Metric Temporal Equilibrium Logic over Timed Traces*

ARVID BECKER

University of Potsdam, Germany

PEDRO CABALAR

University of Corunna, Spain

MARTÍN DIÉGUEZ LERIA, Université d'Angers, France

TORSTEN SCHAUB, ANNA SCHUHMANN

University of Potsdam, Germany

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Abstract

In temporal extensions of Answer Set Programming (ASP) based on linear-time, the behavior of dynamic systems is captured by sequences of states. While this representation reflects their relative order, it abstracts away the specific times associated with each state. However, timing constraints are important in many applications like, for instance, when planning and scheduling go hand in hand. We address this by developing a metric extension of linear-time temporal equilibrium logic, in which temporal operators are constrained by intervals over natural numbers. The resulting Metric Equilibrium Logic provides the foundation of an ASP-based approach for specifying qualitative and quantitative dynamic constraints. To this end, we define a translation of metric formulas into monadic first-order formulas and give a correspondence between their models in Metric Equilibrium Logic and Monadic Quantified Equilibrium Logic, respectively. Interestingly, our translation provides a blue print for implementation in terms of ASP modulo difference constraints.

 $K\!EY\!WORDS\!\!:$ answer set programming, metric temporal logic, equilibrium logic, nonmonotonic reasoning

1 Introduction

Reasoning about actions and change, or more generally about dynamic systems, is not only central to knowledge representation and reasoning but at the heart of Computer Science (Fisher et al. 2005). In practice, this kind of reasoning often requires both qualitative as well as quantitative dynamic constraints. For instance, when planning and scheduling at once, actions may have durations and their effects may need to meet deadlines. On the other hand, any flexible formalism for actions and change must incorporate some form of non-monotonic reasoning to deal with inertia and other types of defaults.

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Over the last years, we addressed qualitative dynamic constraints by combining traditional approaches, like Dynamic and Linear Temporal Logic (DL (Harel et al. 2000) and LTL (Pnueli 1977)), with the base logic of Answer Set Programming (ASP (Lifschitz 1999)) namely, the logic of Here-and-There (HT (Heyting 1930)) and its non-monotonic extension, called Equilibrium Logic (Pearce 1997). This resulted in non-monotonic linear dynamic and temporal equilibrium logics (DEL (Bosser et al. 2018; Cabalar et al. 2019) and TEL (Aguado et al. 2013; Cabalar et al. 2018; Aguado et al. 2023)) that gave rise to the temporal ASP system *telingo* (Cabalar et al. 2019; Cabalar et al. 2020) extending the ASP system *clingo* (Gebser et al. 2016).

A commonality of such dynamic and temporal logics is that they abstract from specific time points when capturing temporal relationships. For instance, in temporal logic, we can use the formula $\Box(use \rightarrow \Diamond clean)$ to express that a machine has to be eventually cleaned after being used. Nothing can be said about the delay between using and cleaning the machine.

A key design decision was to base both aforementioned logics, TEL and DEL, on the same linear-time semantics. We continued to maintain the same linear-time semantics, embodied by sequences of states, when elaborating upon a first "light-weight" metric temporal extension of HT (Cabalar et al. 2020). The "light-weightiness" is due to treating time as a state counter by identifying the next time with the next state. For instance, this allows us to refine our example by stating that, if the machine is used, it has to be cleaned within the next 3 states, viz. $\Box(use \rightarrow \Diamond_{[1..3]} clean)$. Although this permits the restriction of temporal operators to subsequences of states, no fine-grained timing constraints are expressible. In other words, it is as if state transitions were identified with time clicks, and the two things could not be dissociated.

In this paper, we overcome this limitation by dealing with *timed traces* where each state has an associated *time*, as done in Metric Temporal Logic (MTL (Koymans 1990)). This allows us to measure time differences between events. For instance, in our example, we may thus express that whenever the machine is used, it has to be cleaned within 60 to 120 time units, by writing:

$$\Box(use \to \Diamond_{[60..120]} clean)$$

Unlike the non-metric version, this stipulates that once *use* is true in a state, *clean* must be true in some future state whose associated time is at least 60 and at most 120 time units after the time of *use*. The choice of time domain is crucial, and might even lead to undecidability in the continuous case. We rather adapt a discrete approach that offers a sequence of snapshots of a dynamic system.

The definition of the new variant of *Metric (Temporal) Equilibrium Logic* (MEL for short) is done in two steps. We start with the definition of a monotonic logic called *Metric (Temporal) logic of Here-and-There* (MHT), a temporal extension of the intermediate logic of Here-and-There, mentioned above. We then select some models from MHT that are said to be in equilibrium, obtaining in this way a non-monotonic entailment relation.

The rest of the paper is organized as follows. In the next section, we start describing the monotonic basis, MHT, that generalizes (Cabalar et al. 2020) by adding timed traces, and provide some basic properties and useful equivalences in this logic. In Section 3, we study the non-monotonic formalism, MEL, providing the definition of metric equilibrium models as a kind of minimal MHT models. We also illustrate this definition with an example and discuss the property of strong equivalence for metric theories, proving that it coincides with equivalence in the monotonic logic, MHT. Section 4 provides a translation of MHT into a fragment of first-order HT, called *Quantified Here-and-There with Differ*ence Constraints, following a similar spirit to the well-known translation of Kamp (1968) from LTL to first-order logic. Finally, Section 5 contains a discussion and concludes the paper. Appendix A includes all the proofs of the results in the paper.

2 Metric Logic of Here-and-There

In this section, we start describing the metric extension of HT, called MHT, that is used as monotonic basis for defining Metric Equilibrium Logic later on. We begin introducing some notation. Given $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\omega\}$, we let [m..n] stand for the set $\{i \in \mathbb{N} \mid m \leq i \leq n\}$, $m \leq i \leq n\}$, [m..n) for $\{i \in \mathbb{N} \mid m \leq i < n\}$, and (m..n] stand for $\{i \in \mathbb{N} \mid m < i \leq n\}$. We use letters I, J to denote intervals and, since they stand for sets, we apply standard set operations on them, like inclusion $I \subseteq J$ or membership $n \in I$.

Given a set \mathcal{A} of propositional variables (called *alphabet*), a *metric formula* φ is defined by the grammar:

 $\varphi ::= p \mid \bot \mid \varphi_1 \otimes \varphi_2 \mid \bullet_I \varphi \mid \varphi_1 \mathbf{S}_I \varphi_2 \mid \varphi_1 \mathbf{T}_I \varphi_2 \mid \circ_I \varphi \mid \varphi_1 \mathbf{U}_I \varphi_2 \mid \varphi_1 \mathbf{R}_I \varphi_2$

where $p \in \mathcal{A}$ is an atom and \otimes is any binary Boolean connective $\otimes \in \{\rightarrow, \land, \lor\}$. The last six cases above correspond to temporal operators, each of them indexed by some interval I of the form [m..n) with $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\omega\}$. In words, \bullet_I , \mathbf{S}_I , and \mathbf{T}_I are past operators called *previous*, *since*, and *trigger*, respectively; their future counterparts \circ_I , \mathbf{U}_I , and \mathbf{R}_I are called *next*, *until*, and *release*. Strictly speaking, we should differentiate between the syntactic representation of an interval, and its semantic counterpart, the associated set of time points it represents. For simplicity, we just use the same representation for both concepts but, as said above, we restrict the form of intervals that can be used as modal subindices to the case [m..n) where n is possibly ω . Yet, some syntactic abbreviations are allowed in the temporal subindices. For instance, we let subindex [m..n] stand for [m..n+1), provided $n \neq \omega$. Also, we sometimes use the subindices ' $\leq n$ ', ' $\geq m$ ' and 'm' as abbreviations of intervals [0..n], $[m..\omega)$ and [m..m], respectively. Also, whenever $I = [0..\omega)$, we simply omit subindex I.

We also define several common derived operators like the Boolean connectives $\top \stackrel{def}{=} \neg \bot$, $\neg \varphi \stackrel{def}{=} \varphi \rightarrow \bot$, $\varphi \leftrightarrow \psi \stackrel{def}{=} (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$, and the following temporal operators:

		$\perp {f T}_I arphi$	always before			$\perp {\sf R}_{I} arphi$	always afterward
		$ op \mathbf{S}_{I} arphi$	eventually before	* 1 /		$ op {f U}_I arphi$	eventually afterward
-		$\neg ullet \top$	initial			$\neg \circ \top$	final
$\widehat{ullet}_I arphi$	$\stackrel{def}{=}$	$\bullet_I \varphi \vee \neg \bullet_I \top$	weak previous	$\widehat{o}_{I} \varphi$	$\stackrel{def}{=}$	$\mathrm{o}_{I}\varphi \vee \neg \mathrm{o}_{I}\top$	weak next

Note that *initial* and *final* are not indexed by any interval; they only depend on the state of the trace, not on the actual time associated with this state. On the other hand, the weak version of *next* can no longer be defined in terms of *final*, as done in (Cabalar et al. 2018) with non-metric $\hat{\bigcirc}\varphi \equiv \bigcirc\varphi \lor \mathbf{F}$. For the metric case $\hat{\bigcirc}_{I}\varphi$, the disjunction $\bigcirc_{I}\varphi \lor \neg\bigcirc_{I}\top$ must be used instead, in order to keep the usual dualities among operators (the same applies to weak *previous*).

A metric theory is a (possibly infinite) set of metric formulas. As an example of a

metric theory, we may consider the following scenario for modeling the behavior of traffic lights. While the light is red by default, it changes to green within less than 15 time units (say, seconds) whenever the button is pushed; and it stays green for another 30 seconds at most. This can be represented as follows.

$$\Box(red \land green \to \bot) \tag{1}$$

$$\Box(\neg green \to red) \tag{2}$$

$$\Box (push \to \Diamond_{[1..15)} (\Box_{\leq 30} green)) \tag{3}$$

Note that this example combines a default rule (2) with a metric rule (3), describing the initiation and duration period of events. This nicely illustrates the interest in nonmonotonic metric representation and reasoning methods.

A Here-and-There trace (for short HT-trace) of length $\lambda \in \mathbb{N} \cup \{\omega\}$ over alphabet \mathcal{A} is a sequence of pairs $(\langle H_i, T_i \rangle)_{i \in [0..\lambda)}$ with $H_i \subseteq T_i \subseteq \mathcal{A}$ for any $i \in [0..\lambda)$. For convenience, we usually represent an HT-trace as the pair $\langle \mathbf{H}, \mathbf{T} \rangle$ of traces $\mathbf{H} = (H_i)_{i \in [0..\lambda)}$ and $\mathbf{T} = (T_i)_{i \in [0..\lambda)}$. Notice that, when $\lambda = \omega$, this covers traces of infinite length. We say that $\langle \mathbf{H}, \mathbf{T} \rangle$ is total whenever $\mathbf{H} = \mathbf{T}$, that is, $H_i = T_i$ for all $i \in [0..\lambda)$.

Definition 1

A timed trace $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ over $(\mathbb{N}, <)$ is a pair consisting of

- an HT-trace $\langle \mathbf{H}, \mathbf{T} \rangle = (\langle H_i, T_i \rangle)_{i \in [0..\lambda)}$ and
- a function $\tau : [0..\lambda) \to \mathbb{N}$ such that $\tau(i) \le \tau(i+1)$.

A timed trace of length $\lambda > 1$ is called *strict* if $\tau(i) < \tau(i+1)$ for all $i \in [0..\lambda)$ such that $i+1 < \lambda$ and *non-strict* otherwise. We assume w.l.o.g. that $\tau(0) = 0$.

Function τ assigns to each state index $i \in [0..\lambda)$ a time point $\tau(i) \in \mathbb{N}$ representing the number of time units (seconds, miliseconds, etc, depending on the chosen granularity) elapsed since time point $\tau(0) = 0$, chosen as the beginning of the trace. The difference to the variant of MHT presented in (Cabalar et al. 2020) boils down to the choice of function τ . Essentially, the latter corresponds now to the case where τ is the identity function on the interval $[0..\lambda)$.

Given any timed HT-trace, satisfaction of formulas is defined as follows.

Definition 2 (MHT-satisfaction)

A timed HT-trace $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length λ over alphabet \mathcal{A} satisfies a metric formula φ at step $k \in [0.\lambda)$, written $\mathbf{M}, k \models \varphi$, if the following conditions hold:

- 1. $\mathbf{M}, k \not\models \bot$
- 2. $\mathbf{M}, k \models p \text{ if } p \in H_k \text{ for any atom } p \in \mathcal{A}$
- 3. $\mathbf{M}, k \models \varphi \land \psi$ iff $\mathbf{M}, k \models \varphi$ and $\mathbf{M}, k \models \psi$
- 4. $\mathbf{M}, k \models \varphi \lor \psi$ iff $\mathbf{M}, k \models \varphi$ or $\mathbf{M}, k \models \psi$
- 5. $\mathbf{M}, k \models \varphi \rightarrow \psi$ iff $\mathbf{M}', k \not\models \varphi$ or $\mathbf{M}', k \models \psi$, for both $\mathbf{M}' = \mathbf{M}$ and $\mathbf{M}' = (\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$
- 6. $\mathbf{M}, k \models \mathbf{\bullet}_{I} \varphi$ iff k > 0 and $\mathbf{M}, k-1 \models \varphi$ and $\tau(k) \tau(k-1) \in I$
- 7. $\mathbf{M}, k \models \varphi \mathbf{S}_I \psi$ iff for some $j \in [0..k]$ with $\tau(k) \tau(j) \in I$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in (j..k]$
- 8. $\mathbf{M}, k \models \varphi \mathbf{T}_I \psi$ iff for all $j \in [0..k]$ with $\tau(k) \tau(j) \in I$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in (j..k]$
- 9. $\mathbf{M}, k \models \circ_I \varphi$ iff $k + 1 < \lambda$ and $\mathbf{M}, k + 1 \models \varphi$ and $\tau(k+1) \tau(k) \in I$

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- 10. $\mathbf{M}, k \models \varphi \mathbf{U}_I \psi$ iff for some $j \in [k..\lambda)$ with $\tau(j) \tau(k) \in I$, we have $\mathbf{M}, j \models \psi$ and $\mathbf{M}, i \models \varphi$ for all $i \in [k..j)$
- 11. $\mathbf{M}, k \models \varphi \mathbf{R}_I \psi$ iff for all $j \in [k..\lambda)$ with $\tau(j) \tau(k) \in I$, we have $\mathbf{M}, j \models \psi$ or $\mathbf{M}, i \models \varphi$ for some $i \in [k..j)$

Satisfaction of derived operators can be easily deduced:

Proposition 1

Let $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ be a timed HT-trace of length λ over \mathcal{A} . Given the respective definitions of derived operators, we get the following satisfaction conditions:

12. $\mathbf{M}, k \models \mathbf{I} \text{ iff } k = 0$

13. $\mathbf{M}, k \models \widehat{\mathbf{o}}_{I} \varphi$ iff k = 0 or $\mathbf{M}, k-1 \models \varphi$ or $\tau(k) - \tau(k-1) \notin I$ 14. $\mathbf{M}, k \models \mathbf{\phi}_{I} \varphi$ iff $\mathbf{M}, i \models \varphi$ for some $i \in [0..k]$ with $\tau(k) - \tau(i) \in I$ 15. $\mathbf{M}, k \models \mathbf{m}_{I} \varphi$ iff $\mathbf{M}, i \models \varphi$ for all $i \in [0..k]$ with $\tau(k) - \tau(i) \in I$ 16. $\mathbf{M}, k \models \mathbf{F}$ iff $k + 1 = \lambda$ 17. $\mathbf{M}, k \models \widehat{\mathbf{o}}_{I} \varphi$ iff $k + 1 = \lambda$ or $\mathbf{M}, k+1 \models \varphi$ or $\tau(k+1) - \tau(k) \notin I$ 18. $\mathbf{M}, k \models \Diamond_{I} \varphi$ iff $\mathbf{M}, i \models \varphi$ for some $i \in [k..\lambda)$ with $\tau(i) - \tau(k) \in I$ 19. $\mathbf{M}, k \models \Box_{I} \varphi$ iff $\mathbf{M}, i \models \varphi$ for all $i \in [k..\lambda)$ with $\tau(i) - \tau(k) \in I$

A formula φ is a *tautology* (or is valid), written $\models \varphi$, iff $\mathbf{M}, k \models \varphi$ for any timed HTtrace \mathbf{M} and any $k \in [0..\lambda)$. MHT is the logic induced by the set of all such tautologies. For two formulas φ, ψ we write $\varphi \equiv \psi$, iff $\models \varphi \leftrightarrow \psi$, that is, $\mathbf{M}, k \models \varphi \leftrightarrow \psi$ for any timed HT-trace \mathbf{M} of length λ and any $k \in [0..\lambda)$. A timed HT-trace \mathbf{M} is an MHT model of a metric theory Γ if $\mathbf{M}, 0 \models \varphi$ for all $\varphi \in \Gamma$. The set of MHT models of Γ having length λ is denoted as $\text{MHT}(\Gamma, \lambda)$, whereas $\text{MHT}(\Gamma) \stackrel{def}{=} \bigcup_{\lambda=0}^{\omega} \text{MHT}(\Gamma, \lambda)$ is the set of all MHT models of Γ of any length. We may obtain fragments of any metric logic by imposing restrictions on the timed traces used for defining tautologies and models. That is, MHT_f stands for the restriction of MHT to traces of any finite length $\lambda \in \mathbb{N}$ and MHT_{ω} corresponds to the restriction to traces of infinite length $\lambda = \omega$.

We say that a metric theory is *temporal* if all its modal operators are subindex-free. Temporal formulas share the same syntax as LTL, although the absence of an interval in MHT is understood as an abbreviation for the fixed interval $[0..\omega)$. The following result shows that, for temporal theories, MHT satisfaction collapses into THT satisfaction (Aguado et al. 2023). Hence, we can use non-timed traces and ignore function τ in this case.

Proposition 2

Let Γ be a temporal theory. Then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \models \Gamma$ in MHT iff $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$ in THT.

An interesting subset of MHT is the one formed by total timed traces like $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$. In the non-metric version of temporal HT, the restriction to total models corresponds to Linear Temporal Logic (LTL (Pnueli 1977)). In our case, the restriction to total traces defines a metric version of LTL, which we call *Metric Temporal Logic* (or MTL for short). We present next several properties about total traces and the relation between MHT and MTL.

Proposition 3 (Persistence)

Let $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ be a timed HT-trace of length λ over \mathcal{A} and let φ be a metric formula over \mathcal{A} . Then, for any $k \in [0..\lambda)$, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ then $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \varphi$.

Thanks to Proposition 3 and a decidability result in (Ouaknine and Worrell 2007), we get:

Corollary 1 (Decidability of MHT_f) The logic of MHT_f is decidable.

The next result shows that the satisfaction of negated formulas as classical ones also extends from HT to MHT:

Proposition 4

Let $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ be a timed HT-trace of length λ over \mathcal{A} and let φ be a metric formula over \mathcal{A} . Then, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \neg \varphi$ iff $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \not\models \varphi$.

In the non-metric case, LTL models can be obtained from THT by adding a particular axiom schema, we call the *temporal excluded middle* axiom.

Definition 3 (Temporal Excluded Middle)

Given a set of propositional variables \mathcal{A} , we define the theory $\text{EM}(\mathcal{A})$ as

 $\mathrm{EM}(\mathcal{A}) \stackrel{def}{=} \{ \Box (p \lor \neg p) \mid p \in \mathcal{A} \}.$

This same axiom schema can also be used to reduce MHT to MTL, assuming that, in our current context, operator \Box stands for $\Box_{[0..\omega)}$ as detailed above.

Proposition 5

Let \mathcal{A} be a set of atoms. For all MHT interpretation $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ over \mathcal{A} , we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \mathrm{EM}(\mathcal{A})$ iff $\mathbf{H} = \mathbf{T}$.

Corollary 2

Let Γ be a metric theory over alphabet \mathcal{A} . The MTL models of Γ coincide with the MHT models of $\Gamma \cup \text{EM}(\mathcal{A})$.

Interestingly, if an equivalence does not involve implication (or negation), we can just check it by only considering total models:

$Proposition \ 6$

Let φ and ψ be metric formulas without implication (and so, without negation either). Then, $\varphi \equiv \psi$ in MTL iff $\varphi \equiv \psi$ in MHT.

Many tautologies in MHT or its fragments have a dual version depending on the nature of the operators involved. The following pair of duality properties allows us to save space and proof effort when listing interesting valid equivalences. We define all pairs of dual connectives as follows: \top/\bot , \wedge/\lor , $\mathbf{U}_I/\mathbf{R}_I$, $\circ_I/\widehat{\circ}_I$, \Box_I/\Diamond_I , $\mathbf{S}_I/\mathbf{T}_I$, $\bullet_I/\widehat{\bullet}_I$, $\mathbf{\Xi}_I/\mathbf{\Phi}_I$. For any formula φ without implications, we define $\delta(\varphi)$ as the result of replacing each connective by its dual operator.

Then, we get the following corollary of Proposition 6.

 $\mathbf{6}$

Corollary 3 (Boolean Duality)

Let φ and ψ be metric formulas without implication. Then, we have in MHT that $\varphi \equiv \psi$ iff $\delta(\varphi) \equiv \delta(\psi)$.

Let $\mathbf{U}_I/\mathbf{S}_I$, $\mathbf{R}_I/\mathbf{T}_I$, $\mathbf{O}_I/\mathbf{\Phi}_I$, $\widehat{\mathbf{O}}_I/\mathbf{\Phi}_I$, \Box_I/\mathbf{H}_I , and $\Diamond_I/\mathbf{\Phi}_I$ be all pairs of swapped-time connectives and $\sigma(\varphi)$ be the replacement in φ of each connective by its swapped-time version. Then, we have the following result for finite traces.

Lemma 1

There exists a mapping ρ on finite timed HT-traces **M** of the same length $\lambda \geq 0$ such that for any $k \in [0..\lambda)$, $\mathbf{M}, k \models \varphi$ iff $\rho(\mathbf{M}), \lambda - 1 - k \models \sigma(\varphi)$.

Theorem 1 (Temporal Duality Theorem)

A metric formula φ is a MHT_f-tautology iff $\sigma(\varphi)$ is a MHT_f-tautology.

The next properties capture some distributivity laws for temporal operators with respect to conjunction and disjunction.

Proposition 7

For all metric formulas φ , ψ , and χ , the following equivalences hold in MHT:

$$\begin{array}{ll} \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \lor \circ_{I} \psi & \varphi ~ \mathsf{U}_{I} (\chi \lor \psi) \equiv (\varphi ~ \mathsf{U}_{I} ~ \chi) \lor (\varphi ~ \mathsf{U}_{I} ~ \psi) \\ \circ_{I} (\varphi \land \psi) \equiv \circ_{I} \varphi \land \circ_{I} \psi & (\varphi \land \chi) ~ \mathsf{U}_{I} ~ \psi \equiv (\varphi ~ \mathsf{U}_{I} ~ \chi) \land (\varphi ~ \mathsf{U}_{I} ~ \psi) \\ \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \land \circ_{I} \psi & (\varphi \lor \chi) ~ \mathsf{R}_{I} ~ \psi \equiv (\varphi ~ \mathsf{R}_{I} ~ \chi) \land (\varphi ~ \mathsf{R}_{I} ~ \psi) \\ \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \lor \circ_{I} \psi & (\varphi \land \chi) ~ \mathsf{R}_{I} ~ \psi \equiv (\varphi ~ \mathsf{R}_{I} ~ \psi) \lor (\chi ~ \mathsf{R}_{I} ~ \psi) \\ \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \lor \circ_{I} \psi & (\varphi \land \chi) ~ \mathsf{S}_{I} ~ \psi \equiv (\varphi ~ \mathsf{S}_{I} ~ \chi) \lor (\varphi ~ \mathsf{S}_{I} ~ \psi) \\ \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \land \circ_{I} \psi & \circ_{I} \psi & \circ_{I} (\varphi \lor \psi) \equiv \circ_{I} \varphi \lor \circ_{I} \psi \\ \circ_{I} (\varphi \lor \psi) \equiv \diamond_{I} \varphi \lor \diamond_{I} \psi & \mathsf{I} (\varphi \land \psi) \equiv \mathsf{I}_{I} \varphi \land \mathsf{I}_{I} \psi \\ (\varphi \lor \chi) ~ \mathsf{T}_{I} ~ \psi \equiv (\varphi ~ \mathsf{T}_{I} ~ \psi) \lor (\chi ~ \mathsf{T}_{I} ~ \psi) & \varphi ~ \mathsf{T}_{I} (\chi \land \psi) \equiv (\varphi ~ \mathsf{T}_{I} ~ \chi) \land (\varphi ~ \mathsf{T}_{I} ~ \psi) \end{array}$$

We can also prove a kind of De Morgan duality between until and release, and analogously, between since and trigger:

Proposition 8

For all metric formulas φ and ψ , the following equivalences hold in MHT:

$$\neg (\varphi \mathbf{U}_{I} \psi) \equiv \neg \varphi \mathbf{R}_{I} \neg \psi \qquad \neg (\varphi \mathbf{S}_{I} \psi) \equiv \neg \varphi \mathbf{T}_{I} \neg \psi$$
$$\neg (\varphi \mathbf{R}_{I} \psi) \equiv \neg \varphi \mathbf{U}_{I} \neg \psi \qquad \neg (\varphi \mathbf{T}_{I} \psi) \equiv \neg \varphi \mathbf{S}_{I} \neg \psi$$

Another interesting result has to do with the effect of extending or stretching the interval in these operators.

Proposition 9

Let I and J be two intervals satisfying $I \subseteq J$. For all metric formulas φ and ψ , the following expressions are valid in MHT:

$$(\varphi \mathbf{U}_{I} \psi) \to (\varphi \mathbf{U}_{J} \psi)$$

$$(\varphi \mathbf{R}_{J} \psi) \to (\varphi \mathbf{R}_{I} \psi)$$

$$(\varphi \mathbf{R}_{J} \psi) \to (\varphi \mathbf{R}_{I} \psi)$$

$$(\varphi \mathbf{T}_{J} \psi) \to (\varphi \mathbf{T}_{I} \psi)$$

We observe next the effect of the semantics of *always* and *eventually* on truth constants. If $m, n \in \mathbb{N}$, then $\Box_{[m..n]} \bot$ means that there is no state in interval [m..n) and $\Diamond_{[m..n]} \top$ means that there is at least one state in this interval. The formula $\Box_{[m..n]} \top$ is a tautology, whereas $\Diamond_{[m..n]} \bot$ is unsatisfiable. The same applies to past operators $\blacklozenge_{[m..n]}$ and $\blacksquare_{[m..n]}$.

Strict traces

In the rest of this section, we consider a group of results that hold under the assumption of strict traces, namely, that $\tau(i) < \tau(i+1)$ for any pair of consecutive time points. We can enforce metric models to be traces with a strict timing function τ . This can be achieved with the simple addition of the axiom $\Box \neg \Box_0 \top$. In the following, we assume that this axiom is included and consider, in this way, strict timing. For instance, a consequence of strict timing is that one-step operators become definable in terms of other connectives. For non-empty intervals [m..n] with m < n, we get:

$$\begin{split} \bullet_{[m..n)}\varphi &\equiv \blacksquare_{[1..m)} \bot \land \phi_{[h..n)}\varphi \\ \circ_{[m..n)}\varphi &\equiv \square_{[1..m)} \bot \land \Diamond_{[h..n)}\varphi \qquad \text{where } h = \max(1,m); \end{split}$$

whereas for empty intervals with $m \ge n$, we obtain $\bullet_{[m..n)} \varphi \equiv \circ_{[m..n)} \varphi \equiv \bot$.

The following equivalences state that interval [0..0] makes all binary metric operators collapse into their right hand argument formula, whereas unary operators collapse to a truth constant. For metric formulas ψ and φ and for strict traces, we have:

$$\psi \,\mathbf{U}_0 \,\varphi \equiv \psi \,\mathbf{R}_0 \,\varphi \equiv \varphi \tag{4}$$

$$\mathsf{O}_0 \,\varphi \equiv \mathbf{\bullet}_0 \,\varphi \equiv \bot \tag{5}$$

$$\widehat{\mathsf{O}}_0 \, \varphi \equiv \widehat{\bullet}_0 \, \varphi \equiv \top \tag{6}$$

The last two lines are precisely an effect of dealing with strict traces. For instance, $O_0 \varphi \equiv \bot$ tells us that it is always impossible to have a successor state with the same time (the time difference is 0) as the current one, regardless of the formula φ at hand.

The next lemma allows us to unfold metric operators for single-point time intervals [n..n] with n > 0.

Lemma 2

For metric formulas ψ and φ , strict traces and for n > 0, we have:

$$\psi \mathbf{U}_{n} \varphi \equiv \psi \wedge \bigvee_{i=1}^{n} \circ_{i} (\psi \mathbf{U}_{n-i} \varphi) \quad (7) \qquad \qquad \Diamond_{n} \varphi \equiv \bigvee_{i=1}^{n} \circ_{i} \Diamond_{n-i} \varphi \qquad (9)$$
$$\psi \mathbf{R}_{n} \varphi \equiv \psi \vee \bigwedge_{i=1}^{n} \widehat{\circ}_{i} (\psi \mathbf{R}_{n-i} \varphi) \quad (8) \qquad \qquad \Box_{n} \varphi \equiv \bigwedge_{i=1}^{n} \widehat{\circ}_{i} \Box_{n-i} \varphi \qquad (10)$$

The same applies for the dual past operators.

Going one step further, we can also unfold *until* and *release* for intervals of the form [0..n] with the application of the following result.

Lemma 3

For metric formulas ψ and φ , strict traces and for n > 0, we have:

$$\psi \mathbf{U}_{\leq n} \varphi \equiv \varphi \lor (\psi \land \bigvee_{i=1}^{n} \circ_{i} (\psi \mathbf{U}_{\leq (n-i)} \varphi))$$
(11)

$$\psi \,\mathbf{R}_{\leq n} \,\varphi \equiv \varphi \wedge (\psi \vee \bigwedge_{i=1}^{n} \widehat{\mathsf{O}}_{i}(\psi \,\mathbf{R}_{\leq (n-i)} \,\varphi)) \tag{12}$$

Metric Equilibrium Logic

The same applies for the dual past operators.

Finally, the next theorem contains a pair of equivalences that, when dealing with finite intervals, can be used to recursively unfold *until* and *release* into combinations of *next* with Boolean operators (an analogous result applies for *since*, *trigger* and *previous* due to temporal duality).

Theorem 2 (Next-unfolding)

For metric formulas ψ and φ , strict traces and for $m, n \in \mathbb{N}$ such that 0 < m < n - 1, we have:

$$\psi \,\mathbf{U}_{[m.n)} \,\varphi \equiv \psi \wedge \left(\bigvee_{i=1}^{m} \circ_{i} (\psi \,\mathbf{U}_{[m-i..n-i)} \,\varphi) \vee \bigvee_{i=m+1}^{n-1} \circ_{i} (\psi \,\mathbf{U}_{\leq (n-1-i)} \,\varphi)\right) \tag{13}$$

$$\psi \,\mathbf{R}_{[m..n)} \,\varphi \equiv \psi \lor \left(\bigwedge_{i=1}^{m} \widehat{\mathsf{o}}_{i}(\psi \,\mathbf{R}_{[(m-i)..(n-i))} \,\varphi) \land \bigwedge_{i=m+1}^{n-1} \widehat{\mathsf{o}}_{i}(\psi \,\mathbf{R}_{\leq (n-1-i)} \,\varphi)\right) \tag{14}$$

The same applies for the dual past operators.

As an example, consider the metric formula $p \mathbf{U}_{[2..4)} q$.

$$p \mathbf{U}_{[2..4)} q \equiv p \land \left(\bigvee_{i=1}^{2} \circ_{i}(p \mathbf{U}_{[(2-i)..(4-i))} q) \lor \bigvee_{i=2+1}^{3} \circ_{i}(p \mathbf{U}_{\leq(3-i)} q)\right)$$
$$\equiv p \land \left(\circ_{1}(p \mathbf{U}_{[1..3)} q) \lor \circ_{2}(p \mathbf{U}_{\leq 1} q) \lor \circ_{3}(p \mathbf{U}_{0} q)\right)$$
$$\equiv p \land \left(\circ_{1}(p \mathbf{U}_{[1..3)} q) \lor \circ_{2}(q \lor (p \land \circ_{1} q)) \lor \circ_{3} q\right)$$
$$\equiv p \land \left(\circ_{1}(\circ_{1}(q \lor (p \land \circ_{1} q)) \lor \circ_{2} q) \lor \circ_{2}(q \lor (p \land \circ_{1} q)) \lor \circ_{3} q\right)$$

Another useful result that can be applied to unfold metric operators is the following range splitting theorem.

Theorem 3 (Range splitting)

For metric formulas ψ and φ and strict traces, we have

$$\psi \mathbf{U}_{[m..n)} \varphi \equiv (\psi \mathbf{U}_{[m..i)} \varphi) \lor (\psi \mathbf{U}_{[i..n)} \varphi) \qquad \text{for all } i \in [m..n)$$

$$\psi \mathbf{R}_{[m..n)} \varphi \equiv (\psi \mathbf{R}_{[m..i)} \varphi) \land (\psi \mathbf{R}_{[i..n)} \varphi) \qquad \text{for all } i \in [m..n)$$

The same applies for the dual past operators.

A metric formula φ is said to be in *unary normal form*, if intervals only affect unary temporal operators, while binary operators **U**, **R**, **S**, **T** are only used in their temporal form, without any attached intervals. The following proposition, inspired by (D'Souza and Tabareau 2004), allows us to translate any arbitrary metric formula into unary normal form.

Proposition 10

For metric formulas φ and ψ , strict traces, and for m and n such that m > 0, the following

equivalences hold in MHT:

$$\begin{split} \varphi \ \mathbf{U}_{[m..n]} \psi &\equiv \Diamond_{[m..n]} \psi \land \Box_{[0..m}) (\varphi \ \mathbf{U} (\varphi \land \circ \psi)) & \varphi \ \mathbf{U}_{[0..n]} \psi &\equiv \Diamond_{[0..n]} \psi \land \varphi \ \mathbf{U} \psi \\ \varphi \ \mathbf{U}_{[m..n]} \psi &\equiv \Diamond_{[m..n]} \psi \land \Box_{[0..m]} (\varphi \ \mathbf{U} (\varphi \land \circ \psi)) & \varphi \ \mathbf{U}_{[0..n]} \psi &\equiv \Diamond_{[0..n]} \psi \land \varphi \ \mathbf{U} \psi \\ \varphi \ \mathbf{U}_{(m..n]} \psi &\equiv \Diamond_{(m..n]} \psi \land \Box_{[0..m]} (\varphi \ \mathbf{U} (\varphi \land \circ \psi)) & \varphi \ \mathbf{U}_{(0..n]} \psi &\equiv \Diamond_{(0..n]} \psi \land \varphi \ \mathbf{U} (\varphi \land \circ \psi) \\ \varphi \ \mathbf{U}_{(m..n]} \psi &\equiv \Diamond_{(m..n]} \psi \land \Box_{[0..m]} (\varphi \ \mathbf{U} (\varphi \land \circ \psi)) & \varphi \ \mathbf{U}_{(0..n]} \psi &\equiv \Diamond_{(0..n]} \psi \land \varphi \ \mathbf{U} (\varphi \land \circ \psi) \\ \varphi \ \mathbf{R}_{[m..n]} \psi &\equiv \Box_{[m..n]} \psi \lor \Diamond_{[0..m]} (\varphi \ \mathbf{R} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{R}_{[0..n]} \psi &\equiv \Box_{[0..n]} \psi \lor \varphi \ \mathbf{R} \psi \\ \varphi \ \mathbf{R}_{(m..n]} \psi &\equiv \Box_{(m..n]} \psi \lor \Diamond_{[0..m]} (\varphi \ \mathbf{R} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{R}_{(0..n]} \psi &\equiv \Box_{(0..n]} \psi \lor \varphi \ \mathbf{R} \psi \\ \varphi \ \mathbf{R}_{(m..n]} \psi &\equiv \Box_{(m..n]} \psi \lor \Diamond_{[0..m]} (\varphi \ \mathbf{R} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{R}_{(0..n]} \psi &\equiv \Box_{(0..n]} \psi \lor \varphi \ \mathbf{R} (\varphi \lor \widehat{\circ} \psi) \\ \varphi \ \mathbf{R}_{(m..n]} \psi &\equiv \Box_{(m..n]} \psi \land \Box_{[0..m]} (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) & \varphi \ \mathbf{R}_{(0..n]} \psi &\equiv \Box_{(0..n]} \psi \lor \varphi \ \mathbf{R} (\varphi \lor \widehat{\circ} \psi) \\ \varphi \ \mathbf{S}_{[m..n]} \psi &\equiv \phi_{[m..n]} \psi \land \blacksquare_{[0..m]} (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) & \varphi \ \mathbf{S}_{[0..n]} \psi &\equiv \phi_{[0..n]} \psi \land \varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi) \\ \varphi \ \mathbf{S}_{(m..n]} \psi &\equiv \phi_{(m..n]} \psi \land \blacksquare_{[0..m]} (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) & \varphi \ \mathbf{S}_{(0..n)} \psi &\equiv \phi_{[0..n]} \psi \land \varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi) \\ \varphi \ \mathbf{S}_{(m..n]} \psi &\equiv \phi_{(m..n]} \psi \land \blacksquare_{[0..m]} (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) & \varphi \ \mathbf{S}_{(0..n)} \psi &\equiv \phi_{(0..n]} \psi \land \varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi) \\ \varphi \ \mathbf{S}_{(m..n)} \psi &\equiv (m..n) \psi \land \blacksquare_{[0..m]} (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) & \varphi \ \mathbf{S}_{(0..n)} \psi &\equiv (\varphi \ \mathbf{S} (\varphi \land \widehat{\circ} \psi)) \\ \varphi \ \mathbf{T}_{[m..n]} \psi &\equiv \blacksquare_{[m..n]} \psi \land \oplus_{[0..m]} (\varphi \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{T}_{[0..n]} \psi &\subseteq \mathbf{T} (\varphi \land \widehat{\circ} \psi) \\ \varphi \ \mathbf{T}_{(m..n]} \psi &\equiv \blacksquare_{(m..n]} \psi \lor \phi_{[0..m]} (\varphi \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{T}_{(0..n]} \psi &\equiv \blacksquare_{(0..n]} \psi \lor \nabla \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi) \\ \varphi \ \mathbf{T}_{(m..n]} \psi &\equiv \blacksquare_{(m..n]} \psi \lor \phi_{[0..m]} (\varphi \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{T}_{(0..n]} \psi &\equiv \blacksquare_{(0..n]} \psi \lor \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi) \\ \varphi \ \mathbf{T}_{(m..n]} \psi &\equiv \blacksquare_{(m..n]} \psi \lor \phi_{[0..m]} (\varphi \ \mathbf{T} (\varphi \lor \widehat{\circ} \psi)) & \varphi \ \mathbf{T}_{(0$$

Corollary 4

Any metric formula can be translated into unary normal form (assuming strict traces).

3 Metric Equilibrium Logic

As in traditional Equilibrium Logic (Pearce 1997), non-monotonicity is achieved in MEL by a selection among the MHT models of a theory. In what follows, we keep assuming the use of strict traces.

Definition 4 (Metric Equilibrium/Stable Model)

Let \mathfrak{S} be some set of timed HT-traces. A total timed HT-trace $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau) \in \mathfrak{S}$ is a metric equilibrium model of \mathfrak{S} iff there is no other $\mathbf{H} < \mathbf{T}$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \in \mathfrak{S}$. The timed trace (\mathbf{T}, τ) is called a metric stable model of \mathfrak{S} .

We talk about metric equilibrium (or metric stable) models of a theory Γ when $\mathfrak{S} = MHT(\Gamma)$, and we write $MEL(\Gamma, \lambda)$ and $MEL(\Gamma)$ to stand for the metric equilibrium models of $MHT(\Gamma, \lambda)$ and $MHT(\Gamma)$, respectively. *Metric Equilibrium Logic* (MEL) is the non-monotonic logic induced by the metric equilibrium models of metric theories. As before, variants MEL_f and MEL_{ω} refer to MEL when restricted to traces of finite and infinite length, respectively.

Proposition 11

The set of metric equilibrium models of Γ can be partitioned on the trace lengths, namely, $\bigcup_{\lambda=0}^{\omega} \text{MEL}(\Gamma, \lambda) = \text{MEL}(\Gamma).$ Back to our example, suppose we have the theory Γ consisting of formulas (1)-(3), viz.

$$\Box(red \land green \to \bot) \tag{1}$$

$$\Box(\neg green \to red) \tag{2}$$

$$\Box (push \to \Diamond_{[1..15)} (\Box_{\leq 30} \ green)) \tag{3}$$

In the example, we abbreviate subsets of the set of atoms $\{green, push, red\}$ as strings formed by their initials: For instance, pr stands for $\{push, red\}$. For the sake of readability, we represent traces (T_0, T_1, T_2) as $T_0 \cdot T_1 \cdot T_2$. Consider first the total models of Γ : the first two rules force one of the two atoms green or red to hold at every state. Besides, we can choose adding *push* or not, but if we do so, *green* should hold later on according to (3). Now, for any total model $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma$ where *green* or *push* hold at some states, we can always form **H** in which we remove those atoms from all the states and it is not difficult to see that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \Gamma$, so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ is not in equilibrium. As a consequence, metric equilibrium models of Γ have the form $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ being $\mathbf{T} =$ $\langle T_i \rangle_{i \in [0, \lambda)}$ with $T_i = \{red\}$ for all $i \in [0, \lambda)$ and any arbitrary strict timing function τ . To illustrate non-monotonicity, suppose now that we have $\Gamma' = \Gamma \cup \{ O_5 \text{ push} \}$ and, for simplicity, consider length $\lambda = 3$ and traces of the form $T_0 \cdot T_1 \cdot T_2$. Again, it is not hard to see that total models with green or push in state T_0 are not in equilibrium, being the only option $T_0 = \{red\}$. The same happens for green at T_1 , so we get $T_1 = \{push, red\}$ as only candidate for equilibrium model. However, since $push \in T_1$, the only possibility to satisfy the consequent of (3) is having $green \in T_2$. Again, we can also see that adding push at that state would not be in equilibrium so that the only trace in equilibrium is $T_0 = \{red\}, T_1 = \{push, red\}$ and $T_2 = \{green\}$. As for the timing, $\tau(0) = 0$ is fixed, and satisfaction of formula $(o_5 \text{ push})$ fixes $\tau(1) = 5$. Then, from (3) we conclude that green must hold at any moment starting at t between 5+1 and 5+14 and is kept true in all states between t and t + 30 time units, but as $\lambda = 2$, this means just t. To sum up, we get 14 metric equilibrium models with $\tau(0) = 0$ and $\tau(1) = 5$ fixed, but varying $\tau(2)$ between 6 and 19.

We close this section by considering strong equivalence. Two metric theories Γ_1 and Γ_2 are strongly equivalent when $\text{MEL}(\Gamma_1 \cup \Delta) = \text{MEL}(\Gamma_2 \cup \Delta)$ for any metric theory Δ . This means that we can safely replace Γ_1 by Γ_2 in any common context Δ and still get the same set of metric equilibrium models. The following result shows that checking strong equivalence for MEL collapses to regular equivalence in the monotonic logic of MHT.

Theorem 4

Let Γ_1 and Γ_2 be two metric temporal theories built over a finite alphavet \mathcal{A} . Then, Γ_1 and Γ_2 are strongly equivalent iff Γ_1 and Γ_2 are MHT-equivalent.

4 Translation into Monadic Quantified Here-and-There with Difference Constraints

In a similar spirit as the well-known translation of Kamp (1968) from LTL to firstorder logic, we consider a translation from MHT into a first-order version of HT, more precisely, a function-free fragment of the logic of Quantified Here-and-There with static

domains (QHT^s) in (Pearce and Valverde 2008)). The word *static* means that the firstorder domain D is fixed for both worlds, here and there. We refer to our fragment of QHT^s as monadic QHT with difference constraints (or $QHT[\preccurlyeq_{\delta}]$ for short). In this logic, the static domain is a subset $D \subseteq \mathbb{N}$ of the natural numbers containing at least the element $0 \in D$. Intuitively, D corresponds to the set of relevant time points (i.e. those associated with states) considered in each model. Note that the first state is always associated with time $0 \in D$.

The syntax of $QHT[\preccurlyeq_{\delta}]$ is the same as for first-order logic with several restrictions: First, there are no functions other than the 0-ary function (or constant) '0' always interpreted as the domain element 0 (when there is no ambiguity, we drop quotes around constant names). Second, all predicates are monadic except for a family of binary predicates of the form \preccurlyeq_{δ} with $\delta \in \mathbb{Z} \cup \{\omega\}$ where δ is understood as part of the predicate name. For simplicity, we write $x \preccurlyeq_{\delta} y$ instead of $\preccurlyeq_{\delta}(x, y)$ and $x \preccurlyeq_{\delta} y \preccurlyeq_{\delta'} z$ to stand for $x \preccurlyeq_{\delta} y \land y \preccurlyeq_{\delta'} z$. Unlike monadic predicates, the interpretation of $x \preccurlyeq_{\delta} y$ is static (it does not vary in worlds here and there) and intuitively means that the difference x - y in time points is smaller or equal than δ . A first-order formula φ satisfying all these restrictions is called a *first-order metric formula* or *FOM-formula* for short. A formula is a *sentence* if it contains no free variables. For instance, we will see that the metric formula (3) can be equivalently translated into the FOM-sentence:

$$\forall x \left(0 \preccurlyeq_0 x \land push(x) \rightarrow \exists y \left(x \preccurlyeq_{-1} y \preccurlyeq_{14} x \land \forall z \left(y \preccurlyeq_0 z \preccurlyeq_{30} y \rightarrow green(z) \right) \right) \right)$$
(15)

We sometimes handle *partially grounded* FOM sentences where some variables in predicate arguments have been directly replaced by elements from D. For instance, if we represent (15) as $\forall x \ \varphi(x)$, the expression $\varphi(4)$ stands for:

 $0 \preccurlyeq_0 4 \land push(4) \rightarrow \exists y (4 \preccurlyeq_{-1} y \preccurlyeq_{14} 4 \land \forall z (y \preccurlyeq_0 z \preccurlyeq_{30} y \rightarrow green(z)))$

and corresponds to a partially grounded FOM-sentence where the domain element 4 is used as predicate argument in atoms $0 \preccurlyeq_0 4$ and push(4).

A $QHT[\preccurlyeq_{\delta}]$ -signature is simply a set of monadic predicates \mathcal{P} . Given D as above, $Atoms(D, \mathcal{P})$ denotes the set of all ground atoms p(n) for every monadic predicate $p \in \mathcal{P}$ and every $n \in D$. A $QHT[\preccurlyeq_{\delta}]$ -interpretation for signature \mathcal{P} has the form $\langle D, H, T \rangle$ where $D \subseteq \mathbb{N}, 0 \in D$ and $H \subseteq T \subseteq Atoms(D, \mathcal{P})$.

Definition 5 (QHT[\preccurlyeq_{δ}]-satisfaction; (Pearce and Valverde 2008)) A QHT[\preccurlyeq_{δ}]-interpretation $\mathcal{M} = \langle D, H, T \rangle$ satisfies a (partially grounded) FOM-sentence φ , written $\mathcal{M} \models \varphi$, if the following conditions hold:

1. $\mathcal{M} \models \top$ and $\mathcal{M} \not\models \bot$ 2. $\mathcal{M} \models p(t)$ iff $p(t) \in H$ 3. $\mathcal{M} \models t_1 \preccurlyeq_{\delta} t_2$ iff $t_1 - t_2 \leq \delta$ with $t_1, t_2 \in D$ 4. $\mathcal{M} \models \varphi \land \psi$ iff $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$ 5. $\mathcal{M} \models \varphi \lor \psi$ iff $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$ 6. $\mathcal{M} \models \varphi \rightarrow \psi$ iff $\langle D, X, T \rangle \not\models \varphi$ or $\langle D, X, T \rangle \models \psi$ for $X \in \{H, T\}$ 7. $\mathcal{M} \models \forall x \ \varphi(x)$ iff $\mathcal{M} \models \varphi(t)$ for all $t \in D$ 8. $\mathcal{M} \models \exists x \ \varphi(x)$ iff $\mathcal{M} \models \varphi(t)$ for some $t \in D$

We can read the expression $x \preccurlyeq_{\delta} y$ as just another way of writing the difference constraint

 $x - y \leq \delta$. When δ is an integer, we may see it as a lower bound $x - \delta \leq y$ for y or as an upper bound $x \leq y + \delta$ for x. For $\delta = \omega$, $x \preccurlyeq_{\omega} y$ is equivalent to \top since it amounts to the comparison $x - y \leq \omega$. An important observation is that this difference predicate \preccurlyeq_{δ} satisfies the excluded middle axiom, that is, the following formula is a $QHT[\preccurlyeq_{\delta}]$ -tautology:

$$\forall x \,\forall y \,(x \preccurlyeq_{\delta} y \lor \neg (x \preccurlyeq_{\delta} y))$$

for every $\delta \in \mathbb{Z} \cup \{\omega\}$. We provide next several useful abbreviations:

$$\begin{array}{rcl} x \prec_{\delta} y & \stackrel{def}{=} & \neg(y \preccurlyeq_{-\delta} x) \\ x \leq y & \stackrel{def}{=} & x \preccurlyeq_{0} y & x \neq y & \stackrel{def}{=} & \neg(x = y) \\ x = y & \stackrel{def}{=} & (x \leq y) \land (y \leq x) & x < y & \stackrel{def}{=} & (x \leq y) \land (x \neq y) \end{array}$$

For any pair \odot , \oplus of comparison symbols, we extend the abbreviation $x \odot y \oplus z$ to stand for the conjunction $x \odot y \land y \oplus z$. Note that the above derived order relation $x \leq y$ captures the one used in Kamp's original translation (Kamp 1968) for LTL.

Equilibrium models for first-order theories are defined as in (Pearce and Valverde 2008).

Definition 6 (Quantified Equilibrium Model; (Pearce and Valverde 2008))

Let φ be a first-order formula. A total $QHT[\preccurlyeq_{\delta}]$ -interpretation $\langle D, T, T \rangle$ is a first-order equilibrium model of φ if $\langle D, T, T \rangle \models \varphi$ and there is no $H \subset T$ satisfying $\langle D, H, T \rangle \models \varphi$.

Before presenting our translation, we need to remark that we consider non-empty intervals of the form [m..n) with m < n.

Definition 7 (First-order encoding)

Let φ be a metric formula over \mathcal{A} . We define the translation $[\varphi]_x$ of φ for some time point $x \in \mathbb{N}$ as follows:

$$\begin{split} [\bot]_{x} & \stackrel{\text{def}}{=} & \bot \\ [p]_{x} & \stackrel{\text{def}}{=} & p(x) \text{ for any } p \in \mathcal{A} \\ [\varphi \otimes \psi]_{x} & \stackrel{\text{def}}{=} & [\varphi]_{x} \otimes [\beta]_{x} \text{ for any connective } \otimes \in \{\wedge, \vee, \rightarrow\} \\ [\bigcirc_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \left(x < y \land (\neg \exists z \; x < z < y) \land x \preccurlyeq_{-m} y \prec_{n} x \land [\psi]_{y}\right) \\ [\widehat{\bigcirc}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \left(x < y \land (\neg \exists z \; x < z < y) \land x \preccurlyeq_{-m} y \prec_{n} x \land [\psi]_{y}\right) \\ [\varphi \mathbf{U}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \left(x \leq y \land x \preccurlyeq_{-m} y \prec_{n} x \land [\psi]_{y} \land \forall z \left(x \leq z < y \rightarrow [\varphi]_{z}\right)\right) \\ [\varphi \mathbf{R}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \left(y < x \land \neg \exists z \ (y < z < x) \land x \prec_{n} y \preccurlyeq_{-m} x \land [\psi]_{y}\right) \\ [\widehat{\bullet}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \left(y < x \land \neg \exists z \ (y < z < x) \land x \prec_{n} y \preccurlyeq_{-m} x \land [\psi]_{y}\right) \\ [\widehat{\bullet}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \ (y \leq x \land x \prec_{n} y \preccurlyeq_{-m} x \land [\psi]_{y}) \\ [\widehat{\bullet}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \exists y \ (y \leq x \land x \prec_{n} y \preccurlyeq_{-m} x \land [\psi]_{y} \land \forall (y < z \leq x \rightarrow [\varphi]_{z}))) \\ [\varphi \mathbf{T}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \forall y \ ((y \leq x \land x \prec_{n} y \preccurlyeq_{-m} x \land [\psi]_{y} \land \forall (y < z \leq x \land [\varphi]_{z}))) \\ [\varphi \mathbf{T}_{[m..n)}\psi]_{x} & \stackrel{\text{def}}{=} & \forall y \ ((y \leq x \land x \prec_{n} y \preccurlyeq_{-m} x) \rightarrow ([\psi]_{y} \lor \exists z \ (y < z \leq x \land [\varphi]_{z}))) \end{cases} \end{split}$$

Each quantification introduces a new variable. For instance, consider the translation of (3) at point x = 0. Let us denote (3) as $\Box(push \rightarrow \alpha)$ where α is the formula

 $\Diamond_{[1..15)}(\square_{<30} \text{ green})$. Then, if we translate the outermost operator \square , we get:

$$\begin{split} &[\Box(push \to \alpha)]_{0} \\ &= \ [\bot \ \mathbf{R}_{[0..\omega)} \ (push \to \alpha)]_{0} \\ &= \ \forall y \ ((0 \le y \land 0 \preccurlyeq_{-0} y \prec_{\omega} 0) \to ([push \to \alpha]_{y} \lor \exists z \ (0 \le z < y \land \bot))) \\ &\equiv \ \forall y \ (0 \le y \land 0 \le y \land \top \to ([push]_{y} \to [\alpha]_{y}) \lor \bot) \\ &\equiv \ \forall y \ (0 \le y \land push(y) \to [\alpha]_{y}) \\ &\equiv \ \forall x \ (0 \le x \land push(x) \to [\alpha]_{x}) \end{split}$$

where we renamed the quantified variable for convenience. If we proceed further, with α as $\langle 1...15 \rangle \beta$ and letting β be $(\Box_{\leq 30} \text{ green})$, we obtain:

$$\begin{aligned} \alpha]_x &= [\Diamond_{[1..15)}\beta]_x \\ &= [\top \mathbf{U}_{[1..15)} \ \beta]_x \\ &= \exists y \ (x \le y \land x \preccurlyeq_{-1} y \prec_{15} x \land [\beta]_y \land \forall z \ (x \le z < y \to \top)) \\ &\equiv \exists y \ (x \preccurlyeq_{-1} y \prec_{15} x \land [\beta]_y) \equiv \exists y \ (x \preccurlyeq_{-1} y \preccurlyeq_{14} x \land [\beta]_y) \end{aligned}$$

Finally, the translation of β at y amounts to:

$$\begin{bmatrix} \Box_{\leq 30} \ green \end{bmatrix}_{y}$$

$$= \ \begin{bmatrix} \bot \ \mathbf{R}_{[0..30)} \ green \end{bmatrix}_{y}$$

$$= \ \forall y' \ (\ y \leq y' \land y \preccurlyeq_{-0} y' \prec_{30} y \rightarrow green(y') \lor \exists z \ (y \leq z < y' \land \bot) \)$$

$$= \ \forall y' \ (\ y \leq y' \land y \preccurlyeq_{0} y' \land y' \prec_{30} y \rightarrow green(y') \)$$

$$= \ \forall y' \ (\ y \preccurlyeq_{0} y' \prec_{30} y \rightarrow green(y') \)$$

$$= \ \forall z \ (\ y \preccurlyeq_{0} z \prec_{30} y \rightarrow green(z) \)$$

so that, when joining all steps together, we get the formula (15) given above.

The following model correspondence between MHT_f and $QHT[\preccurlyeq_{\delta}]$ interpretations can be established. Given a timed trace $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ of length $\lambda > 0$ for signature \mathcal{A} , we define the first-order signature $\mathcal{P} = \{p/1 \mid p \in \mathcal{A}\}$ and a corresponding $QHT[\preccurlyeq_{\delta}]$ interpretation $\langle D, H, T \rangle$ where $D = \{\tau(i) \mid i \in [0..\lambda)\}, H = \{p(\tau(i)) \mid i \in [0..\lambda) \text{ and } p \in H_i\}$ and $T = \{p(\tau(i)) \mid i \in [0..\lambda) \text{ and } p \in T_i\}$. Under the assumption of strict semantics, the following model correspondence can be proved by structural induction.

Theorem 5

Let φ be a metric temporal formula, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ a timed trace, $\langle D, H, T \rangle$ its corresponding $QHT[\preccurlyeq_{\delta}]$ interpretation and $i \in [0..\lambda)$.

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \quad \text{iff} \quad \langle D, H, T \rangle \models [\varphi]_{\tau(i)}$$

$$(16)$$

$$(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \quad \text{iff} \quad \langle D, T, T \rangle \models [\varphi]_{\tau(i)}$$

$$(17)$$

5 Discussion

Seen from far, we have presented an extension of the logic of Here-and-There with qualitative and quantitative temporal constraints. More closely, our logics MHT and MEL can be seen as metric extensions of the linear-time logics THT and TEL obtained by constraining temporal operators by intervals over natural numbers. The current approach generalizes the previous metric extension of TEL from (Cabalar et al. 2020) by uncoupling the ordinal position i of a state in the trace from its location in the time line $\tau(i)$, which indicates now the elapsed time since the beginning of that trace. Thus, while $\diamond_{[5..5]} p$ meant in (Cabalar et al. 2020) that p must hold exactly after 5 transitions, it means here that there must be some future state (after n > 0 transitions) satisfying pand located 5 time units later. As a first approach, we have considered time points as natural numbers, $\tau(i) \in \mathbb{N}$. Our choice of a discrete rather than continuous time domain is primarily motivated by our practical objective to implement the logic programming fragment of MEL on top of existing temporal ASP systems, like *telingo*, and thus to avoid undecidability.

The need for quantitative time constraints is well recognized and many metric extensions have been proposed. For instance, actions with durations are considered in (Son et al. 2004) in an action language adapting a state-based approach. Interestingly, quantitative time constraints also gave rise to combining ASP with Constraint Solving (Baselice et al. 2005); this connection is now semantically reinforced by our translation advocating the enrichment of ASP with difference constraints. Even earlier, metric extensions of Logic Programming were proposed in (Brzoska 1995). As well, metric extensions of Datalog are introduced in (Wałega et al. 2019) and applied to stream reasoning in (Wałega et al. 2019). An ASP-based approach to stream reasoning is elaborated in abundance in (Beck et al. 2018). Streams can be seen as infinite traces. Hence, apart from certain dedicated concepts, like time windows, such approaches bear a close relation to metric reasoning. Detailing this relationship is an interesting topic of future research. More remotely, metric constructs were used in trace alignment (De Giacomo et al. 2020), scheduling (Luo et al. 2016), and an extension to Golog (Hofmann and Lakemeyer 2019).

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Appendix A Proofs

Proof of Proposition 1.

$$\begin{split} \mathbf{M}, k &\models \mathbf{I} \\ \text{iff } \mathbf{M}, k &\models \neg \mathbf{\bullet} \top & \text{by Definition of } \mathbf{I} \\ \text{iff } \mathbf{M}, k &\models \neg \mathbf{\bullet}_{[0..\omega)} \top & \text{by Definition of } \mathbf{\bullet} \\ \text{iff } \mathbf{M}, k &\models \neg \mathbf{\bullet}_{[0..\omega)} \top & \text{by Definition of } \mathbf{\bullet} \\ \text{iff } \mathbf{M}, k &\models \neg \mathbf{\bullet}_{[0..\omega)} \top & \text{by Proposition 4 and 3} \\ \text{iff } \mathbf{M}, k - 1 &\models \top \text{ or } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin [0..\omega) & \text{by Definition 2(6)} \\ \text{iff } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin [0..\omega) & \mathbf{M}, k \models \top \text{ for all } k \in [0..\lambda) \\ \text{iff } k = 0 & \text{since } \tau(k-1) \leq \tau(k) \end{split}$$

$\mathbf{M},k\models \widehat{\mathbf{\bullet}}_{I}\varphi$	
$\text{iff }\mathbf{M},k\models \mathbf{\bullet}_{I}\varphi \vee \neg \mathbf{\bullet}_{I}\top$	by Definition of $\widehat{\bullet}_I$
iff $\mathbf{M}, k \models ullet_I \varphi \lor \widehat{ullet}_I \neg \top$	$\neg ullet_I \phi \equiv \widehat{ullet}_I \neg \phi$
$\text{iff }\mathbf{M},k\models \mathbf{\bullet}_{I}\varphi \vee \widehat{\mathbf{\bullet}}_{I}\bot$	$\neg \top \equiv \bot$
iff $\mathbf{M}, k-1 \models \varphi$ and $\tau(k) - \tau(k-1) \in I$ or	
$k = 0$ or $\mathbf{M}, k - 1 \models \bot$ or $\tau(k) - \tau(k - 1) \not\in I$	by Definition 2(6) and $\widehat{\bullet}_I$
iff $\mathbf{M}, k-1 \models \varphi$ and $\tau(k) - \tau(k-1) \in I$ or	
$k = 0$ or $\tau(k) - \tau(k-1) \notin I$	$\mathbf{M}, k \not\models \perp \text{ for all } k \in [0\lambda)$
$\text{iff } k = 0 \text{ or } \mathbf{M}, k-1 \models \varphi \text{ or } \text{ or } \tau(k) - \tau(k-1) \not \in I$	by some propositional reasoning

$\mathbf{M}, k \models igoplus_I arphi$	
$\text{iff } \mathbf{M}, k \models \top \mathbf{S}_{I} \varphi$	by Definition of \blacklozenge_I
iff for some $i \in [0k]$ with $\tau(k) - \tau(i) \in I$	
we have $\mathbf{M}, i \models \varphi$ and $\mathbf{M}, j \models \top$ for all $j \in (ik]$	by Definition $2(7)$
iff $\mathbf{M}, i \models \varphi$ for some $i \in [0k]$ with $\tau(k) - \tau(i) \in I$	$\mathbf{M}, k \models \top \text{ for all } k \in [0\lambda)$

$\mathbf{M},k\models \blacksquare_{I}\varphi$	
$\text{iff } \mathbf{M}, k \models \bot \mathbf{T}_{I} \varphi$	by Definition of \blacksquare_I
iff for all $i \in [0k]$ with $\tau(k) - \tau(i) \in I$, we have $\mathbf{M}, i \models \varphi$ or	
$\mathbf{M}, j \models \bot$ for some $j \in (ik]$	by Definition $2(8)$
iff $\mathbf{M}, i \models \varphi$ for all $0 \in [k\lambda)$ with $\tau(i) - \tau(k) \in I$	$\mathbf{M}, k \not\models \perp \text{ for all } k \in [0\lambda)$

For the resp. past cases 16-19 the same reasoning applies.

Proof of Proposition 2. For the complete definition of THT satisfaction, we refer the reader to (Aguado et al. 2023). Here, it suffices to observe that, when we use interval $I = [0..\omega)$ in all operators, all conditions $x \in I$ in Definition 2 (MHT satisfaction) become trivially true, so that the use of τ is irrelevant and the remaining conditions happen to coincide with THT satisfaction.

Proof of Proposition 3. The proof follows by structural induction on the formula φ . Note that universal quantification of $k \in [0..\lambda)$ is part of the induction hypothesis. In what follows, we denote $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$.

- If $\varphi = \bot$, the property holds trivially because $\mathbf{M}, k \not\models \bot$.
- If φ is an atom $p, \mathbf{M}, k \models p$ implies $p \in H_k \subseteq T_k$ and so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models p$

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Metric Equilibrium Logic

- For conjunction, disjunction and implication the proof follows the same steps as with persistence in (non-temporal) HT
- If $\varphi = \circ_I \alpha$ then $k + 1 < \lambda$, $\tau(k+1) \tau(k) \in I$ and $\mathbf{M}, k+1 \models \alpha$. By induction, the latter implies $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k+1 \models \alpha$ so we get the conditions to conclude $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \circ_I \alpha$.
- If $\varphi = \alpha \mathbf{U}_I \beta$ then $\mathbf{M}, k \models \alpha \mathbf{U}_I \beta$ implies that for some $j \in [k..\lambda)$ with $\tau(j) \tau(k) \in I$, we have $\mathbf{M}, j \models \beta$ and $\mathbf{M}, i \models \alpha$ for all $i \in [k..j)$. Since the induction hypothesis applies on any time point, we can apply it to subformulas β and α to conclude for some $j \in [k..\lambda)$ with $\tau(j) \tau(k) \in I$, we have $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \models \beta$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \alpha$ for all $i \in [k..j)$. But the latter amounts to $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \alpha \mathbf{U}_I \beta$.
- The proofs for \bullet_I and \mathbf{S}_I are completely analogous to the two previous steps, respectively.

Proof of Corollary 1. We claim that MHT_f is satisfiable iff MTL_f is. This together with the decidability of MTL_f (Ouaknine and Worrell 2007) would imply that MHT_f is satisfiable.

The claim is proved as follows: from left to right, let us assume that φ is MHT_f-satisfiable. Therefore, there exists a MHT_f model ($\langle \mathbf{H}, \mathbf{T} \rangle, \tau$) such that ($\langle \mathbf{H}, \mathbf{T} \rangle, \tau$), 0 $\models \varphi$. By Proposition 3, ($\langle \mathbf{T}, \mathbf{T} \rangle, \tau$), 0 $\models \varphi$. Therefore, φ is MTL_f-satisfiable.

Conversely, if φ is MTL_f-satisfiable then there exists a MTL_f model (\mathbf{T}, τ) such that $(\mathbf{T}, \tau), 0 \models \varphi$. (\mathbf{T}, τ) can be turned into the MHT_f model $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ satisfying φ at 0. Therefore, φ is MHT_f-satisfiable.

Proof of Proposition 4. Note that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \neg \varphi$ amounts to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi \rightarrow \bot$ and the latter is equivalent to $\mathbf{M}, k \not\models \varphi$ or $\mathbf{M}, k \models \bot$, for both $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ and $\mathbf{M} = (\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$. Since $\mathbf{M}, k \models \bot$ never holds, we get that this condition is equivalent to both $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \not\models \varphi$. However, by Proposition 3 (persistence), the latter implies the former, so we get that this is just equivalent to $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \not\models \varphi$.

Proof of Proposition 5. From left to right, assume by contradiction that $\mathbf{H} \neq \mathbf{T}$, Since $\mathbf{H} \leq \mathbf{T}$, it follows that there exists $0 \leq i < \lambda$ such that $H_i \subset T_i$. This means that there exists $p \in \mathcal{A}$ such that $p \in T_i \setminus H_i$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \not\models p \lor \neg p$. Since $i \geq 0$ and, clearly, $\tau(i) - \tau(0) \in [0..\omega)$, we obtain that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \Box (p \lor \neg p)$. As a consequence we get $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \mathrm{EM}(\mathcal{A})$: a contradiction. Conversely, assume by contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \mathrm{EM}(\mathcal{A})$. Therefore, there exists $0 \leq i < \lambda$ such that $\tau(i) - \tau(0) \in [0..\omega)$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models p \lor \neg p$. This means that $p \in T_i \setminus H_i$ so $H_i \subset T_i$. As a consequence, $\mathbf{H} \neq \mathbf{T}$: a contradiction.

Proof of Proposition 6. The proof follows similar steps to Proposition 10 in (Aguado et al. 2023) for the non-metric case (and LTL instead of MTL). For a proof sketch, note that if no implication or negation is involved, the evaluation of the formula is exclusively performed on trace **H**, while the there-component **T** is never used, becoming irrelevant (we are free to choose any trace $\mathbf{T} \geq \mathbf{H}$). Thus, checking the equivalence on total traces ($\langle \mathbf{H}, \mathbf{H} \rangle, \tau$) does not lose generality, whereas total traces exactly correspond to MTL satisfaction.

Proof of Lemma 1. The proof follows similar steps to Lemma 2 in (Aguado et al. 2023) for the non-metric case. Again, we define $\rho(\mathbf{M})$ has the timed trace $(\langle \mathbf{H}', \mathbf{T}' \rangle, \tau')$ where $H'_i = H_{\lambda-1-i}$ and $T'_i = T_{\lambda-1-i}$ for all $i \in [0.\lambda)$. The only difference here is that we must also "reverse" the time function τ defining $\tau'(i) = \tau(\lambda - 1) - \tau(i)$ to keep the same relative distances but in reversed order. Then, the proof follows from the complete temporal symmetry of satisfaction of operators (when the trace is finite).

Proof of Theorem 1. The proof follows similar steps to Theorem 3 in (Aguado et al. 2023) for the non-metric case but relying here on Lemma 1 instead.

Proof of Proposition 7.

$\mathbf{M}, k \models o_{I} \left(\varphi \lor \psi \right)$	
iff $\mathbf{M}, k+1 \models \varphi \lor \psi$ and $\tau(k+1) - \tau(k) \in I$	by Definition $2(9)$
iff $(\mathbf{M}, k+1 \models \varphi \text{ or } \mathbf{M}, k+1 \models \psi)$ and $\tau(k+1) - \tau(k) \in I$	by Definition $2(4)$
iff $(\mathbf{M}, k+1 \models \varphi \text{ and } \tau(k+1) - \tau(k) \in I)$	by Distributivity
or $(\mathbf{M}, k+1 \models \psi \text{ and } \tau(k+1) - \tau(k) \in I)$	
$\text{iff } \mathbf{M}, k \models \circ_I \varphi \vee \circ_I \psi$	by Definition $2(9)$

$$\begin{split} \mathbf{M}, k &\models \circ_{I} (\varphi \land \psi) \\ \text{iff } \mathbf{M}, k+1 &\models \varphi \land \psi \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(9)} \\ \text{iff } (\mathbf{M}, k+1 &\models \varphi \text{ and } \mathbf{M}, k+1 &\models \psi) \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(3)} \\ \text{iff } (\mathbf{M}, k+1 &\models \varphi \text{ and } \tau(k+1) - \tau(k) \in I) & \text{by Distributivity} \\ \text{and } (\mathbf{M}, k+1 &\models \psi \text{ and } \tau(k+1) - \tau(k) \in I) & \text{iff } \mathbf{M}, k &\models \circ_{I} \varphi \land \circ_{I} \psi & \text{by Definition 2(9)} \end{split}$$

$$\begin{split} \mathbf{M}, k &\models \widehat{\diamond}_{I} \left(\varphi \lor \psi \right) \\ \text{iff } k + 1 &= \lambda \text{ or } \mathbf{M}, k + 1 \models \varphi \lor \psi \text{ or } \tau(k+1) - \tau(k) \not\in I \\ \text{iff } k + 1 &= \lambda \text{ or } (\mathbf{M}, k+1 \models \varphi \text{ or } \mathbf{M}, k+1 \models \psi) \text{ or } \tau(k+1) - \tau(k) \notin I \\ \text{iff } (k+1 &= \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \text{ or } \tau(k+1) - \tau(k) \notin I) \\ \text{or } (k+1 &= \lambda \text{ or } \mathbf{M}, k+1 \models \psi \text{ or } \tau(k+1) - \tau(k) \notin I) \\ \text{iff } \mathbf{M}, k &\models \widehat{\diamond}_{I} \varphi \lor \widehat{\diamond}_{I} \psi \end{split}$$
by Proposition 1(17)

$$\begin{split} \mathbf{M}, k &\models \widehat{\diamond}_{I} \left(\varphi \land \psi \right) \\ \text{iff } k + 1 &= \lambda \text{ or } \mathbf{M}, k + 1 \models \varphi \land \psi \text{ or } \tau(k+1) - \tau(k) \not\in I \\ \text{iff } k + 1 &= \lambda \text{ or } (\mathbf{M}, k+1 \models \varphi \text{ and } \mathbf{M}, k+1 \models \psi) \text{ or } \tau(k+1) - \tau(k) \notin I \\ \text{iff } (k+1 &= \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \text{ or } \tau(k+1) - \tau(k) \notin I) \\ \text{and } (k+1 &= \lambda \text{ or } \mathbf{M}, k+1 \models \psi \text{ or } \tau(k+1) - \tau(k) \notin I) \\ \text{iff } \mathbf{M}, k &\models \widehat{\diamond}_{I} \varphi \land \widehat{\diamond}_{I} \psi \end{split}$$
by Proposition 1(17)

$$\begin{split} \mathbf{M}, k &\models \Diamond_{I} \left(\varphi \lor \psi \right) \\ \text{iff } \mathbf{M}, i &\models \varphi \lor \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{ by Definition 2(18)} \\ \text{iff } (\mathbf{M}, i &\models \varphi \text{ or } \mathbf{M}, i &\models \psi) \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{ by Definition 2(4)} \\ \text{iff } (\mathbf{M}, i &\models \varphi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \text{ by Distributivity} \\ \text{ or } (\mathbf{M}, i &\models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \text{ by Definition 2(18)} \\ \text{iff } \mathbf{M}, k &\models \Diamond_{I} \varphi \lor \Diamond_{I} \psi & \text{ by Definition 2(18)} \end{split}$$

$$\begin{split} \mathbf{M}, k &\models \Box_{I} \left(\varphi \land \psi \right) \\ \text{iff } \mathbf{M}, i &\models \varphi \land \psi \text{ for all } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(19)} \\ \text{iff } \left(\mathbf{M}, i \models \varphi \text{ and } \mathbf{M}, i \models \psi \right) \text{ for all } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(3)} \\ \text{iff } \left(\mathbf{M}, i \models \varphi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \right) & \text{by Distributivity} \\ \text{and } \left(\mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \right) & \text{iff } \mathbf{M}, k \models \Box_{I} \varphi \land \Box_{I} \psi & \text{by Definition 2(19)} \end{split}$$

$$\begin{split} \mathbf{M}, k &\models \varphi \, \mathbf{U}_{I} \, (\chi \lor \psi) \\ \text{iff } \mathbf{M}, i &\models \chi \lor \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\ \text{and } \mathbf{M}, j &\models \varphi \text{ for all } j \in [k..i) & \text{by Definition 2(10)} \\ \text{iff } (\mathbf{M}, i &\models \chi \text{ or } \mathbf{M}, i \models \psi) \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\ \text{and } \mathbf{M}, j &\models \varphi \text{ for all } j \in [k..i) & \text{by Definition 2(3)} \\ \text{iff } \mathbf{M}, i &\models \chi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \text{ or } \\ \mathbf{M}, i &\models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\ \text{and } \mathbf{M}, j &\models \varphi \text{ for all } j \in [k..i) & \text{iff } \mathbf{M}, k \models (\varphi \, \mathbf{U}_{I} \, \chi) \lor (\varphi \, \mathbf{U}_{I} \, \psi) & \text{by Definition 2(10)} \end{split}$$

$\mathbf{M}, k \models (\varphi \land \chi) \mathbf{U}_{I} \psi$	
iff $\mathbf{M}, i \models \psi$ for some $i \in [k\lambda)$ with $\tau(i) - \tau(k) \in I$	
and $\mathbf{M}, j \models \varphi \land \psi$ for all $j \in [ki)$	by Definition $2(10)$
iff $\mathbf{M}, i \models \psi$ for some $i \in [k\lambda)$ with $\tau(i) - \tau(k) \in I$	by Definition $2(3)$
and $\mathbf{M}, j \models \varphi$ and $\mathbf{M}, j \models \psi$ for all $j \in [ki)$	
iff $\mathbf{M}, i \models \psi$ for some $i \in [k\lambda)$ with $\tau(i) - \tau(k) \in I$	by Distributivity
and $\mathbf{M}, j \models \varphi$ for all $j \in [ki)$ and	
$\mathbf{M}, i \models \psi$ for some $i \in [k\lambda)$ with $\tau(i) - \tau(k) \in I$	
and $\mathbf{M}, j \models \chi$ for all $j \in [ki)$	
iff $\mathbf{M}, k \models (\varphi \mathbf{U}_I \psi)$ and $\mathbf{M}, k \models (\chi \mathbf{U}_I \psi)$	by Definition $2(10)$

$\mathbf{M}, k \models \varphi \mathbf{R}_{I} (\chi \wedge \psi)$	
iff for all <i>i</i> with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	by Definition $2(11)$
$\mathbf{M}, i \models \chi \land \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [ki)$	
iff for all <i>i</i> with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	
$\mathbf{M}, i \models \chi \text{ and } \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [ki)$	by Definition $2(3)$
iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	
$\mathbf{M}, i \models \chi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [ki)$	
and iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	
$\mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [ki)$	by Distributivity
iff $\mathbf{M}, k \models (\varphi \mathbf{R}_I \chi)$ and $(\varphi \mathbf{R}_I \psi)$	by Definition $2(11)$
iff $\mathbf{M}, k \models (\varphi \mathbf{R}_I \chi) \land (\varphi \mathbf{R}_I \psi)$	by Definition $2(3)$

$\mathbf{M},k\models\left(\varphi\vee\chi\right)\mathbf{R}_{I}\;\psi$	
iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	by Definition $2(11)$
$\mathbf{M},i\models\psi\text{ or }\mathbf{M},j\models\varphi\vee\chi\text{ for some }j\in[ki)$	
iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	by Definition $2(4)$
$\mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ or } \mathbf{M}, j \models \chi \text{ for some } j \in [ki)$	
iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	by Distributivity
$\mathbf{M},i\models\psi\text{ or }\mathbf{M},j\models\varphi\text{ for some }j\in[ki)\text{ or }$	
iff for all i with $\tau(i) - \tau(k) \in \mathcal{I}$, we have	
$\mathbf{M}, i \models \psi$ or $\mathbf{M}, j \models \chi$ for some $j \in [ki)$	
iff $\mathbf{M}, k \models (\varphi \mathbf{R}_I \psi)$ or $(\chi \mathbf{R}_I \psi)$	by Definition $2(11)$
iff $\mathbf{M}, k \models (\varphi \mathbf{R}_I \psi) \lor (\chi \mathbf{R}_I \psi)$	by Definition $2(4)$

For the resp. past cases 11-20 the same reasoning applies.

Proof of Proposition 8. We consider the first equivalence. From left to right, assume towards a contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \neg \varphi \mathbf{R}_I \neg \psi$. Therefore, there exists $j \in [i..\lambda)$ such that $\tau(j) - \tau(i) \in I$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \neg \psi$ and for all $k \in [i..j)$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \neg \varphi$. By Proposition 4, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. By the semantics of the until operator we obtain that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_I \psi$. By Proposition 4 it follows that $\langle \mathbf{H}, \mathbf{T} \rangle, i \not\models \neg (\varphi \mathbf{U}_I \psi)$: a contradiction.

From right to left, if $\langle \mathbf{H}, \mathbf{T} \rangle$, $i \not\models \neg (\varphi \mathbf{U}_I \psi)$ then, by Proposition 4, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$, $i \models \varphi \mathbf{U}_I \psi$. Therefore there exists $j \in [i..\lambda)$ such that $\tau(j) - \tau(i) \in I$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$, $j \models \psi$ and for all $k \in [i..j)$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$, $k \models \varphi$. Since $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ satisfies the law of excluded middle, it follows that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$, $j \not\models \neg \psi$ and for all $k \in [i..j)$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$, $i \not\models \neg \varphi \mathbf{R}_I \neg \psi$. By persistency, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \not\models \neg \varphi \mathbf{R}_I \neg \psi$.

The remaining equivalences can be verified in a similar way.

Proof of Proposition 9.

$$\begin{split} \mathbf{M}, k &\models (\varphi \, \mathbf{U}_{I} \, \psi) \text{ iff } \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\ & \text{ and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) & \text{ by Definition 2(10)} \\ & \text{ implies } \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in J \\ & \text{ and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) & \text{ since } I \subseteq J \\ & \text{ iff } \mathbf{M}, k \models (\varphi \, \mathbf{U}_{J} \, \psi) & \text{ by Definition 2(10)} \end{split}$$

$$\mathbf{M}, k \models (\varphi \ \mathbf{R}_J \ \psi) \text{ iff for all } j \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in J$$
we have $\mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i)$ by Definition 2(11)
implies for all $j \in [k..\lambda)$ with $\tau(i) - \tau(k) \in I$
we have $\mathbf{M}, i \models \psi$ or $\mathbf{M}, j \models \varphi$ for some $j \in [k..i)$ since $I \subseteq J$
iff $\mathbf{M}, k \models (\varphi \ \mathbf{R}_I \ \psi)$ by Definition 2(11)

The cases 2 and 4 work analogously

Proof of Proposition 10. We assume that we are dealing with strict traces. We consider first the equivalence $\varphi \mathbf{U}_{[m..n)} \psi \equiv \Diamond_{[m..n)} \psi \wedge \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \wedge \circ \psi)).$

From left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n)} \psi$ then there exists $j \ge i$ such that $\tau(j) - \tau(i) \in [m..n), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) \in [m..n), j \ge i$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[m..n)} \psi$. Moreover, since $m \ne 0, \tau(j) - \tau(i) \ne 0$ so $j \ne i$, which implies that j > i. As a consequence $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \land \bigcirc \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \le t < j - 1$.

Take any arbitrary $y \ge i$. If $y \ge j$ then $\tau(y) - \tau(i) \ge m$ because $\tau(y) \ge \tau(j)$ and $\tau(j) - \tau(i) \ge m$. Therefore, $\tau(y) - \tau(i) \notin [0..m)$. If y < j then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), y \models \varphi \mathbf{U} (\varphi \land \circ \psi)$. Since y was arbitrary chosen, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \land \circ \psi))$.

For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \land \Diamond \psi))$ it follows that there exists j > i such that $\tau(j) - \tau(i) \ge m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j]$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[m..n)} \psi$ there exists j' > i such that $\tau(j') - \tau(i) \ge m$, $\tau(j') - \tau(i) < n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$.

If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n]} \psi$. If $j' \geq j$ then

 $\tau(j') \ge \tau(j)$. Since $\tau(j') - \tau(i) < n$ and $\tau(j') > \tau(j) \tau(j) - \tau(i) < n$ so $\tau(j) - \tau(i) \in [m..n)$, which leads to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n)} \psi$.

For the second equivalence $\varphi \mathbf{U}_{[m..n]} \psi \equiv \Diamond_{[m..n]} \psi \wedge \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \wedge \Diamond \psi)),$ from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n]} \psi$ then there exists $j \geq i$ such that $\tau(j) - \tau(i) \in [m..n], (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) \in [m..n], j \geq i$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[m..n]} \psi$. Moreover, since $m \neq 0, \tau(j) - \tau(i) \neq 0$ so $j \neq i$, which implies that j > i. As a consequence $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \land \Diamond \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \leq t < j - 1$.

Take any arbitrary $y \ge i$. If $y \ge j$ then $\tau(y) - \tau(i) \ge m$ because $\tau(y) \ge \tau(j)$ and $\tau(j) - \tau(i) \ge m$. Therefore, $\tau(y) - \tau(i) \notin [0..m)$. If y < j then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), y \models \varphi \mathbf{U} (\varphi \land \circ \psi)$. Since y was arbitrary chosen, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \land \circ \psi))$.

For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m)} (\varphi \mathbf{U} (\varphi \land \Diamond \psi))$ it follows that there exists j > i such that $\tau(j) - \tau(i) \ge m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[m..n]} \psi$ there exists j' > i such that $\tau(j') - \tau(i) \ge m$, $\tau(j') - \tau(i) \le n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$.

If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n]} \psi$. If $j' \ge j$ then $\tau(j') \ge \tau(j)$. Since $\tau(j') - \tau(i) \le n$ and $\tau(j') > \tau(j), \tau(j) - \tau(i) \le n$ so $\tau(j) - \tau(i) \in [m..n]$, which leads to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n]} \psi$.

For the third equivalence $\varphi \mathbf{U}_{(m..n)} \psi \equiv \Diamond_{(m..n)} \psi \wedge \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \wedge \Diamond \psi))$, from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n)} \psi$ then there exists $j \ge i$ such that $\tau(j) - \tau(i) \in (m..n), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) \in (m..n), j \ge i$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(m..n)} \psi$. Moreover, since $m \ne 0, \tau(j) - \tau(i) \ne 0$ so $j \ne i$, which implies that j > i. As a consequence $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \land \Diamond \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \le t < j - 1$.

Take any arbitrary $y \ge i$. If $y \ge j$ then $\tau(y) - \tau(i) > m$ because $\tau(y) \ge \tau(j)$ and $\tau(j) - \tau(i) > m$. Therefore, $\tau(y) - \tau(i) \notin [0..m]$. If y < j then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), y \models \varphi \mathbf{U}(\varphi \land \circ \psi)$. Since y was arbitrary chosen, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \land \circ \psi))$.

For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \land \Diamond \psi))$ it follows that there exists j > i such that $\tau(j) - \tau(i) > m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j]$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(m..n)} \psi$ there exists j' > i such that $\tau(j') - \tau(i) > m$, $\tau(j') - \tau(i) < n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$.

If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n)} \psi$. If $j' \ge j$ then $\tau(j') \ge \tau(j)$. Since $\tau(j') - \tau(i) < n$ and $\tau(j') > \tau(j), \tau(j) - \tau(i) < n$, so $\tau(j) - \tau(i) \in (m..n)$, which leads to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n)} \psi$.

For the fourth equivalence $\varphi \mathbf{U}_{(m..n]} \psi \equiv \Diamond_{(m..n]} \psi \wedge \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \wedge \circ \psi))$, from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n]} \psi$ then there exists $j \ge i$ such that $\tau(j) - \tau(i) \in (m..n], (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) \in (m..n], j \ge i$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(m..n]} \psi$. Moreover, since $m \ne 0, \tau(j) - \tau(i) \ne 0$ so $j \ne i$, which implies that j > i. As a consequence $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \land \circ \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \le t < j - 1$.

Take any arbitrary $y \ge i$. If $y \ge j$ then $\tau(y) - \tau(i) > m$ because $\tau(y) \ge \tau(j)$ and $\tau(j) - \tau(i) > m$. Therefore, $\tau(y) - \tau(i) \notin [0..m]$. If y < j then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), y \models \varphi \mathbf{U}(\varphi \land \circ \psi)$. Since y was arbitrary chosen, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \land \circ \psi))$. For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Box_{[0..m]} (\varphi \mathbf{U} (\varphi \land \Diamond \psi))$ it follows that there exists j > i such that $\tau(j) - \tau(i) > m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(m..n]} \psi$ there exists j' > i such that $\tau(j') - \tau(i) > m$, $\tau(j') - \tau(i) \le n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$.

If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n]} \psi$. If $j' \ge j$ then $\tau(j') \ge \tau(j)$. Since $\tau(j') - \tau(i) \le n$ and $\tau(j') > \tau(j), \tau(j) - \tau(i) \le n$, so $\tau(j) - \tau(i) \in (m..n]$, which leads to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(m..n]} \psi$.

For the fifth equivalence $\varphi \mathbf{U}_{[0..n)} \psi \equiv \Diamond_{[0..n)} \psi \wedge \varphi \mathbf{U} \psi$,

from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n)} \psi$ then there exists $j \ge i$ such that $\tau(j) - \tau(i) \in [0..n), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. This already implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n)} \psi$. Furthermore, since $[0..n) \subseteq [0..\omega)$, we can also derive $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} \psi$ from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} \psi$ from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n)} \psi$. Putting both implications together we get $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n)} \psi \land \varphi \mathbf{U} \psi$.

For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n)} \psi \land \varphi \mathbf{U} \psi$, we can derive $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} \psi$ and therefore there exists $j \ge i$ s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $\tau(j) - \tau(i) \in [0..\omega)$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n)} \psi$ implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$ for some $j' \ge i$ with $\tau(j') - \tau(i) \in [0..n)$. Now there are two cases to consider. If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n)} \psi$. If $j' \ge j$ then $\tau(j) - \tau(i) \in [0..n)$ and therefore $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n)} \psi$.

For the sixth equivalence $\varphi \mathbf{U}_{[0..n]} \psi \equiv \Diamond_{[0..n]} \psi \land \varphi \mathbf{U} \psi$,

from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n]} \psi$ then there exists $j \ge i$ such that $\tau(j) - \tau(i) \in [0..n], (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. This already implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n]} \psi$. Furthermore, since $[0..n] \subseteq [0..\omega)$, we can also derive $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} \psi$ from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n]} \psi$. Putting both implications together we get $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n]} \psi \land \varphi \mathbf{U} \psi$.

For the converse direction, from $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n]} \psi \land \varphi \mathbf{U} \psi$, we can derive $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} \psi$ and therefore there exists $j \ge i$ s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $\tau(j) - \tau(i) \in [0..\omega)$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{[0..n]} \psi$ implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$ for some $j' \ge i$ with $\tau(j') - \tau(i) \in [0..n]$. Now there are two cases to consider. If j' < j we can easily conclude that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n]} \psi$. If $j' \ge j$ then $\tau(j) - \tau(i) \in [0..n]$ and therefore $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[0..n]} \psi$.

For the seventh equivalence $\varphi \mathbf{U}_{(0..n)} \psi \equiv \Diamond_{(0..n)} \psi \land \varphi \mathbf{U} (\varphi \land \Diamond \psi),$

from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$, $i \models \varphi \mathbf{U}_{(0..n)} \psi$ then there exists j > i such that $\tau(j) - \tau(i) \in (0..n)$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$, $j \models \psi$ and for all $k \in [i..j)$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$, $k \models \varphi$. This already implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$, $i \models \Diamond_{(0..n)} \psi$. Furthermore, since j > i, we know

 $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \land \circ \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \leq t < i - 1$ and therefore $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(0,.n)} \psi \land \varphi \mathsf{U} (\varphi \land \circ \psi).$

For the converse direction, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} (\varphi \land \circ \psi)$ implies that there is j > i s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$ which implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{>0} \psi$. Furthermore, since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(0..n)} \psi$ there exists j' > j s.t. $\tau(j') - \tau(j) \in (0..n)$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$. This together with $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{>0} \psi$ implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(0..n)} \psi$ by following similar reasoning as in the previous cases.

For the eighth equivalence $\varphi \mathbf{U}_{(0..n]} \psi \equiv \Diamond_{(0..n]} \psi \wedge \varphi \mathbf{U} (\varphi \wedge \Diamond \psi)$, from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(0..n]} \psi$ then there exists j > i such that $\tau(j) - \tau(i) \in (0..n], (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j), (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. This already implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(0..n]} \psi$. Furthermore, since j > i, we know $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j - 1 \models \varphi \wedge \Diamond \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), t \models \varphi$ for all $i \leq t < i - 1$ and therefore

 $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(0..n)} \psi \land \varphi \mathbf{U} (\varphi \land \circ \psi).$

For the converse direction, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U} (\varphi \land \circ \psi)$ implies that there is j > i s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$ which implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{>0} \psi$. Furthermore, since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \Diamond_{(0..n]} \psi$ there exists j' > j s.t. $\tau(j') - \tau(j) \in (0..n]$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j' \models \psi$. This together with $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{>0} \psi$ implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{>0} \psi$ implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{(0..n]} \psi$ by following similar reasoning as in the previous cases.

The case for Release can be proven by applying Corollary 3 (Boolean Duality) and uniform substitution to the respective Until cases. The cases for Since and Trigger follow from applying Theorem 1 (Temporal Duality) to the Until and Release cases respectively.

Proof of equivalences (4)-(6).

• Equivalence (4): Take any $i \in [0, \lambda)$. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_0 \psi$ iff there exists $j \in [i, \lambda)$ such that $\tau(j) - \tau(i) = 0$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $i \leq k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) = 0$ it follows that $\tau(j) = \tau(i)$. Under strict semantics, it follows j = i. From this we get the iff $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$. Furthermore,

 $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{R}_0 \psi$ iff for all $j \in [i, \lambda)$ if $\tau(j) - \tau(i) = 0$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \psi$ then there exists $i \leq k < j$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$.

From $\tau(j) - \tau(i) = 0$ it follows that $\tau(j) = \tau(i)$. Under strict semantics, it follows j = i. From this we get the iff $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$.

- Equivalence (5): For the case of $O_0 \varphi$ we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models O_0 \varphi$ iff $i + 1 < \lambda$, $\tau(i + 1) - \tau(i) = 0$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i + 1 \models \varphi$. Since we are considering strict semantics, we get that $\tau(i + 1) - \tau(i) \neq 0$ and we can derive \perp . We can follow a similar reasoning for the case of $\bullet_0 \varphi$.
- Equivalence (6): For the case of $\widehat{\circ}_0 \varphi$ we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \widehat{\circ}_0 \varphi$ iff if $i+1 < \lambda$ and $\tau(i+1) - \tau(i) = 0$ then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \varphi$. Since we are considering strict semantics, we get that $\tau(i+1) - \tau(i) \neq 0$ and we can derive \top . We can follow a similar reasoning for the case of $\widehat{\bullet}_0 \varphi$.

Proof of Lemma 2.

• Equivalence (7): from left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \mathbf{U}_n \varphi$ then there exists $j \in [i, \lambda)$ such that $\tau(j) - \tau(i) = n$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \varphi$ and for all $i \leq k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi$. Since $n > 0, \tau(j) - \tau(i) > 0$ implies that $j \geq i + 1 > i$ and, under strict semantics, $\tau(j) \geq \tau(i+1) > \tau(i)$. If we denote by $m \stackrel{def}{=} \tau(i+1) - \tau(i)$, we can conclude that $0 \leq m < n$ and, moreover, $\tau(j) - \tau(i+1) = n - m$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i + 1 \models \psi \mathbf{U}_{n-m} \varphi$. Since $\tau(i+1) - \tau(i) = m$, it follows $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models o_m (\psi \mathbf{U}_{n-m} \varphi)$. Since m is bounded we conclude that

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \bigvee_{0 < m < n} \circ_m (\psi \, \mathbf{U}_{n-m} \, \varphi).$$

Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$ we get

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \land \bigvee_{0 < m \le n} \circ_m (\psi \mathbf{U}_{n-m} \varphi).$$

Conversely, if

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \land \bigvee_{0 < m \le n} \circ_m (\psi \mathbf{U}_{n-m} \varphi),$$

then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$ and there exists $0 < m \le n$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \circ_m (\psi \mathbf{U}_{n-m} \varphi)$. Therefore, $\tau(i+1) - \tau(i) = m$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \psi \mathbf{U}_{n-m} \varphi$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \psi \mathbf{U}_{n-m} \varphi$, there exists $j \ge i+1$ such that $\tau(j) - \tau(i+1) = n - m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \varphi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi$ for all $i+1 \le k < j$. From $\tau(j) - \tau(i+1) = n - m$ and $\tau(i+1) - \tau(i) = m$ we get that $\tau(j) - \tau(i) = n$. Also, since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi$ for all $i \le k < j$ leading to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \mathbf{U}_n \varphi$

• Equivalence (8): from right to left assume towards a contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi \mathbf{R}_n \varphi$ then there exists $j \in [i, \lambda)$ such that $\tau(j) - \tau(i) = n$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \varphi$ and for all $i \leq k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \psi$. Since $n > 0, n = \tau(j) - \tau(i) > 0$ implies that $j \geq i + 1 > i$ and, under strict semantics, $\tau(j) \geq \tau(i+1) > \tau(i)$. If we denote by $m \stackrel{def}{=} \tau(i+1) - \tau(i)$, we can conclude that $0 < m \leq n$. Furthermore, it follows that $\tau(j) - \tau(i+1) = n - m$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i + 1 \not\models \psi \mathbf{R}_{n-m} \varphi$. Since $\tau(i+1) - \tau(i) = m$, it follows $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \widehat{\circ}_m (\psi \mathbf{R}_{n-m} \varphi)$. Since m is bounded we conclude that

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \bigwedge_{0 < m < n} \widehat{o}_m (\psi \mathbf{R}_{n-m} \varphi).$$

Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi$ we reach the contradiction

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi \lor \bigwedge_{0 < m \le n} \widehat{\mathbf{o}}_m \left(\psi \, \mathbf{R}_{n-m} \, \varphi \right).$$

Conversely, assume towards a contradiction that

$$(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi \lor \bigwedge_{0 < m \le n} \widehat{\mathbf{o}}_m \left(\psi \, \mathbf{R}_{n-m} \, \varphi \right),$$

then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi$ and there exists $0 < m \leq n$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \widehat{O}_m (\psi \mathbf{R}_{n-m} \varphi)$. Therefore, $\tau(i+1) - \tau(i) = m$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \not\models \psi \mathbf{R}_{n-m} \varphi$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \not\models \psi \mathbf{U}_{n-m} \varphi$, there exists $j \geq i+1$ such that $\tau(j) - \tau(i+1) = n - m$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \varphi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \psi$ for all $i+1 \leq k < j$. From $\tau(j) - \tau(i+1) = n - m$ and $\tau(i+1) - \tau(i) = m$ we get that $\tau(j) - \tau(i) = n$. Also, since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi$, it follows that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \psi$ for all $i \leq k < j$ leading to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \psi \mathbf{R}_n \varphi$: a contradiction.

• Equivalences (9)-(10): by definition, $\Diamond_n \varphi \stackrel{def}{=} \top \mathbf{U}_n \varphi$ and $\Box_n \varphi \stackrel{def}{=} \bot \mathbf{R}_n \varphi$. Therefore the proof follows directly from equivalences (7) and (8).

Proof of Lemma 3.

• Equivalence (11): from left to right, assume $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi \mathbf{U}_{\leq n} \varphi$, then there is $i \geq k$ with $\tau(i) - \tau(k) \leq n$ s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ for all $k \leq j < i$. Lets further assume towards a contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi \lor (\psi \land \bigvee_{i=1}^{n} \circ_i (\psi \mathbf{U}_{\leq (n-i)} \varphi))$ This implies that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi \lor (\psi \land \bigvee_{i=1}^{n} \circ_i (\psi \mathbf{U}_{\leq (n-i)} \varphi))$ For the latter to hold either $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \psi$ first. To be consistent with the original assumption $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$ is needed. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \bigvee_{i=1}^{n} \circ_i (\psi \mathbf{U}_{\leq (n-i)} \varphi)$ leads to a contradiction. Considering $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \bigvee_{i=1}^{n} \circ_i (\psi \mathbf{U}_{\leq (n-i)} \varphi)$ leads to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \bigvee_{i=1}^{n} \circ_i (\psi \mathbf{U}_{\leq (n-i)} \varphi)$. This implies that there is no i > k with $\tau(i) - \tau(k) \leq n$ s.t. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi$. Together with $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$ this implies that there is no occurrence of φ in the interval which is contradictory to the original assumption.

From right to left, assume $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi \lor (\psi \land \bigvee_{i=1}^{n} \circ_{i}(\psi \mathbf{U}_{\leq (n-i)} \varphi))$, then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ or $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi \land \bigvee_{i=1}^{n} \circ_{i}(\psi \mathbf{U}_{\leq (n-i)} \varphi)$. If $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ then obviously $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi \mathbf{U}_{\leq n} \varphi$. From the second disjunct we get that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \circ_{i}(\psi \mathbf{U}_{\leq (n-i)} \varphi)$ for some $1 \leq i \leq n$. Then there is a next state that satisfies $\psi \mathbf{U} \varphi$ s.t. φ holds somewhere within the next interval and, due to $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi, \psi$ holds until then. This implies $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \psi \mathbf{U}_{\leq n} \varphi$.

• Equivalence (12) follows directly from Equivalence (11), Corollary 3 (Boolean Duality) and uniform substitution.

Proof of Theorem 2.

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Equivalence (13): from left to right, if M, k ⊨ ψ U_{[m..n)} φ with the restriction m > 0 and m < n - 1, then by Definition 2(10) M, i ⊨ φ for some i ∈ (k..λ) with τ(i) - τ(k) ∈ I and M, j ⊨ φ for all j ∈ [k..i). Since i > k and ψ has to hold since k, it follows that M, k ⊨ ψ and therefore we get: M, k ⊨ ψ and M, i ⊨ φ for some i ∈ (k..λ) with τ(i) - τ(k) ∈ I and M, j ⊨ φ for all j ∈ (k..i). Notice that j ∈ (k..i) since k was separated by M, k ⊨ ψ. Now, considering the two options for the distance of the next state: τ(k + 1) - τ(k) ≤ m or τ(k + 1) - τ(k) ∈ (m..n), we get: M, k ⊨ ψ and M, k + 1 ⊨ ψ U_{[(m-p)..(n-p))} φ if p ≤ m or M, k + 1 ⊨ ψ U_{<(n-p)} φ if p ∈ (m..n), where p = τ(k + 1) - τ(k). The case p ≤ m with M, k + 1 ⊨ ψU_{[(m-p)..(n-p))} can be expressed by:

$$\bigvee_{1 \le i \le m} \circ_m \left(\psi \, \mathbf{U}_{[m-i..n-i]} \, \varphi \right)$$
 larly the case $p = \tau(k+1) - \tau(k)$ can be represented by

m

$$\bigvee_{\substack{+1 \le i \le n-1}} \circ_i \left(\psi \, \mathbf{U}_{\le (n-1-i)} \, \varphi \right)$$

Taking into account those two options and the already performed conclusion $\mathbf{M}, k \models \psi$, we arrive at:

$$\mathbf{M}, k \models \psi \land \bigvee_{1 \le i \le m} \circ_i \left(\psi \, \mathbf{U}_{[m-i..n-i]} \, \varphi \right) \lor \bigvee_{m+1 \le i \le n-1} \circ_i \left(\psi \, \mathbf{U}_{\le (n-1-i)} \, \varphi \right)$$

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Conversely, if

$$\mathbf{M}, k \models \psi \land \bigvee_{1 \leq i \leq m} \circ_i \left(\psi \, \mathbf{U}_{[m-i..n-i]} \, \varphi \right) \lor \bigvee_{m+1 \leq i \leq n-1} \circ_i \left(\psi \, \mathbf{U}_{\leq (n-1-i)} \, \varphi \right),$$

then it follows that $\mathbf{M}, k \models \psi$ and $\mathbf{M}, k+1 \models \psi \mathbf{U}_{[m-i..n-i)} \varphi$ for some $i \in [1..m]$ or $\mathbf{M}, k \models \psi$ and $\mathbf{M}, k+1 \models \psi \mathbf{U}_{\leq (n-1-i)} \varphi$ for some $i \in (1..m)$ since each of the both disjunctions would be satisfied in case of one matching next-formula with the respective *i*. From this, together with the assumption m > 0 and m < n-1 we can conclude that $\mathbf{M}, k \models \psi$ and $\mathbf{M}, i \models \varphi$ for some $i \in (k..\lambda)$ with $\tau(i) - \tau(k) \in [m..n)$ and $\mathbf{M}, j \models \psi$ for all $j \in (k..i)$. Finally, by applying Definition 2(10) we arrive at: $\mathbf{M}, i \models \psi \mathbf{U}_{[m..n)} \varphi$

• Equivalence (14): from left to right, if $\mathbf{M}, k \models \psi \mathbf{R}_{[m..n]} \varphi$, then by Definition 2(11) for all $i \in [k.\lambda)$ with $\tau(i) - \tau(k) \in [m.n]$ we have $\mathbf{M}, i \models \varphi$ or $\mathbf{M}, j \models \psi$ for some $j \in [k..i)$. Let's consider the easy case first, where $\mathbf{M}, k \models \psi$. In this case it is obvious to see that $\mathbf{M}, k \models \psi \lor \left(\bigwedge_{i=1}^{m} \widehat{\mathbf{O}}_{i}(\psi \mathbf{R}_{[(m-i)..(n-i))} \varphi) \land \bigwedge_{i=m+1}^{n-1} \widehat{\mathbf{O}}_{i}(\psi \mathbf{R}_{\leq (n-1-i)} \varphi) \right).$ If $\mathbf{M}, k \not\models \psi$ there has to be a later releasing ψ for all occurrences of $\neg \psi \in [m.n]$. In this case the next state, if there is one before the end of the interval, satisfies a Release formula that considers the time elapsed since k. If the linked time point of the next state is $\in (k..m]$, this state satisfies $\bigwedge_{i=1}^{m} \widehat{O}_i(\psi \mathbf{R}_{[(m-i)..(n-i))} \varphi)$. If the linked time point of the next state is $\in (m..n)$, the next state would satisfy $\bigwedge_{i=m+1}^{n-1} \widehat{\circ}_i(\psi \mathbf{R}_{\leq (n-1-i)} \varphi)$. Considering the possibility of both cases we can conclude that $\mathbf{M}, k \models \psi \lor \left(\bigwedge_{i=1}^{m} \widehat{\circ}_{i}(\psi \mathbf{R}_{[(m-i)..(n-i))} \varphi) \land \bigwedge_{i=m+1}^{n-1} \widehat{\circ}_{i}(\psi \mathbf{R}_{\leq (n-1-i)} \varphi) \right).$ Conversely, assume $\mathbf{M}, k \models \psi \lor \left(\bigwedge_{i=1}^{m} \widehat{o}_{i}(\psi \mathbf{R}_{[(m-i)..(n-i))} \varphi) \land \bigwedge_{i=m+1}^{n-1} \widehat{o}_{i}(\psi \mathbf{R}_{\leq (n-1-i)} \varphi) \right).$ Again, if $\mathbf{M}, k \models \psi$ it follows directly that $\mathbf{M}, k \models \psi \mathbf{R}_{[m..n]} \varphi$. If $\mathbf{M}, k \models \bigwedge_{i=1}^{m} \widehat{\mathbf{O}}_{i}(\psi \mathbf{R}_{[(m-i)..(n-i))}\varphi) \land \bigwedge_{i=m+1}^{n-1} \widehat{\mathbf{O}}_{i}(\psi \mathbf{R}_{\leq (n-1-i)}\varphi)$, we can also follow that $\mathbf{M}, k \models \psi \mathbf{R}_{[m..n)}\varphi$, since both the satisfaction of $\bigwedge_{i=1}^{m} \widehat{\mathbf{O}}_{i}(\psi \mathbf{R}_{[(m-i)..(n-i))}\varphi)$ in a next state $\in (k..m]$ and the satisfaction of $\bigwedge_{i=m+1}^{n-1} \widehat{o}_i(\psi \mathbf{R}_{\leq (n-1-i)}\varphi)$ in a next state $\in (m..n)$ would guarantee it. Taking all possible cases together, we finally arrive at $\mathbf{M}, k \models \psi \mathbf{R}_{[m..n]} \varphi$.

Proof of Theorem 3.

- If $\mathbf{M}, k \models \psi \mathbf{U}_{[m..n)} \varphi$, it follows by applying Definition 2(10) that $\mathbf{M}, i \models \varphi$ for some $i \in [k..\lambda)$ with $\tau(i) - \tau(k) \in [m..n)$ and $\mathbf{M}, j \models \psi$ for all $j \in [k..i)$. From $\tau(i) - \tau(k) \in [m..n)$ we get $\tau(i) - \tau(k) \in [m..s)$ or $\tau(i) - \tau(k) \in [s..n)$ for all $s \in [m..n)$. This implies that $\mathbf{M}, i \models \varphi$ for some $i \in [k..\lambda)$ with $\tau(i) - \tau(k) \in [m..s)$ and $\mathbf{M}, j \models \psi$ for all $j \in [k..i)$ or $\mathbf{M}, i \models \varphi$ for some $i \in [k..\lambda)$ with $\tau(i) - \tau(k) \in [m..s)$ and $\mathbf{M}, j \models \psi$ for all $j \in [k..i)$, for all $s \in [m..n)$. Applying again Definition 2(10) and Definition 2(4) it follows that $\mathbf{M}, k \models (\psi \mathbf{U}_{[m..i)} \varphi) \lor (\psi \mathbf{U}_{[i..n)} \varphi)$ for all $i \in [m..n)$. As every step of the proof works in the converse direction as well, both directions are provided.
- For the respective Release-case the same reasoning applies.

Proof of Theorem 5. The proof goes by structural induction.

Base case let us consider first the case of a propositional variable p. From left to right, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi$ then $p \in H_i$. By definition, $\tau(i) \in D$ and $p(\tau(i)) \in H$. Therefore $\langle (D, \sigma), H, T \rangle \models [p]_{\tau_i}$.

Inductive case: propositional connectives

• Case $\varphi \wedge \psi$:

$$\begin{split} (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i &\models \varphi \land \psi \quad \text{iff} \quad (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \text{ and } (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \\ \stackrel{(16)}{\text{iff}} \quad \langle (D, \sigma), H, T \rangle \models [\varphi]_{\tau_i} \text{ and } \langle (D, \sigma), H, T \rangle \models [\psi]_{\tau_i} \\ \text{iff} \quad \langle (D, \sigma), H, T \rangle \models [\varphi \land \psi]_{\tau_i}. \end{split}$$

$$\begin{split} (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \land \psi & \text{iff} \quad (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \text{ and } (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \psi \\ (17) & \text{iff} \quad \langle (D, \sigma), T, T \rangle \models [\varphi]_{\tau_i} \text{ and } \langle (D, \sigma), T, T \rangle \models [\psi]_{\tau_i} \\ & \text{iff} \quad \langle (D, \sigma), T, T \rangle \models [\varphi \land \psi]_{\tau_i}. \end{split}$$

• Case $\varphi \lor \psi$:

$$\begin{split} (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \lor \psi & \text{iff} \quad (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \text{ or } (\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi \\ (16) \\ \text{iff} & \langle (D, \sigma), H, T \rangle \models [\varphi]_{\tau_i} \text{ or } \langle (D, \sigma), H, T \rangle \models [\psi]_{\tau} \\ \text{iff} & \langle (D, \sigma), H, T \rangle \models [\varphi \lor \psi]_{\tau_i}. \end{split}$$

$$\begin{split} (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i &\models \varphi \lor \psi \quad \text{iff} \quad (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \text{ or } (\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \psi \\ \stackrel{(17)}{\text{iff}} \quad \langle (D, \sigma), T, T \rangle \models [\varphi]_{\tau_i} \text{ or } \langle (D, \sigma), T, T \rangle \models [\psi]_{\tau_i} \\ \text{iff} \quad \langle (D, \sigma), T, T \rangle \models [\varphi \lor \psi]_{\tau_i}. \end{split}$$

• Case $\varphi \to \psi$: In the first case, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \to \psi$ iff for all $\otimes \in \{\mathbf{H}, \mathbf{T}\}$, either $\langle \otimes, \mathbf{T}, \tau \rangle, i \not\models \varphi$ or $\langle \otimes, \mathbf{T}, \tau \rangle, i \models \psi$. By the induction hypothesis (16) and (17) we get iff for all $\oplus \in \{H, T\}$, either $\langle (D, \sigma), \oplus, T \rangle \not\models [\varphi]_{\tau_i}$ or $\langle (D, \sigma), \oplus, T \rangle \models [\psi]_{\tau_i}$. Therefore, $\langle (D, \sigma), H, T \rangle \models [\varphi \to \psi]_{\tau_i}$. In the second case, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \to \psi$ iff either $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \not\models \varphi$ or $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \psi$. By the induction hypothesis (17) we get iff either $\langle (D, \sigma), T, T \rangle \not\models [\varphi]_{\tau_i}$ or $\langle (D, \sigma), T, T \rangle \models [\psi]_{\tau_i}$.

Inductive case: metric temporal operators For simplicity we will consider intervals of the form [m, n) where $m \neq \omega$.

• Case $\circ_{[m..n)}\varphi$: in the first case, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \circ_{[m..n)}$ then there exists $i+1 < \lambda$ such that $m \leq \tau(i+1) - \tau(i) < n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \varphi$. By (16) we get $\langle (D, \sigma), H, T \rangle \models [\varphi]_{\tau_{i+1}}$ From $m \leq \tau(i+1) - \tau(i) < n$ we conclude that $\tau(i) \preccurlyeq_{-m} \tau(i+1) \prec_n \tau(i)$. Since we are dealing with strict traces, $\tau(i) < \tau(i+1)$ and, moreover, there is no other $\tau(j)$ in between. Therefore, for all $d \in D$, not $\tau(i) < d < \tau(i+1)$, so $\langle (D,\sigma), H, T \rangle \models \exists z \ \tau(i) < z < \tau(i+1)$; Since $\tau(i+1) \in D$, we conclude that there exists $d \in D$ such that

$$\langle (D,\sigma), H, T \rangle \models \tau(i) < d \land (\neg \exists z \ \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i) \land [\varphi]_d$$

Therefore,

$$\langle (D,\sigma), H, T \rangle \models \exists y \ (\tau(i) < y \land (\neg \exists z \ \tau(i) < z < y) \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i) \land [\varphi]_y).$$

From this we conclude $\langle (D, \sigma), H, T \rangle \models [\circ_{[m..n]} \varphi]_{\tau(i)}$. Conversely, if $\langle (D, \sigma), H, T \rangle \models [\circ_{[m..n]} \varphi]_{\tau(i)}$ then, by definition,

$$\langle (D,\sigma), H, T \rangle \models \exists y \ (\tau(i) < y \land (\neg \exists z \ \tau(i) < z < y) \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i) \land [\varphi]_y),$$

Therefore, there exists $d \in D$ such that

$$\langle (D,\sigma), H, T \rangle \models \tau(i) < d \land (\neg \exists z \ \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i) \land [\varphi]_d.$$

From $\langle (D,\sigma), H, T \rangle \models \tau(i) < d \land (\neg \exists z \ \tau(i) < z < d)$ and the construction of $\langle (D,\sigma), H, T \rangle$ we conclude that $d = \tau(i)$. Since $\langle (D,\sigma), H, T \rangle \models \tau(i) \preccurlyeq_{-m} \tau(i + 1) \prec_n \tau(i)$, we conclude that $m \leq \tau(i+1) - \tau(i) < n$. By the induction hypothesis (16), $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i + 1 \models \varphi$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \circ_{[m..n)} \varphi$. The second item of this case follows a similar reasoning.

• Case $\widehat{\circ}_{[m..n)}\varphi$: in the first case, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \widehat{\circ}_{[m..n)}$ then $i + 1 < \lambda$ such that $m \leq \tau(i+1) - \tau(i) < n$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i + 1 \not\models \varphi$. By (16) we get $\langle (D, \sigma), H, T \rangle \not\models [\varphi]_{\tau_{i+1}}$ From $m \leq \tau(i+1) - \tau(i) < n$ we conclude that $\tau(i) \preccurlyeq_{-m} \tau(i+1) \prec_n \tau(i)$. Since we are dealing with strict traces, $\tau(i) < \tau(i+1)$ and, moreover, there is no other $\tau(j)$ in between. Therefore, for all $d \in D$, not $\tau(i) < d < \tau(i+1)$, so $\langle (D, \sigma), H, T \rangle \models \exists z \tau(i) < z < \tau(i+1)$; Since $\tau(i+1) \in D$, we conclude that there exists $d \in D$ such that

$$\langle (D,\sigma), H, T \rangle \not\models (\tau(i) < d \land (\neg \exists z \ \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i)) \to [\varphi]_d.$$

Therefore,

$$\langle (D,\sigma), H, T \rangle \not\models \forall y \ \left((\tau(i) < y \land (\neg \exists z \ \tau(i) < z < y) \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i) \right) \rightarrow [\varphi]_y \right).$$

From this we conclude $\langle (D, \sigma), H, T \rangle \not\models [\widehat{\circ}_{[m..n)} \varphi]_{\tau(i)}$. Conversely, if $\langle (D, \sigma), H, T \rangle \not\models [\widehat{\circ}_{[m..n)} \varphi]_{\tau(i)}$ then, by definition,

$$\langle (D,\sigma), H, T \rangle \not\models \forall y \ ((\tau(i) < y \land (\neg \exists z \ \tau(i) < z < y) \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i)) \rightarrow [\varphi]_y),$$

Therefore, there exists $d \in D$ such that

$$\langle (D,\sigma),H,T\rangle \not\models (\tau(i) < d \land (\neg \exists z \; \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i)) \to [\varphi]_d.$$

We consider two cases:

- 1. $\langle (D,\sigma), H, T \rangle \models (\tau(i) < d \land (\neg \exists z \ \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i)) \text{ and } \langle (D,\sigma), H, T \rangle \not\models [\varphi]_d;$
- 2. $\langle (D,\sigma), T, T \rangle \models (\tau(i) < d \land (\neg \exists z \ \tau(i) < z < d) \land \tau(i) \preccurlyeq_{-m} d \prec_n \tau(i)) \text{ and } \langle (D,\sigma), T, T \rangle \not\models [\varphi]_d;$

In any of the previous cases, we conclude that $d = \tau(i+1), m \leq \tau(i+1) - \tau(i) < n$ and $\langle (D, \sigma), H, T \rangle \not\models [\varphi]_{\tau(i)}$.

By the induction hypothesis (16), $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \not\models \varphi$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \widehat{o}_{[m..n]}\varphi$.

The second item of this case follows a similar reasoning.

• Case $\varphi \mathbf{U}_{[m..n)} \psi$: for the first item, if $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n)} \psi$ then there exists $j \ge i$ such that $m \le \tau(j) - \tau(i) < n$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $i \le k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. Since $m \le \tau(j) - \tau(i) < n$ then $\langle (D, \sigma), H, T \rangle \models \tau(i) \le \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j) \prec_n \tau(i)$. From $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and the induction hypothesis (16) we get $\langle (D, \sigma), H, T \rangle \models [\psi]_{\tau(j)}$. From $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $i \le k < j$ and the induction hypothesis (16) we get $\langle (D, \sigma), H, T \rangle \models [\psi]_{\tau(j)}$. From $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $i \le k < j$ and the induction hypothesis (16) we get $\langle (D, \sigma), H, T \rangle \models [\varphi]_d$ for all $d \in \{\tau(k) \mid \tau(i) \le k < \tau(j)\}$. By the semantics, $\langle (D, \sigma), H, T \rangle \models \forall z (\tau(i) \le z < \tau(j) \rightarrow [\varphi]_z)$. Therefore, $\langle (D, \sigma), H, T \rangle \models [\varphi \mathbf{U}_{[m..n)} \psi]_{\tau(i)}$. Conversely, if $\langle (D, \sigma), H, T \rangle \models [\varphi \mathbf{U}_{[m..n)} \psi]_{\tau(i)}$ then

$$\langle (D, \sigma), H, T \rangle \models \exists y \ (\tau(i) \leq y \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i) \land [\psi]_y \land \forall z \ (\tau(i) \leq z < y \rightarrow [\varphi]_z)) \,.$$

This means that there exists $\tau(j) \in D$ such that $\langle (D, \sigma), H, T \rangle \models (\tau(i) \leq \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j))$, $\langle (D, \sigma), H, T \rangle \models [\psi]_{\tau(j)}$ and $\langle (D, \sigma), H, T \rangle \models \forall z (\tau(i) \leq z < \tau(j) \rightarrow [\varphi]_z)$. From $\langle (D, \sigma), H, T \rangle \models (\tau(i) \leq \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j))$ it follows that $i \leq j$ and $\tau(j) - \tau(i) \in [m, n)$. By induction, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$. From $\langle (D, \sigma), H, T \rangle \models \forall z (\tau(i) \leq z < \tau(j) \rightarrow [\varphi]_z)$ it follows that for all $\tau(k) \in D$, if $\tau(i) \leq \tau(k) < \tau(j)$ then $\langle (D, \sigma), H, T \rangle \models [\varphi]_{\tau(k)}$. By induction we get that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$, for all $i \leq k < j$. From all previous statements it follows $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \mathbf{U}_{[m..n)} \psi$. The second item is proved in a similar way.

• Case $\varphi \mathbf{R}_{[m..n)} \psi$: from left to right, assume by contraposition that $\langle (D, \sigma), H, T \rangle \not\models [\varphi \mathbf{U}_{[m..n)} \psi]_{\tau(i)}$ then

 $\langle (D, \sigma), H, T \rangle \not\models \forall y \ \left((\tau(i) \leq y \land \tau(i) \preccurlyeq_{-m} y \prec_n \tau(i)) \rightarrow ([\psi]_y \lor \exists z \ (\tau(i) \leq z < y \land [\varphi]_z)) \right).$

Therefore, there exists $\tau(j) \in D$ such that

$$\langle (D,\sigma), H, T \rangle \not\models (\tau(i) \le \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j) \prec_n \tau(i)) \to ([\psi]_{\tau(j)} \lor \exists z \, (\tau(i) \le z < y \land [\varphi]_z)) \land$$

From this and a some HT reasoning¹ we can conclude that $\langle (D,\sigma), H, T \rangle \models (\tau(i) \leq \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j) \prec_n \tau(i))$ but $\langle (D,\sigma), H, T \rangle \not\models [\psi]_{\tau(j)}$ and $\langle (D,\sigma), H, T \rangle \not\models \exists z \ (\tau(i) \leq z < y \land [\varphi]_z)$. From $\langle (D,\sigma), H, T \rangle \models (\tau(i) \leq \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j))$ it follows that $i \leq j$ and $\tau(j) - \tau(i) \in [m, n)$. By induction (16), $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \psi$.

¹ Using persistency and the fact that the satisfaction of the expression $\tau(i) \leq \tau(j) \wedge \tau(i) \preccurlyeq -m \tau(j) \prec_n \tau(i)$ is not HT-dependent.

From $\langle (D,\sigma), H, T \rangle \not\models \exists z \, (\tau(i) \leq z < \tau(j) \land [\varphi]_z)$ it follows that for all $\tau(k) \in D$, if $\langle (D,\sigma), H, T \rangle \models \tau(i) \leq \tau(k) < \tau(j)$ then $\langle (D,\sigma), H, T \rangle \not\models [\varphi]_{\tau(k)}$. By induction (16) we get that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$, for all $i \leq k < j$. From all previous statements it follows $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \varphi \mathbf{R}_{[m..n)} \psi$: a contradiction.

For the converse direction assume by contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \varphi \mathbf{R}_{[m..n]}$ ψ . Then, there exists $j \geq i$ such that $m \leq \tau(j) - \tau(i) < n$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \psi$ and for all $i \leq k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$. Since $m \leq \tau(j) - \tau(i) < n$ then $\langle (D, \sigma), H, T \rangle \models \tau(i) \leq \tau(j) \land \tau(i) \preccurlyeq_{-m} \tau(j) \prec_n \tau(i)$. From $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \psi$ and the induction hypothesis (16) we get $\langle (D, \sigma), H, T \rangle \not\models [\psi]_{\tau(j)}$. From $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \varphi$, for all $i \leq k < j$, and the induction hypothesis (16) we get $\langle (D, \sigma), H, T \rangle \not\models [\psi]_d$ for all $d \in \{\tau(k) \mid \tau(k) \in D \text{ and } \tau(i) \leq \tau(k) < \tau(j)\}$. By the semantics, $\langle (D, \sigma), H, T \rangle \not\models \forall z (\tau(i) \leq z < \tau(j) \rightarrow [\varphi]_z)$. Therefore, $\langle (D, \sigma), H, T \rangle \not\models [\varphi \mathbf{R}_{[m..n]}, \psi]_{\tau(i)}$: a contradiction. The second item is proved in a similar way.

- The case of $\bullet_{[m..n)}\psi$ is similar to the case of $\circ_{[m..n)}\psi$.
- The case of $\widehat{\bullet}_{[m..n]}\psi$ is similar to the case of $\widehat{\circ}_{[m..n]}\psi$.
- The case of $\varphi \mathbf{S}_{[m..n)} \psi$ is similar to the case of $\varphi \mathbf{U}_{[m..n)} \psi$.
- The case of $\varphi \mathbf{T}_{[m..n]} \psi$ is similar to the case of $\varphi \mathbf{R}_{[m..n]} \psi$.

Proof of Theorem 4. From right to left, if Γ_1 and Γ_2 are MHT equivalent then Γ_1 and Γ_2 have the same MHT models. As a consequence, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same MHT models. Therefore, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ have the same MEL models. Since Γ is any arbitrary temporal theory, Γ_1 and Γ_2 are strongly equivalent.

For the converse direction let us assume that Γ_1 and Γ_2 are strongly equivalent but they are not MHT equivalent. We consider two cases:

- 1. Γ_1 and Γ_2 are not MTL equivalent. Assume, without loss of generality, that there exists a total MHT model $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ such that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1$ but $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \not\models \Gamma_2$. Since $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ is total, it follows that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1 \cup \text{EM}(\mathcal{A})$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \not\models \Gamma_2 \cup \text{EM}(\mathcal{A})$. Moreover, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ is an equilibrium model of $\Gamma_1 \cup \text{EM}(\mathcal{A})$ (since for any $\mathbf{H} < \mathbf{T}$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \text{EM}(\mathcal{A})$) but not of $\Gamma_2 \cup \text{EM}(\mathcal{A})$.
- 2. Γ_1 and Γ_2 are MTL equivalent. Therefore, without loss of generality, there exists a MHT interpretation $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$ with $\mathbf{H} < \mathbf{T}$ such that
 - (a) $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1$ iff $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_2$ because Γ_1 and Γ_2 are MTL equivalent.
 - (b) $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \Gamma_2$ because Γ_1 and Γ_2 are not MHT equivalent.

Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \Gamma_2$, there exists $\varphi \in \Gamma_2$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \varphi$. Moreover, since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1$ then $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1$ so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_2$ and so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \varphi$.

Let us consider the theory $\Gamma \stackrel{def}{=} \{\varphi \to \psi \mid \psi \in \text{EM}(\mathcal{A})\}$. Since $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \not\models \varphi$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \text{EM}(\mathcal{A})$ then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \Gamma$. Therefore, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), 0 \models \Gamma_1 \cup \Gamma$ so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ is not an equilibrium model of $\Gamma_1 \cup \Gamma$. Since Γ_1 and Γ_2 are strongly equivalent, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ cannot be an equilibrium model of $\Gamma_2 \cup \Gamma$. Since $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models$ Γ_2 and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), 0 \models \Gamma$ then the minimality condition must fail. This means that there must exist $\mathbf{H}' < \mathbf{T}$ such that $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models \Gamma_2 \cup \Gamma$. Since $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models$

 Γ_2 then $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models \varphi$. Since $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models \varphi$ and $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models \Gamma$ then $(\langle \mathbf{H}', \mathbf{T} \rangle, \tau), 0 \models \mathrm{EM}(\mathcal{A})$, which contradicts Proposition 5.