Possible Worlds Semantics for Default Logics

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Abstract. We introduce a uniform semantical framework for various default logics in terms of Kripke structures. This possible worlds approach provides a simple but meaningful instrument for comparing existing default logics in a unified setting. The possible worlds semantics is introduced by means of constrained default logic. Also, it easily deals with Brewka's cumulative default logic. The semantics is then extended to Reiter's original default logic as well as Łukaszewicz' variant. The possible worlds approach remedies several difficulties encountered in former proposals aiming at individual default logics. Notably, it provides the first pure model–theoretic semantics for Łukaszewicz' variant of default logic. Since the semantical framework is presented from the perspective of "commitment to assumptions" we also obtain a very natural modal interpretation of the notion of commitment.

1 Motivation

Recent research on default logic [10] has produced many derivatives of Reiter's original formalism. A common feature of all of these variants is their use of constraints, either on formulas as cumulative default logic [3] or on sets of formulas like justified[1] [7] and constrained default logic [4, 13]. In other words, all of the descendants of Reiter's classical default logic employ more "structure" in order to achieve their desired results such as the existence of extensions and "commitment to assumptions" [9] or the formal properties of semi–monotonicity and cumulativity [8]. In a similar way, Etherington's semantics for classical default logic [6] has been extended in order to account for the additional syntactical structures. As a result, two–folded semantics were proposed [12] whose second component was intended to capture the enriched structure in default logics.

[1] We will refer to Reiter's default logic as *classical* default logic and to Łukaszewicz variant as *justified* default logic.

Although the elements of these two–folded semantics are standard first–order interpretations, splitting the semantical characterizations of the extension and its underlying constraints might appear to be artificial. On the other hand, Kripke structures provide means to establish relations between first–order interpretations: a Kripke structure has a distinguished world, the "actual" world, and a set of worlds accessible from it (each world is a first–order interpretation). As a consequence, a first aim of this work is to avoid two–folded semantics by characterizing extensions in default logics by means of Kripke structures; thereby, providing an elegant semantical representation for the additional syntactical structures used in each variant of classical default logic. In fact, this approach turns out to be very general, so that we obtain a uniform semantical framework for comparing existing default logics in a unified setting.

The idea is roughly as follows. In default logics, our beliefs consist of the conclusions given by the applying default rules, and the constraints on our beliefs stem from the justifications provided by the same default rules. Then, the intuition behind our semantics is very natural and easy to understand: the actual world of a Kripke structure exhibits what we believe and the accessible worlds exhibit what constraints we have imposed upon our beliefs. Hence, the actual world is our envisioning of how things are and, therefore, characterizes an extension, whereas the surrounding worlds additionally deal with the constraints and, therefore, provide a context in which that envisioning takes place.

Let us put this in more concrete terms by means of constrained default logic. In constrained default logic, an extension E relies on a set of constraints C. Given a constrained extension (E,C) and a Kripke structure $\mathfrak{m}$, we stipulate that the actual world be a model of the extension, E, and demand that each world accessible from the actual world be a model of the constraints, C. That is, $\mathfrak{m} \models E \wedge \Box C$.[2]

The rest of the paper is organized as follows. In Section 2, we first reproduce the basic definition of classical and constrained default logic. In Section 3, we introduce our possible worlds semantics for constrained default logic and show how it characterizes the notion of "commitment to assumptions". Furthermore, we show that the semantics is able to capture Brewka's cumulative default logic, too. Eventually, in Sections 4 and 5, we demonstrate that our possible worlds semantics applies to Reiter's classical and Lukaszewicz' justified default logic as well. We can then easily compare default logics and characterize the differences between them. In particular, the semantics reveals that all of the various default logics employ constraints (induced by the consequents and justifications of applied default rules) but differ basically in the extent to which the constraints are taken into account. Since this extent is directly related to the notion of "commitment to assumptions", we also obtain a very natural semantical characterization of this notion in the context of default logics.

2 From classical to constrained extensions

Classical default logic was defined by Reiter in [10] as a formal account of reasoning in the absence of complete information.

It is based on first order logic, whose sentences are hereafter simply referred to as formulas (instead of closed formulas). In what follows, we then assume the reader to be familiar with the basic concepts of first order logic (cf. [5]) as well as some

[2]Given a set of formulas S let $\Box S$ stand for $\wedge_{\alpha\in S}\Box\alpha$.

acquaintance with modal logics (cf. [2]). We shall be dealing with a standard first order language (including $\bot$, the "falsum" symbol) and its extension by the modal operator $\Box$. We denote derivability and entailment by $\vdash$ and $\models$, respectively (whether dealing with the pure first order language or the modal one). We use Th to denote the first order consequence operation, that is $Th(S) = \{\alpha \mid S \vdash \alpha\}$. Further definitions and conventions will be introduced when they occur for the first time.

A *default theory* (D, W) consists of a set of formulas W and a set of *default rules* D. A default rule is any expression of the form

$$\frac{\alpha : \beta}{\gamma}$$

where α, β and γ are formulas. α is called the *prerequisite*, β the *justification*, and γ the *consequent* of the default rule.

Default knowledge is incorporated into the framework as nonmonotonic inference rules by means of default rules. They sanction inferences that rely upon given as well as absent knowledge. Such inferences therefore could not be made in a classical framework. A default rule is applicable, if its prerequisite holds and its justification is consistent, that is adding its negation does not yield a contradiction.

Informally, an *extension* of the initial set of facts W is defined as the set of all formulas derivable from W using classical inference rules and all specified default rules:

Definition 2.1 *Let (D, W) be a default theory. For any set of formulas S let $\Gamma(S)$ be the smallest set of formulas S' such that*

1. $W \subseteq S'$,
2. $Th(S') = S'$,
3. *For any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $\neg\beta \notin S$ *then* $\gamma \in S'$.

A set of formulas E is a classical extension of (D, W) iff $\Gamma(E) = E$.

Classical default logic does not enjoy the desirable features known as "commitment to assumptions" and "cumulativity". The case of cumulativity is postponed to a later part. First, we concentrate on the notion of commitment.

Example 2.1 (non–commitment) *The default theory*

$$\left(\left\{\frac{: B}{C}, \frac{: \neg B}{D}\right\}, \emptyset\right)$$

has only one classical extension, $Th(\{C, D\})$. Both default rules have been applied, although they have contradictory justifications. Informally, there has been no commitment to the assumption B nor $\neg B$ [9].

Brewka [3] restored commitment (and cumulativity) in default logic by strengthening the applicability condition for default rules and making the reasons for believing something explicit. In order to keep track of the assumptions, he introduced *assertions*, that is formulas labelled with the set of justifications and consequents of the default rules that have been applied. In [4, 13], it is shown how to retain commitment (and cumulativity) while dropping the shift of formulas to assertions. For the formal presentation, we focus on constrained default logic as introduced in [13][3]. The approach taken by constrained default logic relies basically on dealing with two sets of formulas of the form (E, C). A default rule is applicable if its prerequisite holds in E and its justification is consistent wrt C (while in classical default logic the justification has to be consistent wrt E).

Since constrained default logic does not alter the language the notion of a default theory remains the same.

Definition 2.2 *Let (D, W) be a default theory. For any set of formulas T let $\Upsilon(T)$ be the pair of smallest sets of formulas (S', T') such that*

1. $W \subseteq S' \subseteq T'$,
2. $S' = Th(S')$ *and* $T' = Th(T')$,
3. *For any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta, \gamma \in T'$.

A pair of sets of formulas (E, C) is a constrained extension of (D, W) iff $\Upsilon(C) = (E, C)$.

Constrained extensions commit to their assumptions as constrained default logic employs a much stronger consistency check than classical default logic.

Example 2.2 (commitment) *The default theory*

$$\left(\left\{\frac{: B}{C}, \frac{: \neg B}{D}\right\}, \emptyset\right)$$

has two constrained extensions, $(Th(\{C\}), Th(\{C, B\}))$ *and* $(Th(\{D\}), Th(\{D, \neg B\}))$.

It turns out that the semantics proposed in [12] is adequate to characterize constrained extensions. A preference relation wrt to a set of default rules was defined which is similar to the one introduced in [6]. Simply, pairs of classes of first order interpretations like $(\Pi, \breve{\Pi})$ — called *focused models structures* — were considered instead of classes of first order interpretations. The idea is that, for a default rule $\frac{\alpha : \beta}{\gamma}$ to "apply" wrt a pair $(\Pi, \breve{\Pi})$, its prerequisite α must be valid in Π whereas the conjunction $\beta \wedge \gamma$ of its justification and consequent must be satisfiable in $\breve{\Pi}$. Taking into account all default rules, a maximal focused models structure $(\Pi, \breve{\Pi})$ is constructed which corresponds to a constrained extension, whose constraints correspond to the focused models $\breve{\Pi}$.

Definition 2.3 *Let $\delta = \frac{\alpha : \beta}{\gamma}$ and Π be a class of first order interpretations. The order $\succcurlyeq_\delta$ on $2^\Pi \times 2^\Pi$ is defined as follows. For all $(\Pi_1, \breve{\Pi}_1), (\Pi_2, \breve{\Pi}_2) \in 2^\Pi \times 2^\Pi$ we have*

$$(\Pi_1, \breve{\Pi}_1) \succcurlyeq_\delta (\Pi_2, \breve{\Pi}_2)$$

iff

1. $\forall \pi \in \Pi_2 . \pi \models \alpha$,
2. $\exists \pi \in \breve{\Pi}_2 . \pi \models \beta \wedge \gamma$,

[3] As constrained extensions are equivalent to [4]'s extensions of J–default logic, all results carry over to their J– and PJ–default logic.

3. $\Pi_1 = \{\pi \in \Pi_2 \mid \pi \models \gamma\}$,

4. $\breve{\Pi}_1 = \{\pi \in \breve{\Pi}_2 \mid \pi \models \beta \wedge \gamma\}$.

The induced order $\succcurlyeq_D$ is defined as the transitive closure of the union of all orders $\succcurlyeq_\delta$ such that $\delta \in D$. Clearly, constrained default logic is directly induced by this semantics. A constrained extension (E, C) is determined since E is formed by all formulas that are valid in the class Π of a $\succcurlyeq_D$–maximal focused model structure $(\Pi, \breve{\Pi})$ whereas the constraints C consist of all formulas valid in the class $\breve{\Pi}$ (the so–called focused models).

3 A modal characterization of constrained default logic

The focused models structures suggest that the ordering induced by a default rule has a modal nature with the corresponding semantical approach being based on Kripke structures. Intuitively, a pair $(\Pi, \breve{\Pi})$ is to be rendered as a class $\mathfrak{M}$ of Kripke structures such that Π is captured by the actual worlds in $\mathfrak{M}$ and $\breve{\Pi}$ by the accessible worlds in $\mathfrak{M}$. That is consider a non–modal formula α: it is valid in Π iff α is valid in $\mathfrak{M}$ and it is valid in $\breve{\Pi}$ iff $\Box\alpha$ is valid in $\mathfrak{M}$.

Correspondingly, the counterpart to a maximal focused models structure happens to be a class $\mathfrak{M}$ of Kripke structures such that

$$(\{\alpha \text{ non–modal } \mid \mathfrak{M} \models \alpha\}, \{\alpha \text{ non–modal } \mid \mathfrak{M} \models \Box\alpha\})$$

forms a constrained extension of the default theory under consideration. As always, the first set establishes the extension whereas the second set characterizes its constraints.

We follow the definitions (cf. Appendix A) in [2] of a Kripke structure (called K–model in the sequel). As in Appendix A, we use $\mathfrak{m}$ to denote K-models, $\mathfrak{M}$ to denote classes of K–models, and $\models$ to denote the modal entailment relation. We extend the modal entailment relation $\models$ to classes of K–models $\mathfrak{M}$ and write $\mathfrak{M} \models \alpha$ to mean that each element in $\mathfrak{M}$ (that is, a K–model) entails α.

In order to characterize constrained extensions semantically, we now define a family of strict partial orders on classes of K–models. Analogously to [6, 12], given a default rule δ, its application conditions and the result of applying it are captured by an order $\succ_\delta$ as follows.

Definition 3.1 *Let $\delta = \frac{\alpha:\beta}{\gamma}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be distinct classes of K–models. We define $\mathfrak{M} \succ_\delta \mathfrak{M}'$ iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box(\gamma \wedge \beta)\}$$

and

1. $\mathfrak{M}' \models \alpha$

2. $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$

Given a set of default rules D, the strict partial order $\succ_D$ amounts to the union of the strict partial orders $\succ_\delta$ as follows. $\mathfrak{M} \succ_D \mathfrak{M}'$ iff there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of some $D' \subseteq D$ such that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for some sequence $\langle \mathfrak{M}_i \rangle_{i \in I}$ of subclasses of $\mathfrak{M}'$ satisfying $\mathfrak{M}' = \mathfrak{M}_0$ and $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$.

Moreover, we define the class of K–models associated with W as $\mathfrak{M}_W = \{\mathfrak{m} \mid \mathfrak{m} \models \gamma \wedge \Box\gamma, \gamma \in W\}$ and refer to $\succ_D$–maximal classes of K–models above $\mathfrak{M}_W$ as the *preferred* classes of K–models wrt (D, W).

As for modal logic, observe that the K–models define the modal system K. It makes sense because the only property needed is distributivity for the modal operator $\Box$ to ensure that the constraints are deductively closed.

As a reminder, we give below the axiom schema (K) and inference rule (NEC) that must be added to a classical first order system in order to obtain K:

$$(K) \qquad \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$$

$$(NEC) \qquad \frac{\alpha}{\Box\alpha}$$

The choice of Condition 2 in Definition 3.1 is also worth discussing. At first glance, it seems more adequate to require $\mathfrak{M}' \not\models \neg\Box(\gamma \wedge \beta)$ since we want to add $\Box(\gamma \wedge \beta)$ and the condition $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$ does not a priori exclude $\mathfrak{M}' \models \neg\Box(\gamma \wedge \beta)$. We illustrate why this is needed by means of the next example.

Example 3.1 *Consider the default theory*

$$\left(\left\{\frac{:A}{A}\right\}, \{\neg A\}\right).$$

With $\mathfrak{M}_W \models \neg A$, we also have $\mathfrak{M}_W \models \Box\neg A$. But using the condition $\mathfrak{M}' \not\models \neg\Box A$ would not prevent the "application" of the only default rule.

Notice that Condition 2 in Definition 3.1 is equivalent to

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond(\gamma \wedge \beta). \qquad (1)$$

That is, the consistency condition in constrained default logic corresponds semantically to the requirement that there is a K–model which has some accessible world that satisfies $\gamma \wedge \beta$.

In the following examples, we show how preferred classes of K–models can characterize constrained extensions. At first, we give a detailed example that illustrates the main idea.

Example 3.2 *Consider the default theory*

$$\left(\left\{\frac{A:B}{C}\right\}, \{A\}\right)$$

that yields the constrained extension $(Th(\{A, C\}), Th(\{A, B, C\}))$.

In order to characterize this semantically, we start with

$$\mathfrak{M}_W \models A \wedge \Box A.$$

Since $\mathfrak{M}_W \models A$ it remains to ensure that $\mathfrak{M}_W \not\models \Box\neg(C \wedge B)$ — which is obvious. Hence, we obtain a class of K–models $\mathfrak{M}$ such that

$$\mathfrak{M} \models A \wedge \Box A \wedge C \wedge \Box(C \wedge B).$$

Thus, the actual worlds of our K–models satisfy the formulas of the extension $Th(\{A, C\})$ whereas the surrounding worlds additionally fulfill the constraints, that is $Th(\{A, B, C\})$.

In order to have a comprehensive example throughout the text, we extend the above commitment example as follows.

Example 3.3 (commitment) *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has three constrained extensions: $(Th(\{C\}), Th(\{B, C\})), (Th(\{D\}), Th(\{\neg B, D\}))$, and $(Th(\{E\}), Th(\{\neg D \wedge \neg C, E\}))$.

$\mathfrak{M}_W$ is the class of all K–models and clearly, we have $\mathfrak{M}_W \not\models \Box\neg(C \wedge B)$, $\mathfrak{M}_W \not\models \Box\neg(D \wedge \neg B)$, and $\mathfrak{M}_W \not\models \Box\neg(E \wedge \neg D \wedge \neg C)$. Therefore, all of the default rules are potentially "applicable".

Let us detail the case of the first constrained extension. We obtain a $\succ_{\left\{\frac{:B}{C}\right\}}$–greater class

$$\mathfrak{M} \models C \wedge \Box(C \wedge B).$$

In order to show that there is a $\succ_{\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}}$–greater class, we would have to show that

$$\mathfrak{M} \not\models \Box\neg(D \wedge \neg B).$$

But since $\Box(C \wedge B) \models \Box B$, we have $\mathfrak{M} \models \Box(B \vee \neg D)$ that prevents us from "applying" the second default rule. Analogously, we do not obtain a $\succ_{\left\{\frac{:B}{C}, \frac{:\neg D \wedge \neg C}{E}\right\}}$–greater class.

The last example shows how our construction copes with self–incoherent default theories.

Example 3.4 *Consider the default theory*

$$\left(\left\{\frac{:\neg A}{A}\right\}, \emptyset\right)$$

whose only constrained extension is $(Th(\emptyset), Th(\emptyset))$. $\mathfrak{M}_W$ is the class of all K–models. But since $\mathfrak{M}_W \models \Box\neg(A \wedge \neg A)$ condition 2 of Definition 3.1 is falsified and, therefore, $\mathfrak{M}_W$ is the only preferred class.

An interesting point concerning Definition 3.1 is that finding a non–empty $\mathfrak{M} \subseteq \mathfrak{M}'$ such that $\mathfrak{M} \models \Box(\gamma \wedge \beta)$ whenever $\mathfrak{M}' \not\models \Box\neg(\gamma \wedge \beta)$ might appear to be impossible, hence the next proposition.

Proposition 3.1 *The empty class of K–models is never preferred wrt (D, W) whenever W is consistent.*

As a corollary we obtain that the existence of constrained extensions is guaranteed.

The notion of a preferred class of *K*–models illustrated above is put into a precise correspondence with constrained extensions in the following theorem.

Theorem 3.2 (Correctness & Completeness) *Let* (D,W) *be a default theory. Let* $\mathfrak{M}$ *be a class of K–models and* E, C *deductively closed sets of formulas such that*

$$\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}.$$

Then,

(E, C) *is a constrained extension of* (D,W) *iff* $\mathfrak{M}$ *is a* $\succ_D$*–maximal class above* $\mathfrak{M}_W$.

Then our possible worlds approach amounts to the focused model semantics [12] presented above: the first order interpretations associated with the accessible worlds take over the role of the focused models.

Corollary 3.3 *Let* (D,W) *be a default theory,* $(\Pi, \breve{\Pi})$ *a* $\succcurlyeq_D$*–maximal focused models structure above* $(\{\pi \mid \pi \models W\}, \{\pi \mid \pi \models W\})$ *and* $\mathfrak{M}$ *a preferred class of K–models wrt* (D,W)*. Then, for* α, β *non–modal*

$$\Pi \models \alpha \text{ iff } \mathfrak{M} \models \alpha \text{ and } \breve{\Pi} \models \beta \text{ iff } \mathfrak{M} \models \Box\beta.$$

In the face of the above corollary, observe that a preferred class of K–models contains "more" different actual worlds than accessible ones. The reason is that focused models structures $(\Pi, \breve{\Pi})$ have the inclusion property $\breve{\Pi} \subseteq \Pi$.

How does our semantics reflect the notion of commitment? As already pointed out, the intuition behind our construction is very natural and easy to understand: The actual world of a K–model captures what we believe and the surrounding worlds capture what commitments we have allowed to adopt our beliefs. Therefore, our semantics reflects the notion of commitment through modal necessity: the commitments correspond to formulas whose necessity holds.

Since it is proved in [12] that the focused model semantics captures cumulative default logic [3], Theorem 3.2 and Corollary 3.3 establish a possible worlds semantics for cumulative default logic as is shown next. First, recall that an assertion is a pair $\langle \alpha, \{\alpha_1, \ldots, \alpha_m\}\rangle$, where $\alpha, \alpha_1, \ldots, \alpha_m$ are formulas. Applied to an assertion ξ, $\mathit{Form}(\xi)$ gives the formula whereas $\mathit{Supp}(\xi)$ gives its label (called the support). Also, Brewka had to extend the first order consequence relation to assertions: for a set of assertions $\mathcal{S}$, $\widehat{Th}(\mathcal{S})$ is the smallest set of assertions such that $\mathcal{S} \subseteq \widehat{Th}(\mathcal{S})$ and if $\xi_1, \ldots, \xi_n \in \widehat{Th}(\mathcal{S})$ and $\mathit{Form}(\xi_1), \ldots, \mathit{Form}(\xi_n) \vdash \gamma$, then $\langle \gamma, \mathit{Supp}(\xi_1) \cup \ldots \cup \mathit{Supp}(\xi_n)\rangle \in \widehat{Th}(\mathcal{S})$. An assertional default theory is a pair $(D, \mathcal{W})$, where D is a set of default rules and $\mathcal{W}$ is a set of assertions.

Definition 3.2 *Let* $(D, \mathcal{W})$ *be an assertional default theory. For any set of assertions* $\mathcal{S}$ *let* $\Omega(\mathcal{S})$ *be the smallest set of assertions* $\mathcal{S}'$ *such that*

1. $\mathcal{W} \subseteq \mathcal{S}'$,
2. $\widehat{Th}(\mathcal{S}') = \mathcal{S}'$,
3. *For any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\langle \alpha, \mathit{Supp}(\alpha)\rangle \in \mathcal{S}'$ *and* $\mathit{Form}(\mathcal{S}) \cup \mathit{Supp}(\mathcal{S}) \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\langle \gamma, \mathit{Supp}(\alpha) \cup \{\beta\} \cup \{\gamma\}\rangle \in \mathcal{S}'$.

A set of assertions $\mathcal{E}$ *is an assertional extension for* $(D, \mathcal{W})$ *iff* $\Omega(\mathcal{E}) = \mathcal{E}$.

Now, [15] shows that if (E,C) is a constrained extension of (D,W) then there is an assertional extension $\mathcal{E}$ of $(D,\{\langle\alpha,\emptyset\rangle \mid \alpha \in W\})$ such that $E = \mathit{Form}(\mathcal{E})$ and $C = \mathit{Th}(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E}))$ and conversely, if $\mathcal{E}$ is an assertional extension of $(D,\{\langle\alpha,\emptyset\rangle \mid \alpha \in W\})$ then $(\mathit{Form}(\mathcal{E}), \mathit{Th}(\mathit{Form}(\mathcal{E}) \cup \mathit{Supp}(\mathcal{E})))$ is a constrained extension of (D,W). Consequently, our possible worlds semantics also characterizes cumulative default logic:

Theorem 3.4 *Let $(D,\mathcal{W})$ be an assertional default theory. Let $\mathfrak{M}_{\mathcal{W}}$ be the class of all K–models of $\{v \wedge \Box\eta \mid v \in \mathit{Form}(\mathcal{W}), \eta \in \mathit{Supp}(\mathcal{W})\}$. Then, there exists a set of assertions $\mathcal{E}$ which is an assertional extension of $(D,\mathcal{W})$ such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models \mathit{Form}(\mathcal{E}) \wedge \Box\mathit{Supp}(\mathcal{E})\}$ iff $\mathfrak{M}$ is a preferred class of K–models above $\mathfrak{M}_{\mathcal{W}}$.*

In the context of cumulative default logic, naturally the question arises how the notion of cumulativity can be characterized by our possible worlds semantics. Intuitively, cumulativity stipulates that the addition of a theorem to the premises does not alter the set of conclusions. Apart from its theoretical interest, cumulativity is of great practical relevance. This is, because a cumulative theory operator allows for the use of lemmata needed for reducing computational efforts.

First, let us look at the failure of cumulativity in classical default logic:

Example 3.5 (non–cumulativity[4]) *The default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right)$$

has one classical extension, $\mathit{Th}(\{A\})$. This extension inevitably contains $A \vee B$.

Adding this nonmonotonic theorem to the premises yields the default theory

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right)$$

that has now two extensions: $\mathit{Th}(\{A\})$ and $\mathit{Th}(\{\neg A, B\})$. Regardless of whether or not we employ a skeptical or a credulous notion of theory formation — in both cases we are changing the set of conclusions.

How assertional default theories restore cumulativity is shown below.

Example 3.6 (cumulativity) *The assertional default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \emptyset\right)$$

has also one extension which contains the assertions $\langle A, \{A\}\rangle$ and $\langle A \vee B, \{A\}\rangle$.

Adding the assertion $\langle A \vee B, \{A\}\rangle$ to the premises yields the assertional default theory

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{\langle A \vee B, \{A\}\rangle\}\right)$$

that has still the same assertional extension and no other.

[4] This example is originally due to David Makinson [8].

In the case of constrained default logic, cumulativity was preserved in [13, 15] by means of *lemma default rules* which are prerequisite–free default rules whose justification consists of the assumptions underlying the actual lemma which is given in the consequent. The major difference between the addition of assertions to the facts and the addition of lemma default rules to the set of default rules is that once we have added an assertion to the premises it is not "retractable" any more whenever an inconsistency arises. Thus, the addition of assertions is stronger than that of lemma default rules. Adding an assertion to the premises eliminates all extensions inconsistent with the asserted formula or even its support. On the contrary, lemma default rules preserve all extensions and, therefore, their purpose is more an abbreviation of default proofs.

How can those differences be envisioned by our semantics? Assume we have a constrained extension (E, C) and the corresponding assertional extension $\mathcal{E}$. Whenever we have a theorem $\ell \in E$ and a minimal set of default rules $D_\ell \subseteq GD_D^{(E,C)}$ $\left(= \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}\right)$ which has been used to derive ℓ, there exists as well an assertion[5] $\xi_\ell \in \mathcal{E}$, where

$$\xi_\ell = \langle \ell, \textstyle\bigcup_{\delta \in D_\ell} \{Justif(\delta), Conseq(\delta)\} \rangle.$$

For a complement, the corresponding lemma default rule is

$$\delta_\ell = \frac{:\ \bigwedge_{\delta \in D_\ell} Justif(\delta) \wedge Conseq(\delta)}{\ell}.$$

Take a default theory (D, W) and its assertional counterpart $(D, \mathcal{W})$, where $\mathcal{W} = \{\langle \alpha, \emptyset \rangle \mid \alpha \in W\}$. Looking at cumulative default logic, we enforce (by adding the assertion ξ_ℓ to $\mathcal{W}$) that all preferred classes of K–models entail the formula

$$\ell \wedge \Box\Big(\ell \wedge \textstyle\bigwedge_{\delta \in D_\ell} Justif(\delta) \wedge Conseq(\delta)\Big). \tag{2}$$

In constrained default logic the addition of the lemma default rule δ_ℓ to the set of default rules only demands the expression (2) to be entailed by those preferred classes of K–models, to which generation the lemma default rule has contributed. That is, we enforce the entailment of (2) only for all preferred classes of K–models $\mathfrak{M}$ for which $\mathfrak{M} \succ_{GD_D^{(E,C)} \cup \{\delta_\ell\}} \mathfrak{M}_W$ holds.

4 A modal characterization of classical default logic

The possible worlds approach to default logic presented above turns out to be very general. The first evidence of this arises from the fact that the above semantical characterization carries over easily to classical default logic. Indeed, the analogue to Definition 3.1 can be defined as follows.[6]

Definition 4.1 *Let $\delta = \frac{\alpha:\beta}{\gamma}$. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be distinct classes of K–models. We define $\mathfrak{M} >_\delta \mathfrak{M}'$ iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta\}$$

and

[5]Applied to a default rule δ, *Justif*(δ) yields its justification whereas *Conseq*(δ) returns its consequent.

[6]Given a set of formulas S let $\Diamond S$ stand for $\wedge_{\alpha \in S} \Diamond\alpha$.

1. $\mathfrak{M}' \models \alpha$

2. $\mathfrak{M}' \not\models \Box\neg\beta$

The order $>_D$ is defined analogously to that in Section 3.

Even though classical default logic does not employ explicit constraints, there is a natural counterpart given by the justifications of the generating default rules over a set of formulas E:

$$C_E = \left\{\beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}^7$$

We obtain a semantical characterization that yields a one–to–one correspondence between consistent extensions and non–empty $>_D$–preferred classes of K–models (an inconsistent extension trivially corresponds to $\mathfrak{M}_W$ being preferred while being empty).

Theorem 4.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K–models and E be a deductively closed set of formulas such that*

$$\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}.$$

Then,

E is a consistent classical extension of (D, W) iff $\mathfrak{M}$ is a $>_D$–maximal non–empty class above $\mathfrak{M}_W$.

Comparing Definition 4.1 with Definition 3.1, we observe two basic differences, reflecting the fact that constrained default logic employs a stronger consistency check than classical default logic. For one thing, the second condition on $\mathfrak{M}'$ is weakened such that only β instead of $\gamma \wedge \beta$ is required to be satisfied by some accessible world of some K–model in $\mathfrak{M}'$. This becomes perfectly clear by comparing the following formulation of Condition 2 in Definition 4.1

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond\beta \tag{3}$$

with the one given in (1). For another thing, Definition 4.1 requires $\Diamond\beta$ to be valid in $\mathfrak{M}$ whereas Definition 3.1 requires $\Box\beta$ to be valid in $\mathfrak{M}$. Stated otherwise, the possible worlds semantics for classical extensions requires only *some* accessible world satisfying the justification β whereas the semantics for constrained default logic requires *all* accessible worlds to satisfy β.

The conclusion is that from the perspective of commitment, constrained extensions adopt their beliefs by committing to all consequents and all justifications of applied default rules whereas classical default logic commits to consequents taken together but only to justifications taken separately.

Example 4.1 (non–commitment) *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has only one classical extension: $Th(\{C, D\})$.

$\mathfrak{M}_W$ is the class of all K–models and clearly, we have $\mathfrak{M}_W \not\models \Box\neg B$, $\mathfrak{M}_W \not\models \Box\neg(\neg B)$, and $\mathfrak{M}_W \not\models \Box\neg(\neg D \wedge \neg C)$. That is, all of the default rules are potentially "applicable".

[7] Observe that the membership qualifying property is exactly the third condition in the definition of a classical extension.

From $\mathfrak{M}_W$ we can construct a class of K–models $\mathfrak{M}$ such that $\mathfrak{M} >_{\{\frac{:B}{C}\}} \mathfrak{M}_W$ and

$$\mathfrak{M} \models C \wedge \Box C \wedge \Diamond B.$$

Accordingly, we can also construct a class of K–models $\mathfrak{M}'$ such that $\mathfrak{M}' >_{\{\frac{:B}{C},\frac{:\neg B}{D}\}} \mathfrak{M}_W$ and

$$\mathfrak{M}' \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond \neg B.$$

But it is impossible to obtain a class $\mathfrak{M}''$ such that $\mathfrak{M}'' >_{\{\frac{:B}{C},\frac{:\neg B}{D},\frac{:\neg D \wedge \neg C}{E}\}} \mathfrak{M}_W$ since

$$\mathfrak{M}' \models \Box\neg(\neg D \wedge \neg C).$$

From $\mathfrak{M}_W$, selecting first the third default rule leads to a $>_{\{\frac{:\neg D \wedge \neg C}{E}\}}$–greater class

$$\dot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C).$$

From $\dot{\mathfrak{M}}$ we can construct a class of K–models $\ddot{\mathfrak{M}}$ such that $\ddot{\mathfrak{M}} >_{\{\frac{:\neg D \wedge \neg C}{E},\frac{:B}{C}\}} \mathfrak{M}_W$ and

$$\ddot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C) \wedge C \wedge \Box C \wedge \Diamond B.$$

So, $\ddot{\mathfrak{M}}$ is the empty set of K–models because $\Diamond(\neg D \wedge \neg C) \models \Diamond\neg C$ and $\Box C \wedge \Diamond \neg C \models \bot$.

In contrast to Proposition 3.1, the possible worlds semantics for classical default logic admits the empty set of K–models above some non–empty $\mathfrak{M}_W$. This is the case whenever a default rule is applied whose consequent contradicts the justification of some default rule which is itself applied. In particular, this reflects the failure of semi–monotonicity in classical default logic whereas constrained default logic enjoys semi–monotonicity (A default logic is said to be semi–monotonic iff enlarging the set of default rules of a default theory can only preserve or enlarge the existing extensions.).

In addition, characterizing extensions in default logic strictly by non–empty $>_D$–maximal elements above $\mathfrak{M}_W$ avoids post–filtering mechanisms such as the *stability* criterion introduced in [6]. Whenever an incoherent default theory arises, our characterization yields an empty set of K–models. The purpose of the stability criterion is to ensure the satisfiability of each justification for a given set of default rules. In other words, the stability criterion guarantees the "continued consistency" of the justifications of the applying default rules. In contrary, we ensure the continued consistency of justifications by requiring the validity of $\Diamond\beta$ in all classes of K–models preferred by a default rule $\frac{\alpha:\beta}{\gamma}$. As a consequence, our characterization yields an empty set of K–models, whenever an incoherent default theory arises.

Example 4.2 *The incoherent default theory*

$$\left(\left\{\frac{:\neg A}{A}\right\},\emptyset\right)$$

of Example 3.4 has no classical extension. $\mathfrak{M}_W$ is the class of all K–models. Clearly, $\mathfrak{M}_W \not\models \Box A$ but the resulting class

$$\{\mathfrak{m} \in \mathfrak{M}_W \mid \mathfrak{m} \models A \wedge \Box A \wedge \Diamond\neg A\}$$

is obviously empty.

Finally, let us examine the failure of cumulativity in classical default logic. In Section 3, we have characterized by means of a modal expression the solutions preserving cumulativity. Taking the expression given in (2) but dropping the requirement of joint consistency yields the following modal expression for classical default logic:

$$\ell \wedge \Box\ell \wedge \Diamond \mathit{Justif}(D_\ell) \tag{4}$$

where ℓ is contained in a classical extension E of a default theory (D, W) and $D_\ell \subseteq GD_D^{(E,E)}$ is a set of default rules used to derive ℓ.

Let us look at the canonical cumulativity example.

Example 4.3 (non–cumulativity) *Consider the default theory*

$$\left(\left\{\frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A}\right\}, \{A \vee B\}\right)$$

obtained from Example 3.5 after adding $A \vee B$ *(so that we are considering* $\ell = A \vee B$*). In addition to the extension* $Th(\{A\})$*, we have obtained a second one:* $Th(\{\neg A, B\})$*. The semantical characterization of the classical extension* $Th(\{\neg A, B\})$ *yields a class of K–models* $\mathfrak{M}$ *that is* $>_{\left\{\frac{A\vee B:\neg A}{\neg A}\right\}}$*–greater than* $\mathfrak{M}_W$ *such that*

$$\mathfrak{M} \models (A \vee B) \wedge \Box(A \vee B) \wedge \neg A \wedge \Box\neg A$$

Since

$$D_\ell = \left\{\frac{:A}{A}\right\},$$

$\mathfrak{M}$ *obviously does not entail our above modal expression (4):*

$$\mathfrak{M} \not\models (A \vee B) \wedge \Box(A \vee B) \wedge \Diamond A.$$

The entailment of the expression (4) in all preferred classes of K–models $\mathfrak{M}$ such that $\mathfrak{M} >_{GD_D^{(E,E)} \cup \{\zeta_\ell\}} \mathfrak{M}_W$ can be enforced through the corresponding lemma default rule for classical default logic (cf. [15]): Given ℓ and $D_\ell = \{\delta_1, \ldots, \delta_n\} \subseteq GD_D^{(E,E)}$ as described above, we obtain

$$\zeta_\ell = \frac{:\mathit{Justif}(\delta_1), \ldots, \mathit{Justif}(\delta_n)}{\ell}.$$

5 A modal characterization of justified default logic

Further evidence for the generality of our approach is that it can easily capture a variant of default logic due to [7], which we refer to as justified default logic. Indeed, the analogue to Definition 3.1 and 4.1 can be defined as follows.

Definition 5.1 *Let* $\delta = \frac{\alpha:\beta}{\gamma}$*. Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be distinct classes of K–models. We define* $\mathfrak{M} \rhd_\delta \mathfrak{M}'$ *iff*

$$\mathfrak{M} = \{\mathfrak{m} \in \mathfrak{M}' \mid \mathfrak{m} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta\}$$

and

1. $\mathfrak{M}' \models \alpha$

2. $\mathfrak{M}' \not\models \Box\neg\beta \vee \Diamond\neg\gamma$

The order $\rhd_D$ is defined analogously to that in Section 3.

Compared to the order $>_\delta$ given for classical default logic, the only difference is that the condition $\mathfrak{M}' \not\models \Box\neg\beta$ has become $\mathfrak{M}' \not\models \Box\neg\beta \vee \Diamond\neg\gamma$, that is, $\mathfrak{M}' \not\models \neg(\Box\gamma \wedge \Diamond\beta)$. Again, this becomes apparent by regarding Condition 2 in Definition 5.1, that is

$$\exists \mathfrak{m} \in \mathfrak{M}'.\ \mathfrak{m} \models \Diamond\beta \wedge \Box\gamma. \tag{5}$$

In classical default logic, there has to be a K–model which has some accessible world satisfying β (see (3) above). In justified default logic, however, all accessible worlds of such a K–model additionally have to satisfy γ.

Indeed, the definition reveals the fact that the same constraints implicitly used in classical default logic (in the form of C_E) are explicitly attached to justified extensions[8] (in the form of J, see below) and, moreover, considered when checking consistency. That is, semantically classical and justified default logic account for the justifications of the applied default rules in form of the modal propositions $\Diamond\beta$. However, in classical default logic they are discarded when checking consistency.

Formally, a justified extension is defined as follows.

Definition 5.2 *Let (D, W) be a default theory. For any pair of sets of formulas (S, T) let $\Psi(S, T)$ be the pair of smallest sets of formulas S', T' such that*

1. *$W \subseteq S'$,*

2. *$Th(S') = S'$,*

3. *For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in S'$ and $\forall \eta \in T \cup \{\beta\}.\ S \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ then $\gamma \in S'$ and $\beta \in T'$.*

A set of formulas E is a justified extension of (D, W) wrt to a set of formulas J iff $\Psi(E, J) = (E, J)$.

Łukaszewicz has shown in [7] that justified default logic guarantees the existence of extensions. Semantically, it is obvious that requiring $\mathfrak{M}' \not\models \neg(\Box\gamma \wedge \Diamond\beta)$ and adding those K–models entailing $\Box\gamma \wedge \Diamond\beta$ makes it impossible to obtain the empty set of K–models (in fact, the analogue to Proposition 3.1 holds). Łukaszewicz has also shown that his variant enjoys semi–monotonicity. In fact, "applying" a default rule $\frac{\alpha:\beta}{\gamma}$ enforces all $\rhd_D$–greater classes of K–models $\dot{\mathfrak{M}}$ to entail $\Box\gamma \wedge \Diamond\beta$. Therefore, a later "application" of a default rule $\frac{\alpha':\beta'}{\gamma'}$ whose consequent γ' contradicts β (eg. $\gamma' = \neg\beta$) is prohibited since its "application" requires $\dot{\mathfrak{M}} \not\models \Box\neg\beta' \vee \Diamond\neg\gamma'$.

Analogously to classical default logic, Definition 5.1 only requires $\Diamond\beta$ to be valid in $\mathfrak{M}$ which is not enough for justified default logic to commit to its assumptions.

Example 5.1 (non–commitment) *The default theory*

$$\left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}, \frac{:\neg D \wedge \neg C}{E}\right\}, \emptyset\right)$$

has two justified extensions, $Th(\{C, D\})$ wrt $\{B, \neg B\}$ and $Th(\{E\})$ wrt $\{\neg D \wedge \neg C\}$.

[8] Originally, Łukaszewicz called his extensions *modified* extensions.

The first one is obtained analogously to that in Example 4.1. That is, we obtain a preferred class

$$\mathfrak{M}' \models C \wedge \Box C \wedge \Diamond B \wedge D \wedge \Box D \wedge \Diamond \neg B.$$

Also, selecting first the third default rule leads to a class $\dot{\mathfrak{M}} \rhd_{\left\{\frac{:\neg D \wedge \neg C}{E}\right\}} \mathfrak{M}_W$ *such that*

$$\dot{\mathfrak{M}} \models E \wedge \Box E \wedge \Diamond(\neg D \wedge \neg C).$$

Since we have $\dot{\mathfrak{M}} \models \Diamond \neg C$ *and* $\dot{\mathfrak{M}} \models \Diamond \neg D$ *none of the other default rules is "applicable". Therefore,* $\dot{\mathfrak{M}}$ *is a (non–empty) preferred class.*

Similarly to the case of classical default logic, there is a natural account of constraints attached to a set of formulas E justified by J: the justifications of the generating default rules over E, as determined by J, which are simply

$$C_{(E,J)} = \left\{\beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D,\; \alpha \in E, \forall \eta \in J \cup \{\beta\}.\; E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot\right\}^9$$

Then, correctness and completeness hold as in the former sections.

Theorem 5.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K–models, E a deductively closed set of formulas, and J a set of formulas such that $J = C_{(E,J)}$ and*

$$\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}.$$

Then,

E is a justified extension of (D, W) wrt J iff $\mathfrak{M}$ is a $\rhd_D$–maximal class above $\mathfrak{M}_W$.

The equality $J = C_{(E,J)}$ simply states that the implicit constraints $C_{(E,J)}$ and the explicit constraints J coincide.

Notably, our possible worlds semantics is the first semantical characterization of justified default logic which is purely model–theoretic. In [7], Łukaszewicz had to characterize justified extension by means of pairs (Π, J), where Π is a class of first order interpretations and J is a set of formulas. The reason why Łukaszewicz did so is that justified default logic allows for inconsistent sets of individually consistent constraints (so that the focused models semantics cannot be adapted there).

Finally, a remark concerning Definition 3.1 and 5.1 is in order. Let us compare the respective consistency condition, that is (1) and (5). We observe that the condition in constrained default logic requires that there is a K–model which has some accessible world satisfying $\gamma \wedge \beta$. In contrast, we are faced with a stronger requirement in justified default logic: there has to be a K–model whose accessible worlds all satisfy γ and some accessible world satisfies β. At first glance, this seems to be unintuitive since constrained default logic has a stronger consistency condition than justified default logic (compare Definition 2.2 and 5.2). However, consistency or satisfiability are always relative to a given set of formulas or class of models, respectively. In fact, we consider a much more restricted class of K–models $\mathfrak{M}'$ in (1), that is constrained default logic, than in (5), that is justified default logic. Given a set of default rules D' such that $\mathfrak{M}' \succ_{D'} \mathfrak{M}_W$ and $\mathfrak{M}' \rhd_{D'} \mathfrak{M}_W$, we have $\mathfrak{M}' \models W \wedge$

[9] Observe that the membership qualifying property is exactly the third condition in the definition of a justified extension.

$Conseq(D') \wedge \Box(W \wedge Conseq(D') \wedge Justif(D'))$ in constrained default logic, whereas we encounter a less restricted class of K–models in justified default logic, that is $\mathfrak{M}' \models W \wedge Conseq(D') \wedge \Box(W \wedge Conseq(D')) \wedge \Diamond Justif(D')$. As a consequence, we have to employ a stronger satisfiability condition in justified default logic which is given in (5).

6 Conclusion

We have presented a uniform semantical framework for various default logics in terms of Kripke structures. That is, we have first introduced a possible worlds semantics for constrained default logic and we have proved that it also captures cumulative default logic. Then, we have provided a simple modification for that possible worlds semantics in order to characterize Reiter's classical default logic and in turn Łukaszewicz' justified default logic.

Moreover, the approach remedies several difficulties encountered in former proposals aiming at individual default logics. First, the approach avoids stability conditions as required in [6] and [7]. Second, the possible worlds semantics avoids two–folded semantical structures such as focused models structures [12] or frames as introduced in [7]. Thirdly, the approach provides the first semantical characterization of Lukaszewicz' justified default logic which is purely model–theoretic.

By adopting the perspective of "commitment to assumptions" we have not only gained a clear criterion on that notion itself but also provided a very natural modal interpretation by which existing default logics can be compared in a simple but deeply meaningful manner. In particular, the semantics has revealed that all of the various default logics employ constraints but differ in the extent to which the constraints are considered when checking consistency. Notably, in terms of modalities we have to switch from $\Diamond$ to $\Box$ whenever we want to preserve "commitment to assumptions".

A Modal logic

We follow the definitions in [2] of a Kripke structure (called K–model in the sequel) as a quadruple $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, where Ω is a non–empty set (also called a set of worlds), $\omega_0 \in \Omega$ a distinguished world, $\mathcal{R}$ a binary relation on Ω (also called the accessibility relation) and $\mathcal{I}$ is a function that defines a first order interpretation $\mathcal{I}_\omega$ for each $\omega \in \Omega$. As usual, a K–model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$ is such that the domain of $\mathcal{I}_\omega$ is a subset of the domain of $\mathcal{I}_{\omega'}$ whenever $(\omega, \omega') \in \mathcal{R}$.

Formulas in K–models are interpreted using a language enriched in the following way: in a K–model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, for each $\omega \in \Omega$, the first order interpretation $\mathcal{I}_\omega$ is extended so that for each $e \in D_\omega$ (the domain of $\mathcal{I}_\omega$), a constant $\overline{e}$ is introduced, letting $\mathcal{I}_\omega(\overline{e}) = e$. In every world ω, each term is mapped into an element of D_ω as follows: $\mathcal{I}_\omega(f(t_1, \ldots, t_n)) = (\mathcal{I}_\omega(f))\,(\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n))\,,\ n \geq 0$.

Given a K–model $\mathfrak{m} = \langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, the modal entailment relation $\omega \models \alpha$ (in $\mathfrak{m}$) is defined by recursion on the structure of α:

$$
\begin{array}{lll}
\omega \models P(t_1, \ldots, t_n) & \text{iff} & (\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n)) \in \mathcal{I}_\omega(P) \\
\omega \models \neg\alpha & \text{iff} & \omega \not\models \alpha \\
\omega \models \alpha \vee \beta & \text{iff} & \omega \models \alpha \text{ or } \omega \models \beta \\
\omega \models \forall x\ \alpha[x] & \text{iff} & \omega \models \alpha[\overline{e}] \text{ for all } e \in D_\omega \\
\omega \models \Box\alpha & \text{iff} & \omega' \models \alpha \text{ whenever } (\omega, \omega') \in \mathcal{R}
\end{array}
$$

We write $\mathfrak{m} \models \alpha$ if $\omega_0 \models \alpha$ (in $\mathfrak{m}$). This means that $\mathfrak{m}$ is a model of α. We denote classes of K–models by $\mathfrak{M}$. We extend the modal entailment relation $\models$ to classes of K–models $\mathfrak{M}$ and write $\mathfrak{M} \models \alpha$ to mean that each element in $\mathfrak{M}$ (that is, a K–model) entails α.

B Proofs of Theorems

Proposition 3.1 *The empty class of K–models is never preferred wrt (D, W) whenever W is consistent.*

Proof 3.1 Assume that $\mathfrak{M}_\emptyset \succ_D \mathfrak{M}_W$. By definition, there then exists a subset $D' = \{\delta_0, \delta_1, \ldots\}$ of D such that $\mathfrak{M}_\emptyset = \{\mathfrak{m} \mid \mathfrak{m} \models W \wedge \Box W \wedge \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$ for all $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}\}$. By compactness, there is a finite set $\{W \wedge \Box W \wedge \gamma_0 \wedge \Box(\gamma_0 \wedge \beta_0) \wedge \ldots \wedge \gamma_k \wedge \Box(\gamma_k \wedge \beta_k)\}$ which is inconsistent. By Corollary C.3, $\{W \wedge \gamma_0 \wedge \ldots \wedge \gamma_k\}$ is inconsistent. That is, $W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1} \models \neg\gamma_k$. By modal logic K, $\Box(W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1}) \models \Box\neg\gamma_k$ and $\Box(W \wedge \gamma_0 \wedge \ldots \wedge \gamma_{k-1}) \models \Box\neg(\gamma_k \wedge \beta_k)$. Then, it cannot be the case that $\mathfrak{M}_{k+1} \succ_{\delta_k} \mathfrak{M}_k$ because $\mathfrak{M}_j = \{\mathfrak{m} \mid \mathfrak{m} \models W \wedge \Box W \wedge \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$ for all $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$ such that $i < j\}$. Therefore, there is no such k and D' is empty. So, $\mathfrak{M}_\emptyset = \mathfrak{M}_W$ and, by Corollary C.3, W is inconsistent, a contradiction. ■

In the sequel, we frequently employ the following definition.

Definition B.1 *Let (D, W) be a default theory. Given a possibly infinite sequence of default rules $\Delta = \langle \delta_0, \delta_1, \delta_2, \ldots \rangle$ in D, also denoted $\langle \delta_i \rangle_{i \in I}$ where I is the index set for Δ, we define a sequence of classes of K–models $\langle \mathfrak{M}_i \rangle_{i \in I}$ as follows:*

$$
\begin{array}{lll}
\mathfrak{M}_0 & = & \mathfrak{M}_W \\
\mathfrak{M}_{i+1} & = & \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \odot\beta_i\}, \quad \textit{where } \delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}.
\end{array}
$$

In constrained default logic, $\odot$ is $\Box$. In classical and justified default logic, $\odot$ is $\Diamond$.

We will be more liberal here about the orders $\succ_\delta, >_\delta, \rhd_\delta$ by relaxing the condition that $\mathfrak{M} \succ_\delta \mathfrak{M}'$ (similarly $\mathfrak{M} >_\delta \mathfrak{M}'$ and $\mathfrak{M} \rhd_\delta \mathfrak{M}'$) holds only if $\mathfrak{M}$ and $\mathfrak{M}'$ are distinct. That is, there will be cases where $\mathfrak{M} \succ_\delta \mathfrak{M}$ (similarly $\mathfrak{M} >_\delta \mathfrak{M}$ and $\mathfrak{M} \rhd_\delta \mathfrak{M}$) be true. Clearly, this does not affect the issues under consideration.

B.1 Proof of correctness and completeness for constrained default logic

Theorem 3.2 (Correctness & Completeness) *Let (D,W) be a default theory. Let $\mathfrak{M}$ be a class of K–models and E, C deductively closed sets of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$. Then,*

(E,C) is a constrained extension of (D,W) iff $\mathfrak{M}$ is a $\succ_D$–maximal class above $\mathfrak{M}_W$.

The unsatisfiable case is easily dealt with, so that we prove below the theorem for E and C being satisfiable.

Proof 3.2 (Correctness) Assume (E,C) is a consistent constrained extension of (D,W). The set of generating default rules for (E,C) wrt D is defined as $GD_D^{(E,C)} = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. As has been shown in [14], then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,C)}$ such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{6}$$

Let $\langle \mathfrak{M}_i \rangle_{i \in I}$ be a sequence of classes of K–models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition B.1. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i \in I} \mathfrak{M}_i$ and is $\succ_D$–maximal above $\mathfrak{M}_W$.

Since (E,C) is a constrained extension, it has been proven in [14] that

$$\begin{aligned} E &= \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big), \\ C &= \mathit{Th}\Big(W \cup \mathit{Justif}\Big(GD_D^{(E,C)}\Big) \cup \mathit{Conseq}\Big(GD_D^{(E,C)}\Big)\Big). \end{aligned}$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$ we have obviously that $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$.

Firstly, let us show that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (6), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg(\gamma_k \wedge \beta_k)$ for $\delta_k = \frac{\alpha_k:\beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\}) \cup \mathit{Justif}(\{\delta_0, \ldots, \delta_{k-1}\})$ by C^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box C^k\}$. Since E is satisfiable, so is E^k. By applying Corollary C.7 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg(\gamma_k \wedge \beta_k)$ we obtain that $C^k \models \neg(\gamma_k \wedge \beta_k)$. That is, $C^k \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$. By monotonicity, $C \cup \{\gamma_k\} \cup \{\beta_k\} \vdash \bot$, contradictory to the fact that $\delta_k \in GD_D^{(E,C)}$.

Therefore, $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i\in I} \mathfrak{M}_i \succ_{GD_D^{(E,C)}} \mathfrak{M}_W$. That is, $\mathfrak{M} \succ_D \mathfrak{M}_W$.

Secondly, assume $\mathfrak{M}$ is not $\succ_D$–maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^{(E,C)}$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg(\gamma \wedge \beta)$. First, applying Corollary C.3 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, since $\mathfrak{M} \models E \wedge \Box C$, we get by monotonicity $\Box C \not\models \Box\neg(\gamma \wedge \beta)$, yielding $C \not\models \neg(\gamma \wedge \beta)$ by modal logic K. Of course, $E \models \alpha$ and $C \not\models \neg(\gamma \wedge \beta)$ implies $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,C)}$, a contradiction. ■

Proof 3.2 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\}$ is a $\succ_D$–maximal class of K–models above $\mathfrak{M}_W$.

Let us first establish a useful characterization of C, that is $\hat{C} = \{\eta \text{ non–modal} \mid \mathfrak{M} \models \Box\eta\}$. Obviously, $C \subseteq \hat{C}$. So, $\hat{C} \models C$. In order to prove the converse, notice that $\mathfrak{M} \models \Box\hat{C}$. Since E is satisfiable, $C \models \hat{C}$, by Corollary C.7. Since C and $\hat{C}$ are deductively closed, $C = \hat{C}$.

According to [15] (E,C) is a constrained extension iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \quad \gamma \quad \middle| \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

$$C_{i+1} = Th(C_i) \cup \left\{ \beta \wedge \gamma \,\middle|\, \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} C_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Firstly, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box C_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box C_0$. Since $\mathfrak{M} \succ_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box C_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box C_i$

Consider $\eta \in E_{i+1} \cup C_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in Th(C_i)$. By the induction hypothesis, $\mathfrak{M} \models \Box\eta$.
3. $\eta \in \left\{\beta, \gamma \,\middle|\, \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, C \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot \right\}$. That is, η is either γ or β such that there is a default rule $\frac{\alpha:\beta}{\gamma} \in D$ with $\alpha \in E_i$ and $\neg(\gamma \wedge \beta) \notin C$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Using the above characterization $\hat{C}$ of C, we have $\mathfrak{M} \not\models \Box\neg(\gamma \wedge \beta)$. Since $\mathfrak{M}$ is $\succ_D$–maximal, then $\mathfrak{M} \models \gamma \wedge \Box(\gamma \wedge \beta)$ must hold and both cases for η are covered.

From the three cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box C_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box C_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Secondly, since $\mathfrak{M}$ is a $\succ_D$–maximal class above $\mathfrak{M}_W$ for (D,W), then $\mathfrak{M} = \bigcap_{i\in I} \mathfrak{M}_i$ where $\langle \mathfrak{M}_i \rangle_{i\in I}$ is a sequence of classes of K–models defined for some $\langle \delta_i \rangle_{i\in I}$ according to Definition B.1 such that $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $E_0 = C_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$

Since $\mathfrak{M}_{i+1} \succ_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg(\gamma_i \wedge \beta_i)$ where $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. By Corollary C.3, $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Box(\gamma_i \wedge \beta_i)$, hence $\mathfrak{M} \models \Box(\gamma_i \wedge \beta_i)$ because $\mathfrak{M} = \bigcap_{i\in I} \mathfrak{M}_i$. So, $\gamma_i \wedge \beta_i \in C$. Since C is satisfiable, $\neg(\gamma_i \wedge \beta_i) \notin C$. From $E_k \models \alpha_i$ and $\neg(\gamma_i \wedge \beta_i) \notin C$, we conclude that $\gamma_i \in E_{k+1}$ and $\gamma_i \wedge \beta_i \in C_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box(\gamma_i \wedge \beta_i)$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box C\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} C_i\}$. As a consequence, $\mathfrak{N} \models E$. By Corollary C.3, $\bigcup_{i=0}^{\infty} E_i \models E$. Clearly, the converse can be proved in a similar way. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Returning to $\mathfrak{M} = \mathfrak{N}$, we have $\mathfrak{N} \models \Box C$. Now, $\bigcup_{i=0}^{\infty} E_i$ is satisfiable since E is. Applying Corollary C.7, $\bigcup_{i=0}^{\infty} C_i \models C$. Again, the converse can be proved in a similar way. Then, $\bigcup_{i=0}^{\infty} C_i = C$ because $\bigcup_{i=0}^{\infty} C_i$ and C are both deductively closed sets of formulas.

Then, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ and according to [15] this means (E,C) is a constrained extension of (D,W). ■

B.2 Proof of correctness and completeness for classical default logic

Theorem 4.1 (Correctness & Completeness) *Let (D,W) be a default theory. Let $\mathfrak{M}$ be a class of K–models and E be a deductively closed set of formulas such that $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$. Then,*

E is a consistent classical extension of (D,W) iff $\mathfrak{M}$ is a $>_D$–maximal non–empty class above $\mathfrak{M}_W$.

Proof 4.1 (Correctness) Assume E is a consistent classical extension of (D,W). The set of generating default rules for E wrt D is defined as $GD_D^E = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ \neg\beta \notin E \right\}$. As has been shown in [16], then there exists an enumeration $\langle\delta_i\rangle_{i\in I}$ of GD_D^E such that for $i \in I$

$$W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Prereq(\delta_i). \tag{7}$$

Let $\langle\mathfrak{M}_i\rangle_{i\in I}$ be a sequence of classes of K–models obtained from the enumeration $\langle\delta_i\rangle_{i\in I}$ according to Definition B.1. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i\in I} \mathfrak{M}_i$ and is $>_D$–maximal above $\mathfrak{M}_W$.

Since E is a classical extension, it has been proven in [10] that

$$E = Th\Big(W \cup Conseq\Big(GD_D^E\Big)\Big).$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$ and $C_E = Justif\Big(GD_D^E\Big)$ we have obviously that $\mathfrak{M} = \bigcap_{i\in I} \mathfrak{M}_i$. Clearly, $E \wedge \beta$ is satisfiable for each $\beta \in C_E$.

Firstly, let us show that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (7), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg\beta_k$ for $\delta_k = \frac{\alpha_k : \beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and $\mathit{Justif}(\{\delta_0, \ldots, \delta_{k-1}\})$ by C^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box E^k \wedge \Diamond C^k\}$. Since $E^k \subseteq E$ and $C^k \subseteq C_E$, we have that $E^k \wedge \eta$ is satisfiable for each $\eta \in C_E$ and we can apply Corollary C.6 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg\beta_k$. We obtain that $E^k \models \neg\beta_k$. By monotonicity, $E \models \neg\beta_k$. Since E is deductively closed we have $\neg\beta_k \in E$, contradictory to the fact that $\delta_k \in GD_D^E$.

Therefore, $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i \in I} \mathfrak{M}_i >_{GD_D^E} \mathfrak{M}_W$. That is, $\mathfrak{M} >_D \mathfrak{M}_W$.

Secondly, assume $\mathfrak{M}$ is not $>_D$–maximal. Then, there exists a default rule $\frac{\alpha : \beta}{\gamma} \in D \setminus GD_D^E$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg\beta$. As noted above, $E \wedge \eta$ is satisfiable for each $\eta \in C_E$. First, applying Corollary C.2 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, since $\mathfrak{M} \models E \wedge \Box E \wedge \Diamond C_E$, we get by monotonicity $\Box E \not\models \Box\neg\beta$, yielding $E \not\models \neg\beta$ by modal logic K. Of course, $E \models \alpha$ and $E \not\models \neg\beta$ implies $\frac{\alpha : \beta}{\gamma} \in GD_D^E$, a contradiction.

Thirdly, assume $\mathfrak{M}$ is empty. Then, $\mathfrak{M} \models \Box\bot$. From the definition of $\mathfrak{M}$ and the fact that $E \wedge \eta$ is satisfiable for each $\eta \in C_E$, Corollary C.6 yields $E \models \bot$. This contradicts the consistency of E. ■

Proof 4.1 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\}$ is a non–empty $>_D$–maximal class of K–models above $\mathfrak{M}_W$.

According to [10] E is a classical extension iff $E = \bigcup_{i=0}^{\infty} E_i$ such that $E_0 = W$ and for $i \geq 0$

$$E_{i+1} = \mathit{Th}(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Define $C_0 = \emptyset$, and for $i \geq 0$

$$C_{i+1} = \qquad \left\{ \beta \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}.$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box \bigcup_{i=0}^{\infty} E_i \wedge \Diamond \bigcup_{i=0}^{\infty} C_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $E = \bigcup_{i=0}^{\infty} E_i$.

Firstly, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box E_0 \wedge \Diamond C_0$. Since $\mathfrak{M} >_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box E_0 \wedge \Diamond C_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box E_i \wedge \Diamond C_i$

Consider $\eta \in E_{i+1} \cup C_{i+1}$. Then, one of the two following cases holds.

1. $\eta \in \mathit{Th}(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in \{\beta, \gamma\}$ for some $\frac{\alpha : \beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $\neg\beta \notin E$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Assume $\mathfrak{M} \models \Box\neg\beta$. Since E is deductively closed, we obtain, by definition of C_E, that $E \wedge \eta$ is satisfiable for each $\eta \in C_E$. So, Corollary C.6 applies to $\mathfrak{M}$ and $\mathfrak{M} \models \Box\neg\beta$. As a result,

$E \models \neg\beta$. Then, it follows that $\neg\beta \in E$, a contradiction. So, $\mathfrak{M} \not\models \Box\neg\beta$. Since $\mathfrak{M}$ is $>_D$–maximal, then $\mathfrak{M} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta$ must hold and both cases for η are covered.

From the two cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box E_{i+1} \wedge \Diamond C_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Secondly, since $\mathfrak{M}$ is a $>_D$–maximal class above $\mathfrak{M}_W$ for (D, W), then $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$ where $\langle \mathfrak{M}_i \rangle_{i \in I}$ is a sequence of classes of K–models defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition B.1 such that $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $C_0 \subseteq E_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$

Since $\mathfrak{M}_{i+1} >_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg\beta_i$ where $\delta_i = \frac{\alpha_i : \beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. Suppose that $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is unsatisfiable for some $\eta \in \bigcup_{i=0}^{\infty} C_i$. Then, there is some k such that $\eta \in C_k$ and $E_k \models \neg\eta$. We have shown above that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. Then, $\mathfrak{M} \models \Box E_k \wedge \Diamond\eta$. From $E_k \models \neg\eta$, modal logic K yields $\Box E_k \models \Box\neg\eta$. Therefore, $\mathfrak{M} \models \Box\neg\eta \wedge \Diamond\eta$. Then, $\mathfrak{M}$ is empty, a contradiction. So, $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is satisfiable for each $\eta \in \bigcup_{i=0}^{\infty} C_i$. Since $\mathfrak{N} \models \alpha_i$, we can now apply Corollary C.2 to obtain that $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Diamond\beta_i$, hence $\mathfrak{M} \models \Diamond\beta_i$ because $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Since $\mathfrak{M}$ is non–empty, it follows from $\mathfrak{M} \models \Diamond\beta_i$ and $\mathfrak{M} \models \Box E$ by modal logic K that $E \not\models \neg\beta_i$. That is, $\neg\beta_i \notin E$. From $E_k \models \alpha_i$ and $\neg\beta_i \notin E$, we conclude that $\gamma_i \in E_{k+1}$ and $\beta_i \in C_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_E\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\bigcup_{i=0}^{\infty} C_i\}$. Since $\mathfrak{M}$ hence $\mathfrak{N}$ is non–empty, $\Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\beta$ is satisfiable for each $\beta \in \bigcup_{i=0}^{\infty} C_i$ (as $\Box p \wedge \Diamond q \rightarrow \Diamond(p \wedge q)$ and $\Diamond\bot \rightarrow \bot$ are valid in modal logic K). By Corollary C.2, $\bigcup_{i=0}^{\infty} E_i \models E$. The converse is proved in a similar way, it is just simpler. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Then, $E = \bigcup_{i=0}^{\infty} E_i$ and according to [10] this means E is a consistent classical extension of (D, W) (if E were not consistent, $\mathfrak{M}$ would be empty). ∎

B.3 Proof of correctness and completeness for justified default logic

Theorem 5.1 (Correctness & Completeness) *Let (D, W) be a default theory. Let $\mathfrak{M}$ be a class of K–models, E a deductively closed set of formulas, and J a set of formulas such that $J = C_{(E,J)}$ and $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$. Then,*

E is a justified extension of (D, W) wrt J iff $\mathfrak{M}$ is a $\rhd_D$–maximal class above $\mathfrak{M}_W$.

The unsatisfiable case is easily dealt with, so that we prove below the theorem for $E \wedge \beta$ being satisfiable for each $\beta \in J$ (equivalently, $\mathfrak{M}$ is non–empty as can be seen from modal logic K).

Proof 5.1 (Correctness) Assume E is a consistent justified extension of (D, W) wrt J. The set of generating default rules for (E, J) wrt D is defined as $GD_D^{(E,J)} = \left\{ \frac{\alpha:\beta}{\gamma} \;\middle|\; \alpha \in E,\ \forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$. As has been shown in [11], then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_D^{(E,J)}$ such that for $i \in I$

$$W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash \mathit{Prereq}(\delta_i). \tag{8}$$

Let $\langle \mathfrak{M}_i \rangle_{i \in I}$ be a sequence of classes of K–models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition B.1. We will show that $\mathfrak{M}$ coincides with $\bigcap_{i \in I} \mathfrak{M}_i$ and is $\rhd_D$–maximal above $\mathfrak{M}_W$.

Since E is a justified extension wrt J, it has been proven in [11] that

$$\begin{aligned} E &= \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_D^{(E,J)}\Big)\Big), \\ J &= \mathit{Justif}\Big(GD_D^{(E,J)}\Big). \end{aligned}$$

Then, since $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$ and $C_{(E,J)} = \mathit{Justif}\Big(GD_D^{(E,J)}\Big)$ we have obviously that $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Clearly, if $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$ then $E \wedge \gamma \wedge \eta$ is satisfiable for each $\eta \in \mathit{Justif}\Big(GD_D^{(E,J)}\Big)$.

Firstly, let us show that $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

- Since $\mathfrak{M}_i \subseteq \mathfrak{M}_W$ and $\mathfrak{M}_W \models W$, then by definition of $\mathfrak{M}_i$ we have $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\delta_{i-1})$ for $i \in I$. Now, $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i$ for $i \in I$ implies that $\mathfrak{M}_i \models W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})$. By (8), it follows that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$.

- Let us assume that $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ fails for some $k \in I$. By definition of $\langle \mathfrak{M}_i \rangle_{i \in I}$ and the fact that we have just proven that $\mathfrak{M}_i \models \mathit{Prereq}(\delta_i)$ for $i \in I$, this means that $\mathfrak{M}_k \models \Box\neg\beta_k \vee \Diamond\neg\gamma_k$ for $\delta_k = \frac{\alpha_k:\beta_k}{\gamma_k}$. Let us abbreviate $W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k and $\mathit{Justif}(\{\delta_0, \ldots, \delta_{k-1}\})$ by J^k. By definition, $\mathfrak{M}_k = \{\mathfrak{m} \mid \mathfrak{m} \models E^k \wedge \Box E^k \wedge \Diamond J^k\}$. Clearly, $E^k \subseteq E$ and $J^k \subseteq J$. So, E^k is satisfiable. Also, if $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$ then $E \wedge \gamma \wedge \eta$ is satisfiable for each $\eta \in J^k$. Thus, we can apply Corollary C.5 to the definition of $\mathfrak{M}_k$ and $\mathfrak{M}_k \models \Box\neg\beta_k \vee \Diamond\neg\gamma_k$ to obtain that $E^k \models \neg\beta_k \vee \neg\gamma_k$. That is, $E^k \cup \{\beta_k\} \cup \{\gamma_k\} \vdash \bot$. By monotonicity, $E \cup \{\beta_k\} \cup \{\gamma_k\} \vdash \bot$, contradictory to the fact that $\delta_k \in GD_D^{(E,J)}$.

Therefore, $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$. As a consequence, $\bigcap_{i \in I} \mathfrak{M}_i \rhd_{GD_D^{(E,J)}} \mathfrak{M}_W$. That is, $\mathfrak{M} \rhd_D \mathfrak{M}_W$.

Secondly, assume $\mathfrak{M}$ is not $\rhd_D$–maximal. Then, there exists a default rule $\frac{\alpha:\beta}{\gamma} \in D \setminus GD_D^{(E,J)}$ such that $\mathfrak{M} \models \alpha$ and $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. As noted above, $E \wedge \eta$ is satisfiable for each $\eta \in C_{(E,J)}$. First, applying Corollary C.2 to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \alpha$ yields $E \models \alpha$. Second, $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$ implies by the definition of $\mathfrak{M}$ and monotonicity that $\Box E \wedge \Diamond C_{(E,J)} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. Then, $\Box E \wedge \Diamond C_{(E,J)} \not\models \Diamond\neg\gamma$. By modal logic K, it follows that $E \wedge \eta \not\models \neg\gamma$ whenever $\eta \in C_{(E,J)}$. So, $E \cup \{\gamma\} \cup \{\eta\}$ is satisfiable for each $\eta \in J$ (because $J = C_{(E,J)}$). Returning to $\Box E \wedge \Diamond C_{(E,J)} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$, another consequence is $\Box E \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. That is, $\Box E \not\models \Box\gamma \rightarrow \Box\neg\beta$. By modal logic K, it follows that $E \not\models \gamma \rightarrow \neg\beta$. So, $E \cup \{\gamma\} \cup \{\beta\}$ is satisfiable. In all, $E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ whenever $\eta \in J \cup \{\beta\}$. Together with $E \models \alpha$, this implies $\frac{\alpha:\beta}{\gamma} \in GD_D^{(E,J)}$, a contradiction. ∎

Proof 5.1 (Completeness) Assume $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\}$ is a non–empty $\rhd_D$–maximal class of K–models above $\mathfrak{M}_W$.

According to [7] E is a justified extension wrt J iff $(E, J) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} J_i)$ such that $E_0 = W$ and $J_0 = \emptyset$ and for $i \geq 0$

$$E_{i+1} = Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall\eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

$$J_{i+1} = \quad J_i \quad \cup \left\{ \beta \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D, \alpha \in E_i, \forall\eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot \right\}$$

Let us abbreviate $\{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\bigcup_{i=0}^{\infty} J_i\}$ by $\mathfrak{N}$. We will show that $\mathfrak{M} = \mathfrak{N}$, in order to show that $E = \bigcup_{i=0}^{\infty} E_i$ and $J = \bigcup_{i=0}^{\infty} J_i$.

Firstly, let us show by induction that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond J_i\}$ for $i \geq 0$.

Base By definition, $\mathfrak{M}_W \models E_0 \wedge \Box E_0 \wedge \Diamond J_0$. Since $\mathfrak{M} \rhd_D \mathfrak{M}_W$, we get $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_0 \wedge \Box E_0 \wedge \Diamond J_0\}$.

Step The induction hypothesis is: $\mathfrak{M} \models E_i \wedge \Box E_i \wedge \Diamond J_i$

Consider $\eta \in E_{i+1} \cup J_{i+1}$. Then, one of the three following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $\mathfrak{M} \models \eta$.
2. $\eta \in J_i$. By the induction hypothesis, $\mathfrak{M} \models \Diamond\eta$.
3. $\eta \in \{\beta, \gamma\}$ for some $\frac{\alpha:\beta}{\gamma} \in D$ such that $\alpha \in E_i$ and $E \cup \{\gamma\} \cup \{v\} \not\vdash \bot$ for all $v \in J \cup \{\beta\}$. By the induction hypothesis, $\mathfrak{M} \models \alpha$. Assume $\mathfrak{M} \models \Box\neg\beta \vee \Diamond\neg\gamma$. By definition of $C_{(E,J)}$, we obtain that $E \wedge v$ is satisfiable for each $v \in C_{(E,J)}$. Also E is satisfiable. So, Corollary C.5 applies to the definition of $\mathfrak{M}$ and $\mathfrak{M} \models \Box\neg\beta \vee \Diamond\neg\gamma$. As a result, $E \models \neg\beta \vee \neg\gamma$. This contradicts the fact that $E \cup \{\gamma\} \cup \{v\} \not\vdash \bot$ for all $v \in J \cup \{\beta\}$. So, $\mathfrak{M} \not\models \Box\neg\beta \vee \Diamond\neg\gamma$. Since $\mathfrak{M}$ is $\rhd_D$–maximal, then $\mathfrak{M} \models \gamma \wedge \Box\gamma \wedge \Diamond\beta$ must hold and both cases for η are covered.

From the three cases, we obtain $\mathfrak{M} \models E_{i+1} \wedge \Box E_{i+1} \wedge \Diamond J_{i+1}$.

Therefore, we have shown that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond J_i\}$ for $i \geq 0$. So, $\mathfrak{M} \subseteq \mathfrak{N}$.

Secondly, since $\mathfrak{M}$ is a $\rhd_D$–maximal class above $\mathfrak{M}_W$ for (D, W), then $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$ where $\langle\mathfrak{M}_i\rangle_{i \in I}$ is a sequence of classes of K–models defined for some $\langle\delta_i\rangle_{i \in I}$ according to Definition B.1 such that $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$.

Let us show by induction that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$.

Base Since $\mathfrak{M}_0 = \mathfrak{M}_W$ and $J_0 \subseteq E_0 = W$, the result is obvious.

Step The induction hypothesis is: $\mathfrak{N} \subseteq \mathfrak{M}_i$

Since $\mathfrak{M}_{i+1} \rhd_{\delta_i} \mathfrak{M}_i$ for $i \in I$ we have $\mathfrak{M}_{i+1} = \{\mathfrak{m} \in \mathfrak{M}_i \mid \mathfrak{m} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i\}$ and $\mathfrak{M}_i \models \alpha_i$ and $\mathfrak{M}_i \not\models \Box\neg\beta_i \vee \Diamond\neg\gamma_i$ where $\delta_i = \frac{\alpha_i:\beta_i}{\gamma_i}$.

By the induction hypothesis, we have $\mathfrak{N} \models \alpha_i$. Suppose that $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is unsatisfiable for some $\eta \in \bigcup_{i=0}^{\infty} J_i$. Then, there is some k such that $\eta \in J_k$ and $E_k \models \neg\eta$. We have shown above that $\mathfrak{M} \subseteq \{\mathfrak{m} \mid \mathfrak{m} \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for

$i \geq 0$. Then, $\mathfrak{M} \models \Box E_k \wedge \Diamond\eta$. From $E_k \models \neg\eta$, modal logic K yields $\Box E_k \models \Box\neg\eta$. Therefore, $\mathfrak{M} \models \Box\neg\eta \wedge \Diamond\eta$. Then, $\mathfrak{M}$ is empty, a contradiction. So, $\bigcup_{i=0}^{\infty} E_i \wedge \eta$ is satisfiable for each $\eta \in \bigcup_{i=0}^{\infty} J_i$. Since $\mathfrak{N} \models \alpha_i$, we can now apply Corollary C.2 to obtain that $\bigcup_{i=0}^{\infty} E_i \models \alpha_i$. By compactness and monotonicity, there exists k such that $E_k \models \alpha_i$. By definition, $\mathfrak{M}_{i+1} \models \Box\gamma_i \wedge \Diamond\beta_i$, hence $\mathfrak{M} \models \Box\gamma_i \wedge \Diamond\beta_i$ because $\mathfrak{M} = \bigcap_{i \in I} \mathfrak{M}_i$. Since $\mathfrak{M}$ is non–empty, it follows from $\mathfrak{M} \models \Box\gamma_i \wedge \Diamond\beta_i$ and $\mathfrak{M} \models \Box E$ by modal logic K that $E \wedge \gamma_i \not\models \neg\beta_i$. That is, $E \cup \{\gamma_i\} \cup \{\beta_i\} \not\vdash \bot$. Also, since $\mathfrak{M}$ is non–empty, it follows from $\mathfrak{M} \models \Box\gamma_i$ and $\mathfrak{M} \models \Box E \wedge \Diamond C_{(E,J)}$ by modal logic K that $\mathfrak{M} \models \Diamond(E \wedge \gamma_i \wedge \eta)$ for $\eta \in C_{(E,J)}$. That is, $E \cup \{\gamma_i\} \cup \{\eta\} \not\vdash \bot$ for $\eta \in J$ (because $J = C_{(E,J)}$). From $E_k \models \alpha_i$ and $E \cup \{\gamma_i\} \cup \{\eta\} \not\vdash \bot$ for $\eta \in J \cup \{\beta_i\}$, we conclude that $\gamma_i \in E_{k+1}$ and $\beta_i \in J_{k+1}$. Hence, $\mathfrak{N} \models \gamma_i \wedge \Box\gamma_i \wedge \Diamond\beta_i$. By the induction hypothesis and the definition of $\mathfrak{M}_{i+1}$ we obtain $\mathfrak{N} \subseteq \mathfrak{M}_{i+1}$.

Therefore, we have shown that $\mathfrak{N} \subseteq \mathfrak{M}_i$ for $i \in I$. That is, $\mathfrak{N} \subseteq \mathfrak{M}$.

In all, $\mathfrak{M} = \mathfrak{N}$. That is, $\{\mathfrak{m} \mid \mathfrak{m} \models E \wedge \Box E \wedge \Diamond C_{(E,J)}\} = \{\mathfrak{m} \mid \mathfrak{m} \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\bigcup_{i=0}^{\infty} J_i\}$. Since $\mathfrak{M}$ hence $\mathfrak{N}$ is non–empty, $\Box\bigcup_{i=0}^{\infty} E_i \wedge \Diamond\beta$ is satisfiable for each $\beta \in \bigcup_{i=0}^{\infty} J_i$ (as $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$ and $\Diamond\bot \to \bot$ are valid in modal logic K). By Corollary C.2, $\bigcup_{i=0}^{\infty} E_i \models E$. The converse is proved in a similar way. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Since $E = \bigcup_{i=0}^{\infty} E_i$, the definitions of $C_{(E,J)}$ and J_i make it easy to verify that $C_{(E,J)} = \bigcup_{i=0}^{\infty} J_i$ That is, $J = \bigcup_{i=0}^{\infty} J_i$.

Then, $E = \bigcup_{i=0}^{\infty} E_i$ and $J = \bigcup_{i=0}^{\infty} J_i$, and according to [7] this means E is a justified extension of (D, W) wrt J. ■

C Proofs of some modal propositions

Proposition C.1 *Let p, q, r, $s_1, \ldots, s_n$ be non–modal formulas such that $q \wedge s_i$ is satisfiable for $i = 1, \ldots, n$.*

If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \to r$ then $\models p \to r$.

Proof C.1 Assume the contrary. Then, $p \wedge \neg r$ is satisfiable. It is thus possible to define the K–model $\mathfrak{m} = \langle \omega_0, \{\omega_i \mid i = 0, \ldots, n\}, \{(\omega_0, \omega_i) \mid i = 1, \ldots, n\}, \mathcal{I} \rangle$ such that $\omega_0 \models p \wedge \neg r$ and $\omega_i \models q \wedge s_i$ for $i = 1, \ldots, n$. Clearly, $\mathfrak{m}$ contradicts the validity of $p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \to r$ even in the limiting case where $n = 0$. ■

Corollary C.2 *Let S, T, U and V be sets of non–modal formulas and $T \wedge u$ is satisfiable for each $u \in U$.*

If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models V$ then $S \models V$.

Proof C.2 Consider $v \in V$. $\mathfrak{M} \models v$ means $S \wedge \Box T \wedge \Diamond U \models v$. By compactness, $S' \wedge \Box T' \wedge \Diamond U' \models v$ for some finite subsets S', T' and U' of S, T and U, respectively. Since the deduction theorem for material implication holds in modal logic K, we get $\models S' \wedge \Box T' \wedge \Diamond U' \to v$. Applying Proposition C.1, $\models S' \to v$. That is, $S' \models v$. By monotonicity, $S \models v$. So, $S \models V$. ■

Corollary C.3 *Let S, T and V be sets of non–modal formulas.*

If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T\}$ and $\mathfrak{M} \models V$ then $S \models V$.

Proof C.3 Apply Corollary C.2 in the limiting case where U is empty ($n = 0$ in Proposition C.1). ■

Proposition C.4 *Let p, q, r, $s_1, \ldots, s_n$, t be non–modal formulas, with p and $q \wedge s_i \wedge \neg t$ satisfiable for $i = 1, \ldots, n$.*
If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Box r \vee \Diamond t$ then $\models q \rightarrow r \vee t$.

Proof C.4 Assume the contrary. Then, $q \wedge \neg r \wedge \neg t$ is satisfiable. Define the K–model $\mathfrak{m} = \langle \omega_0, \{\omega_i \mid i = 0, \ldots, n+1\}, \{(\omega_0, \omega_i) \mid i = 1, \ldots, n+1\}, \mathcal{I} \rangle$ with $\mathcal{I}$ as follows. Let $\omega_0 \models p$. Let $\omega_{n+1} \models q \wedge \neg r \wedge \neg t$. For $i = 1, \ldots, n$, let $\omega_i \models q \wedge s_i \wedge \neg t$. Then, $\mathfrak{m}$ contradicts the validity of $p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Box r \vee \Diamond t$ even in the limiting case where $n = 0$. ■

Corollary C.5 *Let S, T and U be sets of non–modal formulas and let p and q be non–modal formulas such that S is satisfiable and $T \wedge u \wedge \neg q$ is satisfiable for each $u \in U$.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models \Box p \vee \Diamond q$ then $T \models p \vee q$.

Proof C.5 $\mathfrak{M} \models \Box p \vee \Diamond q$ means $S \wedge \Box T \wedge \Diamond U \models \Box p \vee \Diamond q$. By compactness, $S' \wedge \Box T' \wedge \Diamond U' \models \Box p \vee \Diamond q$ for some finite subsets S', T' and U' of S, T and U, respectively. Since the deduction theorem for material implication holds in modal logic K, we get $\models S' \wedge \Box T' \wedge \Diamond U' \rightarrow \Box p \vee \Diamond q$. Applying Proposition C.4, $\models T' \rightarrow p \vee q$. That is, $T' \models p \vee q$. Accordingly, $T \models p \vee q$. ■

Corollary C.6 *Let S, T, U and V be sets of non–modal formulas such that S is satisfiable and $T \wedge u$ is satisfiable for each $u \in U$.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T \wedge \Diamond U\}$ and $\mathfrak{M} \models \Box V$ then $T \models V$.

Proof C.6 Consider $v \in V$. Then, $\mathfrak{M} \models \Box v$. Since $\bot$ and $\Diamond \bot$ are equivalent in modal logic K, $\mathfrak{M} \models \Box v \vee \Diamond \bot$. Applying Corollary C.5, $T \models v \vee \bot$. That is, $T \models v$. Accordingly, $T \models V$. ■

Corollary C.7 *Let S, T and U be sets of non–modal formulas such that S is satisfiable.*
If $\mathfrak{M} = \{\mathfrak{m} \mid \mathfrak{m} \models S \wedge \Box T\}$ and $\mathfrak{M} \models \Box V$ then $T \models V$.

Proof C.7 Apply Corollary C.6 in the limiting case where U is empty ($n = 0$ in Proposition C.4). ■

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