that hold in $\mathcal{S}$ are mapped to facts or choice rules, while conditions on defined ones are enforced by means of integrity constraints. As a result, $T$-stable models are understood as regular stable models, yet relative to extensions of a given program $P$ determined by underlying $T$-solutions. Notably, the concept of $T$-stable models also carries on to logic programs allowing for further constructs, such as weight constraints and disjunction, which have not been discussed here for brevity (cf. [26]).

### 4 Algorithmic Characterization

As detailed in [20], a ground program $P$ induces completion and loop nogoods, given by $\Delta_P = \Delta_B(P) \cup \Delta_{A \cup (T \setminus h(P))}$ or $\Delta_P = \bigcup_{B \subseteq U \subseteq A \cup (T \setminus h(P))} \{\lambda(a, U) \mid a \in U\}$, respectively, where $B(P)$ is the set of rule bodies occurring in $P$. Note that both sets of nogoods are restricted by regular atoms and defined theory atoms, while external theory atoms can a priori be assigned freely, although any occurrences in rule bodies are subject to evaluation via respective nogoods in $\Delta_B(P)$. A (partial) assignment $\mathcal{A}$ is a consistent set of (signed) literals of the form $T v$ or $F v$ for $v \in (A \cup T) \cup B(P)$, i.e., $\{T v, F v\} \not\subseteq \mathcal{A}$ for all $v \in (A \cup T) \cup B(P)$; $\mathcal{A}$ is total if $\{T v, F v\} \cap \mathcal{A} = \emptyset$ for all $v \in (A \cup T) \cup B(P)$. We say that some nogood $\delta$ is violated by $\mathcal{A}$ if $\delta \subseteq \mathcal{A}$. When $T = \emptyset$, so that $T = \emptyset$ as well, each total assignment $\mathcal{A}$ that does not violate any nogood $\delta \in \Delta_P \cup \Delta_P$ yields a regular stable model of $P$, and such an assignment $\mathcal{A}$ is called a solution (for $\Delta_P \cup \Delta_P$).

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3 Although we omit formal details, atoms in Satisfiability Modulo Theories (SMT; [4]) belong to first-order predicates interpreted in a theory $T$, and the ones that hold in some model of $T$ provide a $T$-solution $\mathcal{S}^T \subseteq \mathcal{T}^T$. 

5
Figure 1: Basic algorithm for Conflict-Driven Constraint Learning (CDCL) modulo theories

We now extend the concept of a solution to \( \mathcal{T} \)-stable models. To this end, we follow the idea of external propagators in [12] and identify a theory \( T \in \mathcal{T} \) with a set \( \Delta_T \subseteq 2^{(\mathcal{F}_a \cup \mathcal{F} \cup \mathcal{T})} \) of theory nogoods such that, given a total assignment \( A \), we have that \( \delta \subseteq A \) for some \( \delta \in \Delta_T \) if there is no \( T \)-solution \( S_T \) such that \( \{a \in T \mid \mathcal{F}_a \in A\} \subseteq S_T \) and \( \{a \in \mathcal{T} \mid \mathcal{F}_a \in A\} \cap S_T = \emptyset \). That is, the nogoods in \( \Delta_T \) must reject \( A \) iff no \( T \)-solution (i) includes all theory atoms in \( T \) that are assigned to true by \( A \) and (ii) excludes all strict theory atoms in \( T \) assigned to false by \( A \). This semantic condition establishes a (one-to-one) correspondence between \( T \)-stable models of \( P \) and solutions for \((\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in \mathcal{T}} \Delta_T \).

**Proposition 1** Let \( P \) be a program, \( \mathcal{T} \) be a collection of theories, \( X \subseteq A \cup \mathcal{T} \), and \( B \subseteq B(P) \) the set of rule bodies that hold w.r.t. \( X \). Then, we have that \( X \) is a \( \mathcal{T} \)-stable model of \( P \) iff \( A = \{Tv \mid v \in X \cup B\} \cup \{Fv \mid v \in ((A \cup \mathcal{T}) \setminus X) \cup (B(P) \setminus B)\} \) is a solution for \((\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in \mathcal{T}} \Delta_T \).

The nogoods in \((\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in \mathcal{T}} \Delta_T \) provide the logical fundament for the Conflict-Driven Constraint Learning (CDCL) procedure (cf. [24, 20]) outlined in Figure 1. While the completion nogoods in \( \Delta_P \) are usually made explicit and subject to unit propagation, the loop nogoods in \( \Delta_T \) as well as theory nogoods in \( \Delta_T \) are typically handled by dedicated propagators and particular members are selectively recorded, i.e., when a respective propagator identifies some nogood \( \delta \) such that \( |\delta \setminus A| \leq 1 \) and \( \{Tv \mid Fv \in \delta\} \cup \{Fv \mid Tv \in \delta\} \cap A = \emptyset \), and we say that such a nogood is unit. In fact, a unit nogood \( \delta \) yields either a conflict, if \( \delta \) is violated by \( A \), or otherwise a literal to be assigned by unit propagation.

While the dedicated propagator for loop nogoods is built-in in systems like clingo 5, those for theories are provided via the interface detailed in Section 5. To utilize custom propagators, Figure 1 includes an initialization step in Line (I). In addition to the “registration” of a propagator for a theory \( T \) as an extension of the basic CDCL procedure, common tasks performed in this step include setting up internal data structures and so-called watches for (a subset of) the theory atoms in \( T \), so that the propagator will be invoked (only) when some watched literal gets assigned.

The main CDCL loop starts with unit propagation on completion and loop nogoods, the latter handled by the respective built-in propagator, as well as any nogoods already recorded. If this results in a non-total assignment without conflict, theory propagators for which some of their watched literals have been assigned are invoked in Line (P). A propagator for a theory \( T \) can then inspect the current assignment, update its data structures accordingly, and most importantly, perform theory propagation determining theory nogoods \( \delta \in \Delta_T \) to record. Usually, any such nogood \( \delta \) is unit in order to trigger a conflict or unit propagation, although this is not a necessary condition. The interplay of unit and theory propagation continues until a conflict or total assignment arises, or no (further) watched literals of theory propagators get assigned by unit propagation. In the latter case, some non-deterministic decision is made to extend the partial assignment at hand and then to proceed with unit and theory propagation.

If no conflict arises and an assignment \( A \) is total, in Line (C), theory propagators are called, one by one, for a final check of \( A \). The idea is that, e.g., a “lazy” propagator for a theory \( T \) that does not
exhaustively test violations of its theory nogoods by partial assignments can make sure that $A$ is indeed a solution for $\Delta_T$, or record some violated nogood(s) from $\Delta_T$ otherwise. Even in case theory propagation on partial assignments is exhaustive and a final check is not needed to detect conflicts, the information that search led to a total assignment can be useful in practice, e.g., to store values for integer variables like $\text{start}(1)$, $\text{start}(2)$, $\text{end}(1)$, and $\text{end}(2)$ in Listing 2 that witness the existence of a $T$-solution.

Finally, in case of a conflict, i.e., some completion or recorded nogood is violated by the current assignment, provided that some non-deterministic decision is involved in the conflict, a new conflict constraint is recorded and utilized to guide backjumping in Line (U), as usual with CDCL. In a similar fashion as the assignment of watched literals serves as trigger for theory propagation, theory propagators are informed when they become unassigned upon backjumping. This allows them to undo earlier operations, e.g., internal data structures can be reset to return to a state taken prior to the assignment of watches.

In summary, the basic CDCL procedure is extended in four places to account for custom propagators: initialization, propagation of (partial) assignments, final check of total assignments, and undo steps upon backjumping.

5 Propagator Interface

We now turn to the implementation of theory propagation in clingo 5 and detail the structure of its interface depicted in Figure 2. The interface Propagator has to be implemented by each custom propagator. After registering such a propagator with clingo, its functions are called during initialization and search as indicated in Figure 1. Function Propagator.init is called once before solving (Line (I) in Figure 1) to allow for initializing data structures used during theory propagation. It is invoked with a PropagateInit object providing access to symbolic (SymbolicAtom) as well as theory (TheoryAtom) atoms. Both kinds of atoms are associated with program literals, which are in turn associated with solver literals. Program as well as solver literals are identified by non-zero integers, where positive and negative numbers represent positive or negative literals, respectively. In order to get notified about assignment changes, a propagator can set up watches on solver literals during initialization.

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4For brevity, we below drop the qualification Propagator and use its function names unqualified.
5Program literals are also used in the aspif format (see A).
6Note that clasp’s preprocessor might associate a positive or even negative solver literal with multiple atoms.
#script (python)

class Pigeonator:
    def __init__(self):
        self.place = {}  # shared state
        self.state = []  # per thread state

    def init(self, init):
        for atom in init.symbolic_atoms.by_signature("place", 2):
            lit = init.solver_literal(atom.literal)
            self.place[lit] = atom.symbol.args[1]
        init.add_watch(lit)
        self.state = [{} for _ in range(init.num_threads)]

    def propagate(self, control, changes):
        holes = self.state[control.thread_id]
        for lit in changes:
            hole = self.place[lit]
            prev = holes.setdefault(hole, lit)
            if prev != lit and not control.add_nogood([lit, prev]):
                return

    def undo(self, thread_id, assignment, changes):
        holes = self.state[thread_id]
        for lit in changes:
            hole = self.place[lit]
            if holes.get(hole) == lit:
                del holes[hole]

def main(prg):
    prg.register_propagator(Pigeonator())
    prg.ground({"base", []})
    prg.solve()

#end.

1 { place(P,H) : H = 1..h } 1 :- P = 1..p.
% { place(P,H) : P = 1..p } 1 :- H = 1..h.

Listing 3: Propagator for the pigeon-hole problem

During search, function propagate is called with a PropagateControl object and a (non-empty) list of watched literals that got assigned in the recent round of unit propagation (Line (P) in Figure 1). The PropagateControl object can be used to inspect the current assignment, record nogoods, and trigger unit propagation. Furthermore, to support multi-threaded solving, its thread_id property identifies the currently active thread, each of which can be viewed as an independent instance of the CDCL algorithm in Figure 1. Function undo is the counterpart of propagate and called whenever the solver retracts assignments to watched literals (Line (U) in Figure 1). In addition to the list of watched literals that have been retracted (in chronological order), it receives the identifier and the assignment of the active thread. Finally, function check is similar to propagate, yet invoked without a list of changes. Instead, it is (only) called on total assignments (Line (C) in Figure 1), independently of watches. Overriding the empty default implementations of propagator methods is optional. For brevity, we below focus on implementations of the methods in Python, while Lua or C could be used as well.

For illustration, consider Listing 3 giving a propagator for (half of) the pigeon-hole problem. Although this setting is constructed, it showcases central aspects that are also relevant when implementing more...
complex propagators, e.g., the Pigeonator is both stateful and can be used with multiple threads. The underlying ASP encoding is given in Line 37: A (choice) rule generates solution candidates by placing each of the \( p \) pigeons in exactly one among \( h \) holes. While the rule commented out in Line 38 would ensure that there is at most one pigeon per hole, this constraint is handled by the Pigeonator class implementing the Propagator interface (except for check) in Lines 8–28. Whenever two pigeons are placed in the same hole, it adds a binary nogood forbidding the placement. To this end, it maintains data structures for, given a newly placed pigeon, detecting whether there is a conflict. More precisely, the propagator has two data members: The self.state dictionary in Line 5 maps solver literals for \( \text{place}/2 \) atoms to their corresponding holes, and the self.state list in Line 6 stores for each solver thread its current placement of pigeons as a mapping from holes to true solver literals for \( \text{place}/2 \) atoms.

Function init in Lines 8–13 sets up watches as well as the dictionaries in self.place and self.state. To this end, it traverses (symbolic) atoms over \( \text{place}/2 \) in Lines 9–12. Each such atom is associated with a solver literal, obtained in Line 10. The mapping from the solver literal to its corresponding hole is then stored in the self.place dictionary in Line 11. In the last line of the loop, a watch is added for each solver literal at hand, so that the solver calls propagate whenever a pigeon is placed. Finally, in Line 13, the self.state list of placements per thread, subject to change upon propagation and backjumping, is initialized with empty dictionaries.

Function propagate, given in Lines 15–21, accesses control.thread_id in Line 16 to obtain the holes dictionary storing the active thread’s current placement of pigeons. The loop in Lines 17–21 then iterates over the list of changes, i.e., solver literals representing newly placed pigeons. After in Line 18 determining the hole associated with a recently assigned literal, Python’s setdefault function is used to update the state: Depending on whether hole already appears as a key in the holes dictionary, the function either retrieves its associated literal or inserts the new literal under key hole. While the latter case amounts to updating the placement of pigeons, the former signals a conflict, triggered by recording a binary nogood in Line 20. Given that the solver has to resolve the conflict and backjump, the call to add_nogood always yields false, so that propagation stops without processing remaining changes any further.\(^8\)

Function undo in Lines 23–28 resets a thread’s placement of pigeons upon backjumping. Similar to propagate, the active thread’s current placement is obtained in Line 24, and changes are traversed in Lines 25–28. The latter correspond to retracted solver literals, for which the condition in Line 27 makes sure that exactly those stored in Line 19 before are cleared, thus reflecting that the hole determined in Line 26 is free again. Finally, function main in Lines 30–33 first registers the Pigeonator propagator in Line 31, and then initiates grounding and solving with clingo.

6 Experiments

Our approach aims at a simple yet general framework for incorporating theory reasoning into ASP solving. Hence, it leaves room for various ways of encoding a problem and of implementing theory propagation. To reflect this from a practical perspective, we empirically explore several options for solving problems with difference logic (DL) constraints. To be more precise, we contrast an encoding relying on defined theory atoms with one leaving them external (cf. Section 3), and a stateless with a stateful propagator implementation. As a non-strict interpretation of DL constraints is sufficient for the problems given below, we stick to this option and do not vary it.

The consistency of a set \( C \) of DL constraints can be checked by mapping them to a weighted directed graph \( G(C) \). The nodes of \( G(C) \) are the (integer) variables occurring in \( C \), and for each \( x_1 - x_2 \leq k \) in \( C \), \( G(C) \) includes an edge from \( x_1 \) to \( x_2 \) with weight \( k \). Then, \( C \) is DL-consistent iff \( G(C) \) contains no cycle whose sum of edge weights is negative. The difference between a stateless and stateful DL-propagator amounts to whether the corresponding graph is built from scratch upon each invocation or only once and updated subsequently. In our experiments, we use the Bellman-Ford algorithm [6, 16] as basis for a stateless propagator, and the one in [11] for update operations in the stateful case. Both propagator implementations detect negative cycles and record (solver) literals corresponding to their weighted edges as nogoods.

\(^8\)The optional arguments tag and lock of add_nogood can be used to control the scope and lifetime of recorded nogoods. Furthermore, in a propagator that does not add violated nogoods only, function control.propagate can be invoked to trigger unit propagation.
Table 1: Comparison between different encodings and DL-propagators for scheduling problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>#</th>
<th>T</th>
<th>TO</th>
<th>T</th>
<th>TO</th>
<th>T</th>
<th>TO</th>
<th>T</th>
<th>TO</th>
<th>T</th>
<th>TO</th>
<th>T</th>
<th>TO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow shop</td>
<td>120</td>
<td>569</td>
<td>110</td>
<td>283</td>
<td>40</td>
<td>382</td>
<td>70</td>
<td>177</td>
<td>30</td>
<td>281</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Job shop</td>
<td>80</td>
<td>600</td>
<td>80</td>
<td>600</td>
<td>80</td>
<td>600</td>
<td>80</td>
<td>37</td>
<td>0</td>
<td>43</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Open shop</td>
<td>60</td>
<td>405</td>
<td>40</td>
<td>214</td>
<td>20</td>
<td>213</td>
<td>20</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>260</td>
<td>525</td>
<td>230</td>
<td>366</td>
<td>140</td>
<td>398</td>
<td>170</td>
<td>72</td>
<td>30</td>
<td>109</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Theory atoms corresponding to DL constraints are formed as described in Section 2. The difference between using defined and external theory atoms boils down to their occurrence in the head of a rule, as in Line 19 of Listing 1, viz.

\[
\&\text{diff} \{ \text{end}(T) - \text{start}(T) \} \leq D : - \text{duration}(T,D).
\]
or in the body, as in

\[
- \text{duration}(T,D), \not \&\text{diff} \{ \text{end}(T) - \text{start}(T) \} \leq D.
\]

Note that the defining usage constrains DL-atoms firmer than the external one: A defined DL-atom is true iff at least one of its bodies holds, while an external one may vary whenever its truth is DL-consistent yet not imposed by integrity constraints (with further problem-specific literals).

To evaluate the different options, we expressed (decision versions of) several scheduling problems [27], typically aiming at the minimization of schedules’ makespan, by logic programs in the language of Section 2. *Flow shop*: A schedule corresponds to a permutation of \( n \) jobs, each including \( m \) sequential subtasks allocating machines 1,...,m for specific amounts of time. *Job shop*: Again considering \( n \) jobs with \( m \) sequential subtasks each, where the order in which subtasks allocate machines 1,...,m for given amounts of time is job-specific, a schedule arranges the subtasks of different jobs in one sequence per machine. *Open shop*: Given the same setting as in the job shop problem, the sequential order of the subtasks of a job is not fixed, but augments a schedule arranging the subtasks of different jobs per machine. For reasons of scalability, we refrain from optimizing the makespan of subtasks, but are only interested in some feasible schedule per instance along with the corresponding earliest start times of subtasks.

The results of our experiments, run sequentially under Linux on an Intel Xeon E5520 2.27 GHz machine equipped with 24 GB main memory, are summarized in Table 1. Each clingo 5 run was restricted to 600 seconds wall-clock time, while memory was never exceeded. Subcolumns headed by ‘T’ report average runtimes, taking timeouts as 600 seconds, and those with ‘TO’ numbers of timeouts over ‘#’ instances of each scheduling problem and in total. Respective results in the column headed by ‘ASP’ reflect the bottom-line performance obtained with plain ASP encodings, which is obviously not competitive due to the ineffectiveness of grounding problems over large numeric domains. The remaining columns consider the four combinations of encoding and DL-propagator features of interest. First, we observe that the stateful propagator (on the right) has a clear edge over its stateless counterpart (in the middle). Second, with both propagator implementations, the firm encoding using defined DL-atoms outperforms the one leaving them external on instances of the flow shop problem. While this experiment is not meant to be universal, it demonstrates that different features have an impact on the resulting performance. In how far the tuning of theory propagators matters also depends on the use case at hand, e.g., solving a challenging application problem versus rapid prototyping of dedicated reasoning procedures.

7 Discussion

The clingo 5 system provides a comprehensive infrastructure for enhancing ASP with theory reasoning. This ranges from generic means for expressing theories along with their support by gringo, over a theory-aware intermediate format, to simple yet powerful interfaces in C, Lua, and Python. In each case, a propagator can specify (up to) four basic functions to customize its integration into clasp’s propagation, where an arbitrary number of (independent) theory propagators can be be incorporated. Logically, ASP encodings may build upon defined or external theory atoms, and their associated conditions may be
strict or non-strict. In practice, clingo 5 supports stateless and stateful theory propagators, which can be controlled in a fine-grained way. For instance, propagators are thread-sensitive, watches can be set to symbolic as well as theory literals, and the scope and lifetime of nogoods stemming from theory propagation can be configured.

A first step toward a more flexible ASP infrastructure was done with clingo 4 [18] by introducing Lua and Python APIs for multi-shot solving. Although this allows for fine-grained control of complex ASP reasoning processes, the functionality provided no access to clasp’s propagation and was restricted to inspecting (total) stable models. The extended framework for theory propagation relative to partial assignments (cf. Figure 1) follows the canonical approach of SMT [4]. While dlvhex implicitly provides access to clasp’s propagation, this is done on the more abstract level of higher-order logic programs. Also, dlvhex as well as many other systems, such as clingcon or inca, implement specialized propagation via clasp’s internal interfaces, whose usage is more involved and subject to change with each release. Although the new high-level interfaces may not yet fully cover all desired features, they provide a first step toward easing the development of such dedicated systems and putting them on a more stable basis. Currently, clingo 5’s infrastructure is already used as a basis for clingcon 3 [3], le2casp [9], and its integration with SWI-Prolog. Finally, we believe that the extended grounding capacities along with the intermediate format supplemented in A. will also be beneficial for non-native approaches and ease the overall development of ASP-oriented solvers. This applies to systems like dingo, mingo, and aspmt [5], the latter implementing ASP with theory reasoning by translation to SMT, which so far had to resort to specific input formats and meta-programming to bypass the grounder.

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References


A Intermediate Language

To accommodate the richer input language, a more general grounder-solver interface is needed. Although this could be left internal to clingo 5, it is good practice to explicate such interfaces via an intermediate language. This does not only provide a transparent interface, but also allows for using alternative downstream solvers or transformations.

Unlike the block-oriented smodels format, the aspif format is line-based. Notably, it abolishes the need of using symbol tables as in smodels’ format for passing along meta-expressions and rather allows gringo 5 to output information as soon as it is grounded. An aspif file starts with a header, beginning with the keyword asp along with version information and optional tags:

```
asp v m n r t 1 ... t k
```

where v, m, n, r are non-negative integers representing the version in terms of major, minor, and revision numbers, and each t_i is a tag for k ≥ 0. Currently, the only tag is incremental, meant to set up the underlying solver for multi-shot solving. An example header is given in Line 1 of Figure 3(b) and 4. The rest of the file is comprised of one or more logic programs. Each logic program is a sequence of lines of aspif statements followed by a 0, one statement or 0 per line, respectively. Positive and negative integers are used to represent positive or negative literals, respectively. Hence, 0 is not a valid literal.

Let us now briefly describe the format of aspif statements and illustrate them with a simple logic program in Figure 3 as well as the result of grounding a subset of Listing 1 in Figure 4.

```
{a}.
2 b :- a.
3 c :- not a.
```

Rule statements have form

```
1 H B
```

in which head H has form

```
h a_1 ... a_m
```

where h ∈ {0, 1} determines whether the head is a disjunction or choice, m ≥ 0 is the number of head elements, and each a_i is a positive literal.

Body B has one of two forms:

- Normal bodies have form

```
0 a_1 ... a_n
```

where n ≥ 0 is the length of the rule body, and each l_i is a literal.

- Weight bodies have form

```
1 a_1 w_1 ... a_n w_n
```

where l is a positive integer to denote the lower bound, n ≥ 0 is the number of literals in the rule body, and each l_i and w_i are a literal and a positive integer.

All types of ASP rules are included in the above rule format. Heads are disjunctions or choices, including the special case of singular disjunctions for representing normal rules. As in the smodels format, aggregates are restricted to a singular body, just that in aspif cardinality constraints are taken as special weight constraints. Otherwise, a body is simply a conjunction of literals.

The three rules in Figure 3(a) are represented by the statements in Line 2-4 of Figure 3(b). For instance, the four occurrences of 1 in Line 2 capture a rule with a choice in the head, having one element, identified by 1. The two remaining zeros capture a normal body with no element. For another example, Line 2-7 of Figure 4 represent the four facts in Line 1 and 2 of Listing 2 along with the ones (comprising theory atoms) in Line 6 of Listing 2.

---

9 ASP Intermediate Format
10 http://www.tcs.hut.fi/Software/smodels
Minimize statements have form

\[ \min \sum_{i=1}^{n} w_i l_i \]

where \( p \) is an integer priority, \( n \geq 0 \) is the number of weighted literals, each \( l_i \) is a literal, and each \( w_i \) is an integer weight. Each of the above expressions gathers weighted literals sharing the same priority \( p \) from all \#minimize directives and weak constraints in a logic program. As before, maximize statements are translated into minimize statements.

Projection statements result from \#project directives and have form

\[ \proj \]

where \( n \geq 0 \) is the number of atoms, and each \( a_i \) is a positive literal.

Output statements result from \#show directives and have form

\[ \text{Output statements can print the symbolic representation of atoms } a, b, \text{ or } c, \text{ whenever the corresponding atom is true. For instance, the string 'a' is printed if atom '1' holds. Unlike this, the statements in Line 8-11 of Figure 4 unconditionally print the symbolic representation of the atoms stemming from the four facts in Line 1 and 2 of Listing 2.} \]

External statements result from \#external directives and have form

\[ \text{External statements are of the form } \text{av} \]

where \( a \) is a positive literal, and \( v \in \{0, 1, 2, 3\} \) indicates free, true, false, and release.

Assumption statements have form

\[ \ass \]

where \( n \geq 0 \) is the number of literals, and each \( l_i \) is a literal. Assumptions instruct a solver to compute stable models such that \( l_1, \ldots, l_n \) hold. They are only valid for a single solver call.

Heuristic statements result from \#heuristic directives and have form

\[ \text{Heuristic statements are of the form } \text{amakpn} \]

where \( m \in \{0, \ldots, 5\} \) stands for the \((m+1)\)th heuristic modifier among level, sign, factor, init, true, and false, \( a \) is a positive literal, \( k \) is an integer, \( p \) is a non-negative integer priority, \( n \geq 0 \) is the number of literals in the condition, and the literals \( l_i \) are the condition under which the heuristic modification should be applied.

Edge statements result from \#edge directives and have form

\[ \text{Edge statements are of the form } \text{uvln} \]

where \( u \) and \( v \) are integers representing an edge from node \( u \) to node \( v \), \( n \geq 0 \) is the length of the condition, and the literals \( l_i \) are the condition for the edge to be present.

Let us now turn to the theory-specific part of aspif. Once a theory expression is grounded, gringo 5 outputs a serial representation of its syntax tree. To illustrate this, we give in Figure 4 the (sorted) result of grounding all lines of Listing 1 related to difference constraints, viz. Line 2/3, 11, 15/16, and 19, as well as Line 1 and 13.

Theory terms are represented using the following statements:

\[ \text{Theory terms are represented using the following statements:} \]

\[ w \]

where \( n \geq 0 \) is a length, index \( u \) is a non-negative integer, integer \( w \) represents a numeric term,
string s of length n represents a symbolic term (including functions) or an operator, integer t is either -1, -2, or -3 for tuple terms in parentheses, braces, or brackets, respectively, or an index of a symbolic term or operator, and each u_i is an integer for a theory term. Statements (3), (4), and (5) capture numeric terms, symbolic terms, as well as compound terms (tuples, sets, lists, and terms over theory operators).

Fifteen theory terms are given in Line 12-20 of Figure 4. Each of them is identified by a unique index in the third spot of each statement. While Line 12-20 stand for primitive entities of type (3) or (4), the ones beginning with ‘9,2’ represent compound terms. For instance, Line 21 and 22 represent end(1) or start(1), respectively, and Line 23 corresponds to end(1)-start(1).

Theory atoms are represented using the following statements:

\[
9. v \cdot \ldots \cdot v_n \cdot j_1 \cdot \ldots \cdot j_m
\]

\[
9. v \cdot \ldots \cdot v_n
\]

\[
9. v \cdot \ldots \cdot v_n \cdot j_1
\]

where \( n \geq 0 \) and \( m \geq 0 \) are lengths, index \( v \) is a non-negative integer, \( a \) is a positive literal or 0 for directives, each \( u_i \) is an integer for a theory term, each \( i_1 \) is an integer for a literal, integer \( p \) refers to a symbolic term, each \( v_i \) is an integer for a theory atom element, and integer \( q \) refers to a theory operator. Statement (6) captures elements of theory atoms and directives, and statements (7) and (8) refer to the latter.

For instance, Line 27 captures the (single) theory element in ‘{ end(1)-start(1) }’, and Line 29 represents the theory atom ‘\&diff { end(1)-start(1) } <= 200’.

Comments have form

\[ \text{10_s} \]

where s is a string not containing a newline.

The aspif format constitutes the default output of gringo 5. With clasp 3.2, ground logic programs can be read in both smodels and aspif format.

### B Proof of Proposition 1

**Proposition 1** Let \( P \) be a program, \( T \) be a collection of theories, \( X \subseteq A \cup T \), and \( B \subseteq B(P) \) the set of rule bodies that hold w.r.t. \( X \). Then, we have that \( X \) is a \( T \)-stable model of \( P \) iff \( A = \{ Tv \mid v \in X \cup B \} \cup \{ Fv \mid v \in \{(A \cup T) \setminus X \} \cup (B(P) \setminus B) \} \) is a solution for \( (\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in T} \Delta_T \).

**Proof 1 (Proof (Sketch))** As detailed in [20, Lemma 3.1], \( A \) is the unique solution for \( \Delta_{B(P)} \) such that \( \{ a \in A \cup T \mid Ta \in A \} = X \). For any \( T \)-solution \( S \) and theory \( T \in T \), let \( S^T = S \cap T^T \) below.

\( \implies \) Let \( S \) be some \( T \)-solution such that \( X \) is a stable model of the program in (1) and (2). Since all rules in \( P \) are satisfied by \( X \) and there is no non-empty unfounded subset of \( X \cap (A \cup T(h(P))) \) for \( P \) w.r.t. \( X \), we have that \( A \) is a solution for \( (\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in T} \Delta_T \). In view of the rules added to \( P \) in (1) and (2), for any \( T \in T \), \( X \) is such that \( X \cap T_{cT} = S \cap T_{cT} \) and \( X \cap T^T \subseteq S \cap T^T \). Thus, \( S^T \) is a \( T \)-solution satisfying \( \{ a \in T^T \mid Ta \in A \} = X \cap T^T \subseteq S^T \) and \( \{ a \in T_{cT} \mid Fa \in A \} \cap S^T = (T_{cT} \setminus X) \cap S^T = \emptyset \), so that \( A \) is a solution for \( \Delta_T \) as well.

\( \Leftarrow \) Let \( A \) be a solution for \( (\Delta_P \cup \Lambda_P) \cup \bigcup_{T \in T} \Delta_T \). Then, there is some \( T \)-solution \( S \) such that, for any \( T \in T \), \( \{ a \in T^T \mid Ta \in A \} = X \cap T^T \subseteq S^T \) and \( \{ a \in T_{cT} \mid Fa \in A \} \cap S^T = (T_{cT} \setminus X) \cap S^T = \emptyset \). Thus, \( X \cap T_{cT} = S \cap T_{cT} \) and \( X \cap T^T \subseteq S \cap T^T \), which yields that \( X \) satisfies all rules added to \( P \) in (1) and (2) for \( S \), where atoms in \( X \cap (T \setminus h(P)) \) do not belong to any unfounded set for the program in (1) and (2) w.r.t. \( X \). Since \( A \) is a solution for \( \Delta_P \cup \Lambda_P \), we also have that all rules in \( P \) are satisfied by \( X \) and no non-empty subset of \( X \cap (A \cup T(h(P))) \) is unfounded for \( P \) w.r.t. \( X \), so that \( X \) is indeed a stable model of the program in (1) and (2).