AN APPROACH TO CONTEXT-BASED DEFAULT REASONING*

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Abstract

In this paper, we elaborate the idea that contexts provide an important and meaningful notion in default reasoning. We demonstrate this by looking at Reiter's default logic that has been the prime candidate for formalizing consistency-based default reasoning ever since its introduction in 1980. This results in a new context-based approach to default logic, called contextual default logic. The approach extends the notion of a default rule and supplies each default extension with a context. In particular, contextual default logic provides a unified framework for default logics. That is, it allows for embedding existing variants of default logic along with more traditional approaches like the closed world assumption. Since this is accomplished in a homogeneous way, we gain additional expressiveness by combining the diverse approaches. A key advantage of contextual default logic is that it provides a syntactical instrument for comparing existing default logics in a unified setting. In particular, the approach reveals that existing default logics mainly differ in the way they deal with an explicit or implicit underlying context.

1 Introduction

In real life, we are quite often faced with incomplete information. Yet we are not paralyzed by missing information—rather we reason in the absence of information—and still arrive at plausible conclusions.

A versatile way of reasoning in the absence of information is to reason *by default*. Default reasoning puts faith in standard situations. It relies on general rules expressing expected states of affairs. In fact, whenever we are reasoning by default we are implicitly making assumptions about the situation at hand. In this way, our reasoning is driven

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from certain *contexts*[1] induced by the adopted assumptions. Accordingly, we arrive at different conclusions depending on which kind of context we consider.

First of all, let us look at two examples reflecting different notions of contexts.

Let us first turn to a variation of the so-called "broken-arms" example described in [16]. This example involves a robot, say Roby. Among other rules, we have a default rule saying that a robot's arm is usable unless it is broken. Now, suppose we are told that one of Roby's arms, either the left one or the right one, is broken. And we are given no other detail.

Then, being asked about Roby's arms, we clearly answer that only one arm is usable but not both.[2] This is so because the default conclusion that an arm is usable relies on the underlying assumption that it is not broken. After we have drawn the conclusion that an arm is usable we keep relying on the implicit assumption that it is not broken. In this way, the underlying assumption becomes a part of our current context of reasoning. Since we encounter two such assumptions, which are contradictory to each other, our reasoning can be seen as being driven by two distinct contexts. According to the first one, we may conclude that the left arm is usable while assuming that the right arm is broken. And in the second context, we imagine that the left arm is broken and therefore conclude that the right arm is usable. Reconciling these two alternative views directs us to conclude that only one of the arms is actually usable.

For a second example, let us look at the so-called "holidays" example given in [7]. Assume we are set about packing our bag for a trip to Vancouver. As usual, we do so by relying on rules of thumb, like if it is possible that it will be warm then take a T-shirt; if it is possible that it will be cold then take a sweater. Of course, if we know nothing about the weather, apart from the fact that warm and cold weather are mutually exclusive, we are willing to take both, our T-shirt and our sweater.

In this example, it makes sense to draw both conclusions simultaneously. We want to take a T-shirt if it is merely possible for the weather to be fine, and similarly for the sweater if there is a chance for cold weather. Of course, it is both possible that the weather will be warm and possible that the weather will be cold. Hence, we take both clothes with us to Vancouver. Unlike the "broken-arms" example, we thus encounter a different situation in the "holidays" example. Here, our reasoning is rather driven by a single context which is composed of two incompatible subcontexts: One subcontext captures the possible scenario that it is going to be warm in Vancouver, whereas the other one deals with a possible cold weather scenario. In this way, warm and cold weather provide merely possible scenarios of the city of Vancouver.

Here, the underlying assumptions delineate the contexts of reasoning but do not act as hypothetical premises within a context, as in the "broken-arms" example. There, an underlying assumption enforced a certain context of reasoning. The account of an underlying assumptions used in the "holidays" example is weaker. Here, assumptions generate certain (sub)contexts but they cannot inhibit others and thus tolerate further incompatible (sub)contexts.

[1] So far, there has been no relationship established between the conception of contexts studied in this paper and the one investigated in [13].

[2] Unlike well-known approaches to default reasoning, like autoepistemic [15] and default logic [18]. Cf. [16] for a detailed discussion on this phenomenon.

The two previous examples have shown how differently structured contexts may influence the process of reasoning by default. In particular, the examples have revealed the significance of various forms of contexts. In this paper, we pursue further the idea that contexts provide an important and meaningful notion in default reasoning. We will demonstrate this by looking at Reiter's default logic [18], which has been the prime candidate for formalizing consistency-based default reasoning ever since its introduction in 1980.

In Reiter's default logic, standard first-order logic is augmented by non-standard inference rules, called *default rules*. These rules differ from standard inference rules in sanctioning inferences that rely upon given as well as absent information. Such inferences therefore could not be made in a standard framework. In fact, default rules can be seen as rules of conjecture whose role is to augment a given incomplete first-order theory. A set of conclusions sanctioned by a set of default rules is called an *extension* of an initial set of facts. Informally, an extension of a set of facts is the set of all formulas derivable from these facts using standard inference rules and all specified default rules. Or in Reiter's words [18], the "... *intuitive idea which must be captured is that of a set of defaults inducing an extension of some underlying incomplete set of first-order wffs*".

As argued above, contexts govern the drawing of default conclusions. Accordingly, contexts should take an important part in forming extensions in default logic. So, starting out from Reiter's default logic, the basic idea of our approach to context-based default reasoning becomes twofold. First, we supply each default extension with an underlying (sometimes structured) context according to the intuitions sketched above. Second, we extend the concept of a default rule in order to allow for a variety of different application conditions which arise naturally from the distinction between an initial set of facts, a default extension at hand, and its underlying context.

In what follows, we will demonstrate that the context-based approach to default logic is a very powerful one. In particular, we will prove that it allows for embedding existing variants of default logic. Notably, we will show that the different uses of contexts are actually what makes the difference between these variants. As a result of applying the notion of a context to default logic, we obtain a more general but uniform default reasoning system, which combines the expressiveness of former approaches.

Our approach is in accord with the one taken by Marek and Truszczyński in [12]. There, they adopt the view that context-dependent reasoning is the heart of default and even nonmonotonic reasoning. Their *"idea is to relativize the concept of a proof using a context to control the applicability of rules. Speaking more precisely, a context determines what is and what is not a valid derivation"* [12, p. 2]. They apply this idea in turn to Reiter's original default logic, logic programming, truth maintenance systems, and McDermott and Doyle's modal nonmonotonic logic [14]. Hence, this paper might in some respects be regarded as an extension of [12] for capturing different notions of contexts encountered in existing variants of default logics.

The rest of the paper is organized as follows. Section 2 addresses the question how the aforementioned variety of contexts along with their induced forms of reasoning can be formalized. As a result, we will see that a proper account of consistency provides an appropriate answer to this question. This is guided by a thorough investigation into the different conceptions of consistency used in certain variants of default logic.

In Section 3, we introduce *contextual default logic*, as our formal approach to context-based default reasoning. Furthermore, we show how the aforementioned examples are formally dealt with in contextual default logic. Section 4 contains a collection of further examples illustrating the expressiveness enjoyed by contextual default logic. In Section 5, we elaborate on the formal theory of contextual default logic. We give alternative characterizations of extensions in contextual default logic and examine the formal notion of contexts in more detail. Section 6 continues the formal elaboration by giving a possible worlds semantics for contextual default logic, which nicely reflects the interplay of contexts through possible worlds. Section 7 demonstrates the power behind the notion of contexts by showing that existing variants of default logics constitute certain fragments of contextual default logic. The formal analysis is completed in Section 8, where we describe the formal properties of contextual default logic relying on some results obtained in the previous section.

2 Notions of consistency in default logics

We have seen in the introductory section how contexts may influence the default reasoning process. So far, however, it is far from clear how different notions of contexts along with their induced forms of reasoning can be formalized. In what follows, we will argue that a proper account of consistency provides an appropriate answer to this question. In particular, we will show how consistency allows for distinguishing between multiple contexts as opposed to different subcontexts of a wider common context.

Since we explore the notion of contexts in connection with default logic, we have to account first for some formal preliminaries: Classical default logic was defined by Reiter in [18] as a formal account of reasoning in the absence of complete information. It is based on first-order logic, whose sentences are hereafter simply referred to as formulas (instead of closed formulas). In what follows, we then assume the reader to be familiar with the basic concepts of first order logic (cf. [9]) as well as some acquaintance with modal logics (cf. [4]) in Section 6. We shall be dealing with a standard first order language (including $\bot$ and $\top$, the symbols for "falsum" and "verum") and its extension by the modal operator $\Box$. We denote derivability and entailment by $\vdash$ and $\models$, respectively (whether dealing with the pure first order language or the modal one). We use Th to denote the first order consequence operation, that is $Th(S) = \{\alpha \mid S \vdash \alpha\}$. Further definitions and conventions will be introduced when they occur for the first time.

In default logics, default knowledge is incorporated by means of so-called *default rules*. A default rule is any expression of the form $\frac{\alpha:\beta}{\gamma}$, where α, β and γ are formulas. α is called the *prerequisite*, β the *justification*, and γ the *consequent* of the default rule. Accordingly, a *default theory* (D, W) consists of a set of formulas W and a set of default rules D. Informally, an *extension*[3] of the initial set of facts W is the set of all formulas derivable from W by applying classical inference rules and all applicable default rules. Usually, a default rule $\frac{\alpha:\beta}{\gamma}$ is applicable, if its prerequisite α is derivable and its justification β is *consistent* in a certain way. Intuitively, a default rule may thus be interpreted as: "If α is known and $\neg\beta$ is unknown, then conclude γ by default".

[3] Formal definitions for extensions in existing default logics are given in Section 7.

In all existing default logics, the prerequisite α of a default rule $\frac{\alpha:\beta}{\gamma}$ is checked wrt an extension E by requiring $\alpha \in E$. However, the aforementioned variants differ in the way they account for the consistency of the justification β. For instance, in classical default logic [18] the consistency of the justification β is checked wrt the extension E by verifying $\neg\beta \notin E$, whereas in constrained default logic [20, 7] the same is done wrt a set of constraints C, containing the extension E, by checking $\neg\beta \notin C$.

In default logics, there are thus two extreme notions of consistency: Individual and joint consistency. The former is employed in classical default logic, whereas the latter can be found in constrained default logic. Individual consistency requires that no justification of an applying default rule is contradictory with a given extension, whereas joint consistency stipulates that all justifications of all applying default rules are jointly consistent with the extension at hand.

Now, the interesting question is how these notions of consistency deal with the variety of contexts described in the introductory section. In order to illustrate this let us consider the formalizations of the two examples given there.

As a first example, let us take a look at the formalization of the "broken-arms" example given in [16]. Consider the default theory[4]

$$\left(\left\{\frac{:\neg \mathsf{Bl}}{\mathsf{Ul}}, \frac{:\neg \mathsf{Br}}{\mathsf{Ur}}\right\}, \{\mathsf{Bl} \vee \mathsf{Br}\}\right). \tag{1}$$

The set of facts asserts that either the left arm, Bl, or the right arm, Br, is broken. The default rules express that an arm is usable, Ul or Ur, unless it is broken, ie. if we can consistently assume that it is not broken, $\neg\mathsf{Bl}$ or $\neg\mathsf{Br}$. We observe that altogether the facts and the justifications of the two default rules are inconsistent.

In classical default logic, default theory (1) has one extension containing Ul and Ur along with the fact $\mathsf{Bl} \vee \mathsf{Br}$, as depicted in Figure 1 (the circle represents the deductive closure of the given formulas). That is, classical default logic directs us to conclude that both arms are usable even though one of them is known to be broken. Both de-

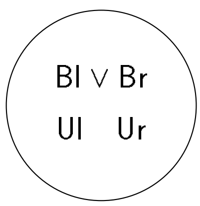

Figure 1: The classical extension in the "broken-arms" example.

fault rules apply, although they have contradictory justifications according to the set of facts. This is so because each justification, $\neg\mathsf{Bl}$ and $\neg\mathsf{Br}$, is separately consistent with the extension.

[4]The original formulation given in [16] uses so-called *semi-normal default rules*, like $(:\mathsf{Ul} \wedge \neg\mathsf{Bl}/\mathsf{Ul})$. For presentation, we have simplified the justifications. This leaves the default theory's formal behavior unaffected.

In this case, the extension is somehow embedded in an implicit context which gathers two incompatible subcontexts: A subcontext formed while assuming $\neg \mathsf{Bl}$ and another one formed under the assumption $\neg \mathsf{Br}$. In precise terms, this amounts to an extension, $Th(\{\mathsf{Bl} \vee \mathsf{Br}, \mathsf{Ul}, \mathsf{Ur}\})$, which is embedded in a wider context, namely

$$Th(\{\neg \mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}, \mathsf{Ur}\}) \cup Th(\{\mathsf{Bl}, \neg \mathsf{Br}, \mathsf{Ul}, \mathsf{Ur}\})$$

containing two incompatible subcontexts. One subcontext, viz. $Th(\{\neg \mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}, \mathsf{Ur}\})$, is formed by the extension and the justification of the default rule $\frac{:\neg \mathsf{Bl}}{\mathsf{Ul}}$; and the second one, $Th(\{\mathsf{Bl}, \neg \mathsf{Br}, \mathsf{Ul}, \mathsf{Ur}\})$, is formed by the extension and the justification of the default rule $\frac{:\neg \mathsf{Br}}{\mathsf{Ur}}$.

According to [16], the solution to the "broken-arms" example obtained in classical default logic seems to be rather unintuitive.[5] In particular, the use of contexts in classical default logic does not coincide with the one suggested in the introductory section. There, we have argued in favor of two distinct contexts rather than a wider contexts including two incompatible subcontexts. This is so because in the "broken-arms" example the justifications act in the sense of underlying assumptions or even implicit hypotheses, which seem to suggest distinct contexts in the case of inconsistencies.

A different solution to the "broken-arms" example is offered by constrained default logic. In constrained default logic, default theory (1) yields two so-called constrained extensions, which are extensions supplied with a certain set of constraints. In fact, we obtain one extension containing Ul and another one containing Ur from default theory (1). In the first constrained extension, the constraints consist of the justification of the default rule $\frac{:\neg \mathsf{Bl}}{\mathsf{Ul}}$ along with the default rule's consequent and the world knowledge. In the second constrained extension, the constraints contain the justification of the default rule $\frac{:\neg \mathsf{Br}}{\mathsf{Ur}}$, its consequent, and the set of facts. These two constrained extensions are depicted in Figure 2 (the inner circle represents the extension and the outer circle stands for the deductively closed set of constraints containing the extension along with the given formulas). Intuitively, this amounts to two alternative world-descriptions:

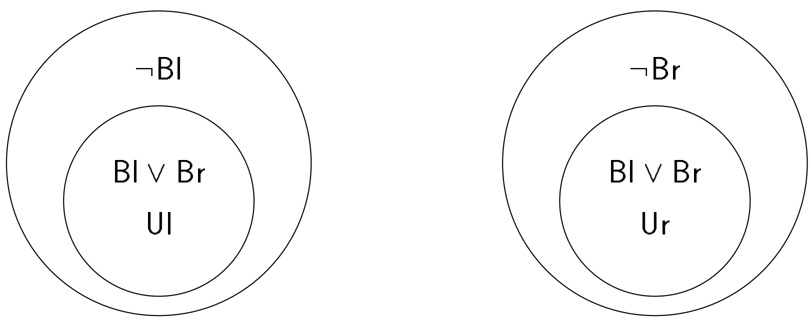

Figure 2: The constrained extensions in the "broken-arms" example.

One asserting that the left arm is usable by relying on the assumption that it is not broken and another one asserting that the right arm is usable based on the assumption that this arm is not broken.

[5] Since we are mainly interested in contexts, we have to refer the reader to the literature [16, 5, 10, 7], for a more detailed analysis why this is an unintuitive solution.

In both cases, the constraints consist of the respective extension along with the justification of the applying default rule. While forming both extensions one of the two default rules is inapplicable since its justification is inconsistent with the justification of the other default rule that has "already" established a certain context of reasoning. In this way, each extension is embedded in a context given by the set of constraints. Hence, in the case of constrained default logic, a context provides an extended world-description enriched by implicit assumptions given by the justifications of the applying default rules.

In our example, this amounts to reasoning under two distinct contexts. More formally, the first extension $Th(\{\mathsf{Bl} \vee \mathsf{Br}, \mathsf{Ul}\})$ is formed while reasoning in the context

$$Th(\{\neg\mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}\}), \tag{2}$$

while the second extension $Th(\{\mathsf{Bl} \vee \mathsf{Br}, \mathsf{Ur}\})$ is embedded in the context

$$Th(\{\mathsf{Bl}, \neg\mathsf{Br}, \mathsf{Ur}\}). \tag{3}$$

It is clear that these two contexts coincide with the ones suggested in the introductory section for the "broken-arms" example.[6] Reconciling the views expressed by the two extensions by intersecting them yields the conclusion $\mathsf{Ul} \vee \mathsf{Ur}$, saying that either the left arm or the right arm is usable. According to [16], this corresponds to be the preferred solution to the "broken-arms" example.

Now, let us turn to our second example in the introductory section: The "holidays" example. Recall that the intuition behind this example suggests the use of a wider context gathering two incompatible subcontexts. But even though the "holidays" example relies on different intuitions than the "broken-arms" example, its formalization is very similar to the one given in (1):

$$\left(\left\{\frac{:\mathsf{W}}{\mathsf{T}}, \frac{:\mathsf{C}}{\mathsf{S}}\right\}, \{\neg\mathsf{W} \vee \neg\mathsf{C}\}\right) \tag{4}$$

Here, the set of facts reflect the common knowledge that warm, W, and cold weather, C, are mutually exclusive. The default rules represent the assertions: If it is possible that it will be warm then take a T-shirt; if it is possible that it will be cold then take a sweater.

As in the case of the "broken-arms" example, we obtain one extension in classical default logic and two extensions in constrained default logic. That is, from default theory (4) we obtain a single classical extension containing $\mathsf{T} \wedge \mathsf{S}$ and two constrained extensions, one containing T and another one containing S. All these extensions are given in Figure 3 and 4 and formed in analogy to the ones given for the "broken-arms" example. Now, however, the intuitively more appealing result is obtained in classical default logic since it directs us to take both a T-shirt and a sweater with us to Vancouver.

Again, it is the use of contexts that makes the difference. In classical default logic, the extension is embedded in a single context being composed of two incompatible subcontexts, namely

[6]Also, they should be compared with the subcontexts encountered in classical default logic in the "broken-arms" example.

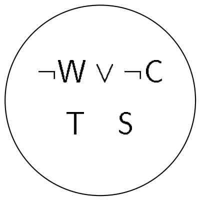

Figure 3: The classical extension in the "holidays" example.

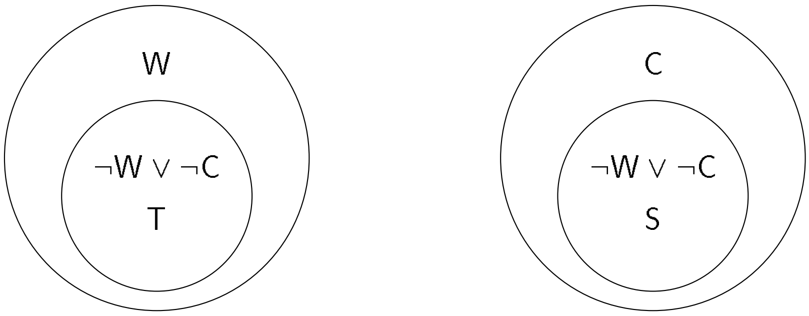

Figure 4: The constrained extensions in the "holidays" example.

$$Th(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\}) \cup Th(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}\}). \qquad (5)$$

This context corresponds to the one suggested in the introductory section. Moreover, it has directed our reasoning to the expected conclusions. Opposed to this solution, obtained in classical default logic, we obtain two alternative extensions based on distinct contexts in constrained default logic, whose reconciliation would prevent us from taking both a T-shirt and a Sweater.

So, classical and constrained default logic have exchanged their roles: In the "broken-arms" example, constrained default logic produces the intuitive result. This has been with reversed roles in the "holidays" example, where classical default logic yields the more appealing solution. Hence, none of them is able to account for the variety of contexts sketched in the introductory section. In this way, classical and constrained default logic are simply not context-sensitive enough; they cannot account for the collection of contexts needed for dealing with both examples at the same time. Rather we observe that both default logics — implicitly or explicitly — use different but fixed notions of contexts.

In particular, we have seen that the notion of contexts is directly related to that of consistency. We have seen in classical default logic that the use of individual consistency requirements in the presence of an inconsistency leads to alternative subcontexts of a common wider context. This is different in constrained default logic, where the use of joint consistency requirements yields distinct contexts as soon as an inconsistency arises. All this will become more apparent in Section 3, where a collection of different "application conditions" for default rules is discussed.

Finally, the question arises how we can account for the whole variety of contexts described in the introductory section. In other words, how can we combine the various notions of contexts in order to provide a more general but uniform setting for default logic. In order to provide an answer to this question, we also have to compromise the notions of individual and joint consistency. In particular, we have to deal with joint consistency requirements in the presence of inconsistent individual consistency requirements. Therefore, we allow for contexts containing contradictory formulas, like Bl and $\neg\mathsf{Bl}$ as in the "broken-arms" example in classical default logic, without containing all possible formulas. Thus, we admit contexts which are not deductively closed. A useful concept is then that of *pointwise closure* $\mathit{Th}_S(T)$.

Definition 2.1 *Let T and S be sets of formulas. If T is non-empty, the pointwise closure of T under S is defined as*

$$\mathit{Th}_S(T) = \bigcup_{\phi \in T} \mathit{Th}(S \cup \{\phi\}).$$

In addition, $\mathit{Th}_S(\emptyset) = \mathit{Th}(S)$.

If S is a singleton set $\{\varphi\}$, we simply write $\mathit{Th}_\varphi(T)$ instead of $\mathit{Th}_{\{\varphi\}}(T)$. Given two sets of formulas T and S, we say that T is *pointwisely closed under* S iff $T = \mathit{Th}_S(T)$. In particular, we simply say that T is pointwisely closed whenever $T = \mathit{Th}_\top(T)$ for any tautology $\top$.

Let us illustrate these definitions and see how the aforementioned contexts can be represented by means of the concept of pointwise closure. Consider first the contexts obtained in the "broken-arms" example in constrained default logic. The context given in (2) can now be described as the pointwise closure of the justification of the default rule $\frac{:\neg\mathsf{Bl}}{\mathsf{Ul}}$ under the facts and the consequent of the same default rule, namely

$$\mathit{Th}_{\{\mathsf{Bl}\vee\mathsf{Br},\mathsf{Ul}\}}(\{\neg\mathsf{Bl}\}) \tag{6}$$

In the same way, we obtain

$$\mathit{Th}_{\{\mathsf{Bl}\vee\mathsf{Br},\mathsf{Ur}\}}(\{\neg\mathsf{Br}\}) \tag{7}$$

for the context in (3). By analogy to (6), (3) is given by the pointwise closure of the justification of the default rule $\frac{:\neg\mathsf{Br}}{\mathsf{Ur}}$ under the world knowledge and the consequent of the same default rule. Both contexts are deductively closed sets of formulas.

The more interesting case is given by the context obtained in the "holidays" example in classical default logic, namely the one given in (5). This context can be described as the pointwise closure of the justifications of the default rules $\frac{:\mathsf{W}}{\mathsf{T}}$ and $\frac{:\mathsf{C}}{\mathsf{S}}$ under the set $\{\neg\mathsf{W}\vee\neg\mathsf{C},\mathsf{T},\mathsf{S}\}$ constituting the single extension obtained above. This yields

$$\mathit{Th}_{\{\neg\mathsf{W}\vee\neg\mathsf{C},\mathsf{T},\mathsf{S}\}}(\{\mathsf{W},\mathsf{C}\}). \tag{8}$$

In contrast to the contexts given in (6) and (7), the previous context is not deductively closed. Rather it consists of two deductively closed subcontexts $\mathit{Th}(\{\mathsf{W},\neg\mathsf{C},\mathsf{T},\mathsf{S}\})$ and $\mathit{Th}(\{\neg\mathsf{W},\mathsf{C},\mathsf{T},\mathsf{S}\})$. In general, such a subcontext is a maximal deductively closed subset of an entire context.

We will take advantage of the concept of pointwise closure in the next section, where we define our formal approach to context-based default reasoning.

3 Contextual default logic

In what follows, we introduce a context-based approach to default logic by supplying default extensions with contexts. Moreover, we introduce an extended concept of a default rule in order to allow for an assortment of application conditions; thereby going beyond those conditions found in existing default logics. The resulting system is called *contextual default logic.*

In our approach, we consider three sets of formulas: A set of facts W, an extension E, and a certain *context* C such that $W \subseteq E \subseteq C$. The set of formulas C is somehow established from the facts, the default conclusions (ie. the consequences of the applied default rules), as well as all underlying consistency assumptions (ie. the justifications of all applied default rules). In fact, this approach trivially captures the above application conditions for existing default rules, eg. $\alpha \in E$ and $\neg\beta \notin E$ in the case of classical default logic.

Yet our approach allows for even more ways of forming application conditions of default rules. Consider a formula φ and three consistent, deductively closed sets of formulas W, E, and C such that $W \subseteq E \subseteq C$. Six more or less strong application conditions are obtained which can be ordered from left to right by decreasing strength; whereby $>$ is read as "implies":

$$\varphi \in W > \varphi \in E > \varphi \in C > \neg\varphi \notin C > \neg\varphi \notin E > \neg\varphi \notin W \tag{9}$$

We can think of W as a deductively closed set of facts, E as a default extension of W, and C as the above mentioned context for E. Then, the first condition $\varphi \in W$ stands for first-order derivability from the facts W. The second condition $\varphi \in E$ stands for derivability from W using first-order logic and certain default rules. This is used in existing default logics as the test for the prerequisite of a default rule. The third condition, $\varphi \in C$, is the strangest one. It expresses "membership in a context of reasoning". We will discuss this condition in more detail in Section 4 and 6. The last three conditions are consistency conditions. The fourth condition $\neg\varphi \notin C$ corresponds to the consistency condition used in constrained default logic, the fifth one $\neg\varphi \notin E$ is used in classical default logic. Finally, the last condition $\neg\varphi \notin W$ is the one used for the closed world assumption [17], where it is restricted to ground negative literals.

This variety of application conditions motivates an extended notion of a default rule.

Definition 3.1 *A contextual default rule δ is an expression of the form*[7]

$$\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \;:\; \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma}$$

where α_W, α_E, α_C, β_C, β_E, β_W, and γ are formulas.

α_W, α_E, α_C are called the W-, E-, and C-prerequisites, also noted $Prereq_W(\delta)$, $Prereq_E(\delta)$, $Prereq_C(\delta)$, β_C, β_E, β_W are called the C-, E-, and W-justifications, also

[7] For simplicity, we restrict ourselves to contextual default rules having only one justification of each type.

noted $Justif_C(\delta)$, $Justif_E(\delta)$, $Justif_W(\delta)$, *and* γ *is called the consequent, also noted* $Conseq(\delta)$. [8]

The six antecedents of a contextual default rule are to be treated along the above intuitions.

A contextual default theory is a pair (D, W), where D is a set of contextual default rules and W is a deductively closed[9] set of formulas. In what follows, we make the above intuitions precise and introduce the notion of a *contextual extension*.

A contextual extension is to be a pair (E, C), where E is a deductively closed set of formulas and C is a pointwisely closed set of formulas. This leads to the following definition.

Definition 3.2 *Let* (D, W) *be a contextual default theory. For any pair of sets of formulas* (T, S) *let* $\nabla(T, S)$ *be the pair of smallest sets of formulas* (T', S') *such that* $W \subseteq T' \subseteq S'$ *and the following condition holds:*

For any

$$\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \;:\; \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D,$$

if

1. $\alpha_W \in W$
2. $\alpha_E \in T'$
3. $\alpha_C \in S'$
4. $\neg\beta_C \notin S$
5. $\neg\beta_E \notin T$
6. $\neg\beta_W \notin W$

then

7. $Th_\gamma(T') \subseteq T'$
8. $Th_{\beta_E}(T') \subseteq S'$
9. $Th_{\beta_C}(S') \subseteq S'$

A pair of sets of formulas (E, C) *is a contextual extension of* (D, W) *iff* $\nabla(E, C) = (E, C)$.

Notice that the operator ∇ is in fact parameterized by (D, W). Furthermore, observe that Conditions *1-6* basically correspond to the six conditions given in (9).

[8] These projections extend to sets of contextual default rules in the obvious way (eg. $Justif_E(\Delta) = \bigcup_{\delta\in\Delta}\{Justif_E(\delta)\}$).

[9] This is no real restriction, but it simplifies matters.

Conditions *7-9* capture the result of applying a contextual default rule. Condition *7* ensures that the consequent of an applied contextual default rule belongs to the final extension and that the final extension is deductively closed. Condition *8* and *9* account for the formation of the final context. Roughly speaking, they aggregate successful consistency checks, namely the ones given in *4* and *5*, respectively. Condition *8* accounts for the individual consistency of β_E by enforcing a subcontext consisting of β_E and the final extension. Condition *9* ensures the joint consistency of β_C by enforcing the consistency of β_C with all final subcontexts.

Intuitively, we start from (W, W) (ie. we take the facts W as our initial version of E and C) and try to apply a contextual default rule by checking Conditions *1-6* and, if we are successful, we enforce *7-9*. That is, we add γ to our current version of E and we add $\phi \wedge \beta_E$ and $\varphi \wedge \beta_C$ to our current version of C, for each ϕ in the final extension E and for each φ in the final context C.

As a first crisp example, let us consider the contextual default theory

$$\left(\left\{\frac{\mathsf{A}\,||:|\,\mathsf{B}\,|}{\mathsf{C}}, \frac{|\,\mathsf{C}\,|:\ \mathsf{E}\,|\,\neg\mathsf{B}\,|}{\mathsf{D}}\right\}, Th(\{\mathsf{A}\})\right) \tag{10}$$

along with its only contextual extension (E,C), where

$$E = Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}\}) \tag{11}$$

$$C = Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{B}\}) \cup Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \neg\mathsf{B}\}). \tag{12}$$

The first component, E, represents the extension, whereas the second one, C, provides its context. This contextual extension is generated from the facts by applying first the first contextual default rule and then the second one.

Now, the contextual default rule

$$\frac{\mathsf{A}\,||:|\,\mathsf{B}\,|}{\mathsf{C}}$$

applies if its prerequisite A is monotonically derivable (ie. if A is derivable without contextual default rules according to Condition 1 in Definition 3.2) and if its E-justification B is consistent with the extension E (according to Condition 5). In other words, B has to be individually consistent. This being the case, we derive C. That is, C is nonmonotonically derivable by means of the first contextual default rule (cf. Condition 2).

Thus, C establishes the prerequisite of the second contextual default rule,

$$\frac{|\,\mathsf{C}\,|:\ \mathsf{E}\,|\,\neg\mathsf{B}\,|}{\mathsf{D}}.$$

In order to derive D, we have to verify the consistency of the two justifications E and ¬B. E has to be jointly consistent (ie. according to Condition 4, it has to be consistent with the context C), whereas ¬B has to be individually consistent (ie. according to Condition 5, it has to be consistent with the extension E). Since this is fulfilled, we obtain the above contextual extension satisfying our consistency requirements.

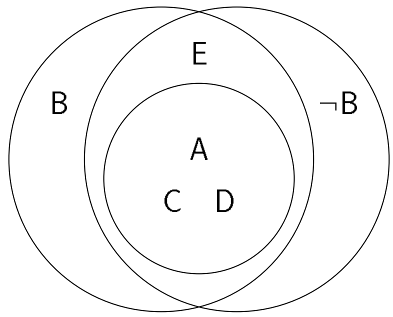

Figure 5: The contextual extension of the preceding contextual default theory.

The preceding contextual extension is illustrated in Figure 5. The extension E is given by the innermost circle, whereas the context C is represented by the two larger outermost circles. Each such circle stands for a deductively closed set of formulas.

Let us examine in detail the context of the contextual extension of (10). The context C is composed of two incompatible subcontexts,

$$Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{B}\}) \qquad \text{and} \qquad Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \neg\mathsf{B}\}).$$

The first subcontext is represented by the left outermost circle in Figure 5, whereas the second one is given by the right outermost circle in the same figure. All such subcontexts contain a common "kernel" given by the extension and all jointly consistent C-justifications, here $Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}\})$ and E. That is, the common kernel is given by

$$Th(\{\mathsf{A}, \mathsf{C}, \mathsf{D}, \mathsf{E}\}).$$

In Figure 5, the kernel consists of the intersection of the two outermost circles.

As mentioned above, incompatible subcontexts arise whenever we deal with inconsistent individual consistency requirements. In this example, the two incompatible subcontexts are originated by the E-justifications B and $\neg\mathsf{B}$. But why is the joint consistency of E not affected by these two contradictory formulas? This is because in our approach joint consistency only requires the consistency of a justification with each subcontext in turn, whereas individual consistency requires the consistency of a justification with at least one such subcontext.

Now, let us further illustrate our approach by returning to our initial examples.

First, let us look at the "broken arms" example. We have argued in the introductory section that this example is best accomplished when using two distinct contexts which in turn lead to two alternative extensions. Furthermore, we have observed in Section 2 that distinct contexts are caused by inconsistent joint consistency requirements. In fact, the "broken arms" example is handled in an intuitive manner in constrained default logic which actually treats justifications as joint consistency requirements.

All this suggests the use of joint consistency assumptions as a formal account of the implicit assumptions present in the "broken arms" example. This gives rise to the following contextual default theory.

$$\left(\left\{\frac{||:\ \neg\mathsf{Bl}\,||}{\mathsf{Ul}}, \frac{||:\ \neg\mathsf{Br}\,||}{\mathsf{Ur}}\right\}, \{\mathsf{Bl} \vee \mathsf{Br}\}\right) \tag{13}$$

As expected, this contextual default theory yields two alternative contextual extensions:

$$(\mathit{Th}(\{\mathsf{Bl} \vee \mathsf{Br}, \mathsf{Ul}\}), \mathit{Th}(\{\neg\mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}\})) \quad \text{and} \quad (\mathit{Th}(\{\mathsf{Bl} \vee \mathsf{Br}, \mathsf{Ur}\}), \mathit{Th}(\{\mathsf{Bl}, \neg\mathsf{Br}, \mathsf{Ur}\}))$$

These two contextual extensions are illustrated in Figure 6. As regards the first contex-

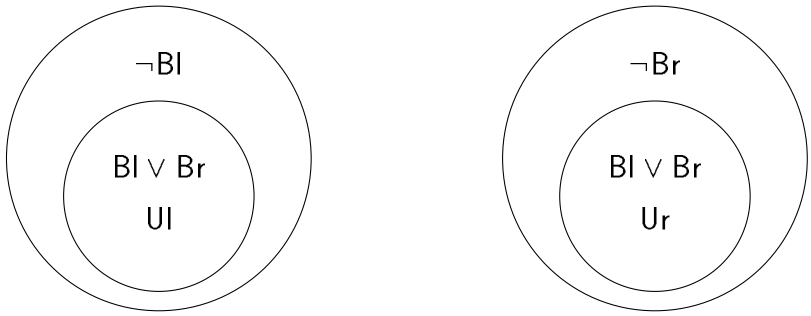

Figure 6: The contextual extension in the "broken-arms" example.

tual extension, we believe that the left arm is usable, Ul, while assuming that it is intact, $\neg\mathsf{Bl}$. In the second contextual extension, the same holds for the right arm. Obviously, these two extensions are identical to the ones obtained in constrained default logic. We will formally account for this relationship in Theorem 7.3 in Section 7. Moreover, the two contexts are deductively closed and they coincide with the ones given in (2) and (3) for constrained default logic. As we will see next, however, contexts are not necessarily deductively closed.

So, let us turn our attention to the "holidays" example. In the introductory section, we have suggested to deal with this example by means of a wider context gathering two incompatible subcontexts. Such a context would then direct our reasoning to a single extension.

On the other hand, we have seen above that subcontexts are originated by individual consistency requirements. Moreover, we have observed in Section 2 that classical default logic yields the more appealing result in the case of the "holidays" example. In fact, classical default logic treats justifications as individual consistency requirements.

All this puts forward the use of individual consistency assumptions for formalizing the "holidays" example in contextual default logic. This results in the following contextual default theory.

$$\left(\left\{\frac{||:|\,\mathsf{W}\,|}{\mathsf{T}}, \frac{||:|\,\mathsf{C}\,|}{\mathsf{S}}\right\}, \{\neg\mathsf{W} \vee \neg\mathsf{C}\}\right) \tag{14}$$

In analogy to classical default logic, we obtain one contextual extension:

$$(\mathit{Th}(\{\neg\mathsf{W} \vee \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\}), \mathit{Th}(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\}) \cup \mathit{Th}(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}\}))$$

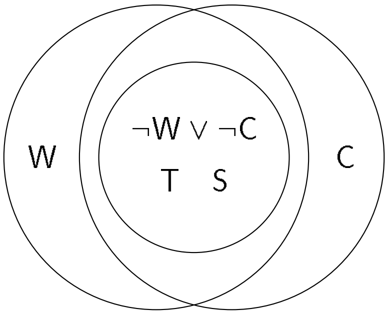

Figure 7: The contextual extension in the "holidays" example.

This contextual extension is illustrated in Figure 7. In fact, the actual extension $Th(\{\neg\mathsf{W} \vee \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\})$ coincides with the extension obtained in classical default logic. That is, both classical and contextual default logic advice us to take both a T-shirt and a sweater on our trip to Vancouver. Moreover, the context $Th(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\}) \cup Th(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}\})$ corresponds to the one given in (5) for classical default logic. In this way, contextual default logic explicates the context directing the reasoning process in classical default logic. That is, in both systems, we deal with a context consisting of two incompatible subcontexts. The first subcontext $Th(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}\})$ is created by the E-justification, W, of the first contextual default rule, whereas the second subcontext $Th(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}\})$ is initiated by the E-justification, C, of the second contextual default rule. The former subcontext representing the warm weather scenario is given by the left outermost circle in Figure 7, whereas the right outermost circle in Figure 7 stands for the latter embodying the cold weather scenario.

Finally, contextual default logic allows for combining the two previous examples. Merging the corresponding contextual default theories results in the following one.

$$\left(\left\{ \begin{array}{c} \frac{|\,|:\neg\mathsf{Bl}\,|\,|}{\mathsf{Ul}}, \frac{|\,|:\neg\mathsf{Br}\,|\,|}{\mathsf{Ur}} \\ \frac{|\,|:|\,\mathsf{W}\,|}{\mathsf{T}}, \frac{|\,|:|\,\mathsf{C}\,|}{\mathsf{S}} \end{array} \right\}, \left\{ \begin{array}{c} \mathsf{Bl} \vee \mathsf{Br} \\ \neg\mathsf{W} \vee \neg\mathsf{C} \end{array} \right\} \right) \tag{15}$$

With this contextual default theory, we arrive at the so-called "holidays with broken arms" example. Notably, this example cannot be treated adequately by any existing default logic. This is so because no existing default logic is able to account for the variety of contexts (or implicit consistency assumptions, respectively) that is needed for an appropriate treatment of this example.

The two contextual extensions of contextual default theory (15) are given in Figure 8; thereby, omitting the set of facts $\{\mathsf{Bl} \vee \mathsf{Br}, \neg\mathsf{W} \vee \neg\mathsf{C}\}$. The first extension

$$Th(\{\neg\mathsf{W} \vee \neg\mathsf{C}, \mathsf{Bl} \vee \mathsf{Br}, \mathsf{T}, \mathsf{S}, \mathsf{Ul}\}) \tag{16}$$

is formed while reasoning in the context

$$Th(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}, \neg\mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}\}) \cup Th(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}, \neg\mathsf{Bl}, \mathsf{Br}, \mathsf{Ul}\}); \tag{17}$$

and the second extension

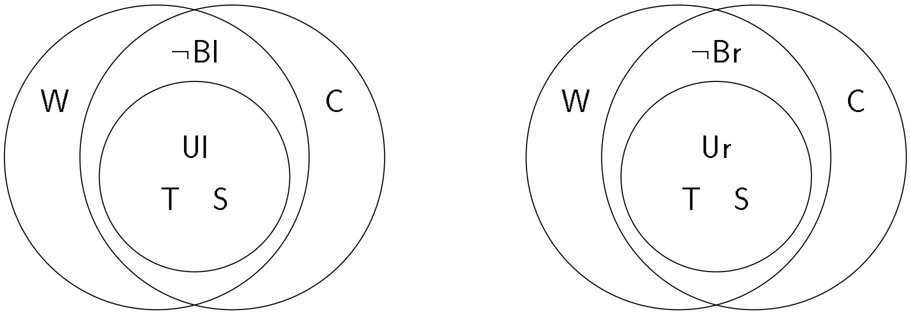

Figure 8: The contextual extension in the "holidays with broken arms" example.

$$Th(\{\neg \mathsf{W} \vee \neg \mathsf{C}, \mathsf{Bl} \vee \mathsf{Br}, \mathsf{T}, \mathsf{S}, \mathsf{Ur}\})$$

is embedded in the context

$$Th(\{\mathsf{W}, \neg\mathsf{C}, \mathsf{T}, \mathsf{S}, \mathsf{Bl}, \neg\mathsf{Br}, \mathsf{Ur}\}) \cup Th(\{\neg\mathsf{W}, \mathsf{C}, \mathsf{T}, \mathsf{S}, \mathsf{Bl}, \neg\mathsf{Br}, \mathsf{Ur}\}).$$

Both extensions direct us to take both a T-shirt and a sweater on our trip to Vancouver. However, in the first extension, we believe that the left arm is usable, while we believe the same for the right arm in the second extension.

Now, let us detail the case of the first contextual extension. This extension is obtained by applying the contextual default rules

$$\frac{|| : \neg\mathsf{Bl}\,||}{\mathsf{Ul}}, \qquad \frac{|| : |\,\mathsf{W}\,|}{\mathsf{T}} \qquad \text{and} \qquad \frac{|| : |\,\mathsf{C}\,|}{\mathsf{S}}. \tag{18}$$

The C-justification of the first contextual default rule, $\neg\mathsf{Bl}$, is a joint consistency requirement. Therefore, it belongs to each subcontext. Each subcontext in turn is originated by the E-justifications of the last two contextual default rules, namely W and C. They have to be individually consistent which is ensured by the respective subcontext that they have in common with the extension at hand.

In the same way, we obtain the second contextual extension.

We present alternative characterizations of contextual extensions and describe their structure in more detail in Section 5. In Section 7, we give general translation schemata for default theories in existing default logics into fragments of contextual default logic. That is, we prove that general default theories in existing default logics correspond to restricted fragments of contextual default logic. Section 7 therefore deals with the application conditions usually used in default logics, namely Conditions *2*, *4* and *5* in Definition 3.2. The remaining ones are discussed in Section 4.

4 Expressiveness of contextual default logic

This section is devoted to the novel application conditions of contextual default rules and how their interplay may influence the contents of extensions. The more "classical" conditions found in existing default logics are discussed in Section 7.

Let us first consider the difference between W- and E-prerequisites of contextual default rules. In general, W-prerequisites should be preferred over E-prerequisites whenever a prerequisite has to be verified, ie. whenever it should not be derivable by default inferences. This cannot be modeled in existing default logics, since they do not distinguish between first-order and default conclusions.

As an example, consider the assertion "usually, we can transplant an organ provided that the person is proven to be dead". Of course, the antecedent of this rule should be more than merely concluded by default. For instance, a person whose body is fully covered with a medical blanket is usually dead, but it takes more evidence for doctors to remove organs. Now, the above rule can be formalized by means of the contextual default rule

$$\frac{\mathsf{D}\,||:\,\mathsf{O}\,||}{\mathsf{O}}, \tag{19}$$

saying that an organ, O, can be transplanted, if this is consistent with the current context, and provided that the death, D, of the person has been verified. Importantly, adding the contextual default rule

$$\frac{|\,\mathsf{C}\,|:|\,\mathsf{D}\,|}{\mathsf{D}}$$

(saying that a person whose body is covered, C, with a blanket is usually dead, D) does not allow $\frac{\mathsf{D}\,||:\,\mathsf{O}\,||}{\mathsf{O}}$ to apply, even in the case where $W = Th(\{\mathsf{C}\})$.

C-prerequisites are a means for weakening antecedents of default rules. This is because a C-prerequisite allows us not only to refer to default conclusions but also to their underlying consistency assumptions: A C-prerequisite is satisfied iff it belongs to some subcontext. Accordingly, certain contextual default rules can only be applied if a certain context has been established. For instance, a contextual default rule $\frac{||:|\,\mathsf{A}\,|}{\mathsf{B}}$ may establish, without actually asserting, a consistency assumption A on which other contextual default rules, like

$$\frac{||\,\mathsf{A}\,:|\,\mathsf{D}\,|}{\mathsf{D}}, \tag{20}$$

rely. This can be useful for modularizing contextual default theories. That is, a set of contextual default rules can be supplied with a certain C-prerequisite A. In this way, these contextual default rules only apply if a subcontext containing A has been established. All this rests in the context and does not enter the actual extension.

In addition, C-prerequisites can be advantageous from the computational point of view. For instance, one can factorize frequently occurring justifications. Consider the contextual default rules $\frac{||:|\,\mathsf{A}\,|}{\mathsf{B}}$, $\frac{|\,\mathsf{B}\,|:|\,\mathsf{A}\,|}{\mathsf{C}}$, and $\frac{|\,\mathsf{C}\,|:|\,\mathsf{A}\,|}{\mathsf{D}}$. Now, proving D in a straightforward way requires that we check the consistency of A three times. This effort for checking consistency reduces to one if we turn each E-justification A into a C-prerequisite and add the contextual default rule $\frac{||:|\,\mathsf{A}\,|}{\top}$. This results in the following contextual default rules.

$$\frac{||:|\,\mathsf{A}\,|}{\top}, \frac{||\,\mathsf{A}\,:||}{\mathsf{B}}, \frac{|\,\mathsf{B}\,|\,\mathsf{A}\,:||}{\mathsf{C}}, \frac{|\,\mathsf{C}\,|\,\mathsf{A}\,:||}{\mathsf{D}}$$

Now, we can prove D with one consistency check and three tests for membership in a subcontext. In this respect, C-prerequisites can serve as a "resource-driven" consistency condition.

Zaverucha takes up the idea of C-prerequisites in [22] in order to address certain anomalous phenomena discussed in the default logic literature. See [22] for details.

Let us now turn to the difference between C- and E-justifications of contextual default rules. As we have seen in (9) in Section 3, C-justifications provide stronger conditions that E-justifications. Interestingly, this can serve for imposing some priorities between two implicit assumptions. This cannot be modeled easily in existing default logics.

Let us consider again the "broken-arms" example. Suppose now that experience shows that the left arm is more reliable than the right one. As a consequence, we might want to focus on scenarios in which the left arm is not broken (without actually asserting this heuristic information within the facts). This amounts to establishing a precedence among the two implicit assumptions $\neg\mathsf{Bl}$ and $\neg\mathsf{Br}$ in contextual default theory (13). Such a direct precedence can be modeled in a very straightforward way by weakening the implicit assumption $\neg\mathsf{Bl}$ wrt $\neg\mathsf{Br}$. In concrete terms, we have to weaken the justification of the default rule in (13) in which $\neg\mathsf{Bl}$ occurs by turning it from a C- into an E-justification. This results in the following contextual default theory:

$$\left(\left\{\frac{||:|\neg\mathsf{Bl}|}{\mathsf{Ul}}, \frac{||:\ \neg\mathsf{Br}||}{\mathsf{Ur}}\right\}, Th(\{\mathsf{Bl}\vee\mathsf{Br}\})\right)$$

This yields the unique contextual extension $(Th(\{\mathsf{Bl}\vee\mathsf{Br},\mathsf{Ul}\}), Th(\{\neg\mathsf{Bl},\mathsf{Br},\mathsf{Ul}\}))$. It corresponds to the first contextual extensions of contextual default theory (13) saying that the left arm is usable, Ul, while assuming that it is not broken, $\neg\mathsf{Bl}$.

The use of W-justifications is closely related to CWA, the closed world assumption [17]. CWA has been introduced in order to complete a given set of facts W. In CWA, a ground negative literal is derivable iff the original atom is not derivable from W. Considering a database about taxpayers, for instance, an individual is not a dead person unless stated otherwise. This is the weakest application condition in (9). Given no other knowledge about an individual, we derive that he is not dead. This can be modeled by means of the contextual default rule

$$\frac{||:||\neg\mathsf{D}}{\neg\mathsf{D}}. \tag{21}$$

The use of W-justifications is much closer to the original formulation of CWA than the usual formalization of CWA by means of E-justifications (cf. [18]). As an example, consider

$$\mathsf{Bl}\vee\mathsf{Br}.$$

Applying the CWA to this disjunction yields an inconsistent set of formulas. This is so because neither Br nor Bl is derivable from $\mathsf{Bl}\vee\mathsf{Br}$. Hence we add both $\neg\mathsf{Br}$ and $\neg\mathsf{Bl}$ to the resultant theory under CWA. This renders the resulting theory inconsistent.

Formalizing the above closed-world reasoning by means of W-justifications leads us to the following contextual default theory.

$$\left(\left\{\frac{||:||\neg\mathsf{Bl}}{\neg\mathsf{Bl}},\frac{||:||\neg\mathsf{Br}}{\neg\mathsf{Br}}\right\},Th(\{\mathsf{Bl}\vee\mathsf{Br}\})\right)$$

This theory yields an inconsistent contextual extension, $(Th(\{\bot\}),Th(\{\bot\}))$. Notice that this result is in accord with CWA but differs from the usual approach of default logic, where we obtain an inconsistent extension iff the facts are inconsistent.

Formalizing our example by means of E-justifications yields the contextual default theory

$$\left(\left\{\frac{||:|\neg\mathsf{Bl}|}{\neg\mathsf{Bl}},\frac{||:|\neg\mathsf{Br}|}{\neg\mathsf{Br}}\right\},Th(\{\mathsf{Bl}\vee\mathsf{Br}\})\right)$$

having two distinct contextual extensions, $(Th(\{\mathsf{Bl}\wedge\neg\mathsf{Br}\}),Th(\{\mathsf{Bl}\wedge\neg\mathsf{Br}\}))$ and $(Th(\{\neg\mathsf{Bl}\wedge\mathsf{Br}\}),Th(\{\neg\mathsf{Bl}\wedge\mathsf{Br}\}))$. Even though this formalization does not comply with CWA, it has the advantage of being consistency-preserving.

In general, we can implement the consistency preservation of W-justifications by adding tautological E-justifications. In this way, we enforce that the given contextual default rule merely contributes to consistent extensions. Consider the contextual default theory

$$\left(\left\{\frac{||:|\top|\neg\mathsf{Bl}}{\neg\mathsf{Bl}},\frac{||:|\top|\neg\mathsf{Br}}{\neg\mathsf{Br}}\right\},Th(\{\mathsf{Bl}\vee\mathsf{Br}\})\right).$$

This theory has no contextual extension, as opposed to the last but one theory, where we obtained an inconsistent contextual extension.

5 Contextual default logic: The formal theory

In the sequel, we give alternative characterizations of contextual extensions and describe their structure in more detail. First, we define the set of generating contextual default rules.

Definition 5.1 *Let (D,W) be a contextual default theory and T and S sets of formulas. The set of generating contextual default rules for (T,S) wrt (D,W) is defined as*

$$GD^{(T,S)}_{(D,W)}=\left\{\frac{\alpha_W\,|\,\alpha_E\,|\,\alpha_C\;:\;\beta_C\,|\,\beta_E\,|\,\beta_W}{\gamma}\in D\;\middle|\;\begin{array}{lll}\alpha_W\in W, & \alpha_E\in T, & \alpha_C\in S,\\ \neg\beta_C\notin S, & \neg\beta_E\notin T, & \neg\beta_W\notin W\end{array}\right\}$$

In fact, the generating contextual default rules allow for characterizing contextual extensions, as we will see in Theorem 5.1. In particular, we can now make precise the claim made before Definition 3.2: In a contextual extension (E,C), the set E is deductively closed and the set C is pointwisely closed.

Theorem 5.1 *Let* (E,C) *be a contextual extension of* (D,W) *and* $\Delta = GD^{(E,C)}_{(D,W)}$. *Then,*

$$Th(W \cup Conseq(\Delta)) = Th(E) = E$$
$$Th_{E \cup Justif_C(\Delta)}(Justif_E(\Delta)) = Th_{\top}(C) = C.$$

The first equation shows that extensions of contextual default theories are formed in the same way as in existing default logics. That is, they consist of the initial facts along with the consequents of all applying contextual default rules. The second equation describes the respective contexts. A context is the pointwise closure of the E-justifications of the applying contextual default rules (corresponding to the individual consistency requirements) under the extension and the C-justifications of the applying contextual default rules (corresponding to the joint consistency requirements). It follows that whenever (E,C) is a contextual extension, C contains the deductive closure of E and all formulas involved in joint consistency requirements. In symbols, $Th\Big(E \cup Justif_C\Big(GD^{(E,C)}_{(D,W)}\Big)\Big) \subseteq C$. Since this set is shared by all subcontexts of a context, we call it the *kernel* of a context. With it, a context can alternatively be described as the pointwise closure of the E-justifications under the kernel.

As an example, let us consider the first contextual extension obtained in the "holidays with broken arms" example, which is presented in (16) and (17). The generating contextual default rules for this contextual extension are given in (18). With these, we can characterize the aforementioned contextual extension as follows. The extension, E, presented in (16), is given by

$$Th(\{\neg \mathsf{W} \vee \neg \mathsf{C}, \mathsf{Bl} \vee \mathsf{Br}\} \cup \{\mathsf{T}, \mathsf{S}, \mathsf{Ul}\}),$$

where $\{\neg \mathsf{W} \vee \neg \mathsf{C}, \mathsf{Bl} \vee \mathsf{Br}\}$ is the set of facts and $\{\mathsf{T}, \mathsf{S}, \mathsf{Ul}\}$ is the set of consequents of the contextual default rules in (18). The context in (17) equals

$$Th_{E \cup \{\neg \mathsf{Bl}\}}(\{\mathsf{W}, \mathsf{C}\}),$$

where $\{\neg \mathsf{Bl}\}$ is the set of C-justifications and $\{\mathsf{W}, \mathsf{C}\}$ is the set of E-justifications of the contextual default rules in (18). Accordingly, the corresponding kernel is given by

$$Th(E \cup \{\neg \mathsf{Bl}\}),$$

which is a deductively closed set of formulas. Also, the extension E is deductively closed, whereas its context is merely pointwisely closed.

In fact, the concept of a kernel provides a meaningful notion in contextual default logic. In the same way as subcontexts are "spanned" by individual E-justifications, a kernel is jointly "spanned" by a set of C-justifications. Thus, intuitively, a kernel provides a syntactic counterpart to the notion of joint consistency, whereas subcontexts are related to individual consistency, as discussed in Section 3. Moreover, we will see in Section 6 that both the concept of a kernel and that of a subcontext have strong semantical underpinnings. But apart from its structural properties, a kernel, say C_K, can be used for defining further application conditions supplementing those given in in (9) in Section 3. That is, for a formula φ, we obtain

$$\varphi \in C_K \qquad \text{and} \qquad \neg\varphi \notin C_K. \tag{22}$$

By referring to the kernel C_K of a context C, both conditions refer to the part of C that is jointly consistent. In this way, the conditions in (22) discard all E-justifications in C. Since $W \subseteq E \subseteq C_K \subseteq C$, we can thus extend equation (9) in the following way:

$$\varphi \in W > \varphi \in E > \varphi \in C_K > \varphi \in C > \neg\varphi \notin C > \neg\varphi \notin C_K > \neg\varphi \notin E > \neg\varphi \notin W \tag{23}$$

The next characterization is similar to the ones usually given to characterize extensions in existing default logics in an inductive format. For clarity, we have factorized the set of contextual default rules applied at each stage.

Theorem 5.2 *Let (D, W) be a contextual default theory and let E and C be sets of formulas. Define*

$$E_0 = W, \qquad C_0 = W$$

and for $i \geq 0$

$$\begin{aligned}
\Delta_i &= \left\{ \frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \;:\; \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D \;\middle|\; \begin{matrix} \alpha_W \in W, & \alpha_E \in E_i, & \alpha_C \in C_i, \\ \neg\beta_C \notin C, & \neg\beta_E \notin E, & \neg\beta_W \notin W \end{matrix} \right\} \\
E_{i+1} &= Th(W \cup Conseq(\Delta_i)) \\
C_{i+1} &= Th_{W \cup Conseq(\Delta_i) \cup Justif_C(\Delta_i)}(Justif_E(\Delta_i))
\end{aligned}$$

Then, (E,C) is a contextual extension of (D,W) iff $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

The extension E is built by successively introducing the consequents of all applying contextual default rules. Also, the deductive closure is computed at each stage. For each partial context C_{i+1}, the partial extension E_{i+1} is unioned with the C-justifications of all applying contextual default rules. This set is unioned in turn with each E-justification of all applying contextual default rules. Again, the deductive closure is computed when appropriate. In this way, each partial context C_{i+1} is built upon its kernel, $Th(E_{i+1} \cup Justif_C(\Delta_i))$.

In order to illustrate this characterization, let us turn our attention to the contextual default theory given in (10). As described in Section 3, the resulting contextual extension, given in (11) and (12), is obtained from the facts, $Th(\{\mathsf{A}\})$ by applying the two given contextual default rules one after another. That is,

$$\Delta_0 = \left\{ \frac{\mathsf{A}\,||\,:\,|\,\mathsf{B}\,|}{\mathsf{C}} \right\}, \quad \Delta_1 = \left\{ \frac{\mathsf{A}\,||\,:\,|\,\mathsf{B}\,|}{\mathsf{C}}, \frac{|\,\mathsf{C}\,|\,:\,\mathsf{E}\,|\,\neg\mathsf{B}\,|}{\mathsf{D}} \right\}, \text{ and } \quad \Delta_i = \Delta_1 \text{ for } i \geq 2.$$

Accordingly, we obtain

$$E_1 = Th(\{\mathsf{A}\} \cup \{\mathsf{C}\}) \qquad \text{and} \qquad C_1 = Th_{\{\mathsf{A}\} \cup \{\mathsf{C}\} \cup \emptyset}(\{\mathsf{B}\})$$

and in turn

$$E_2 = Th(\{\mathsf{A}\} \cup \{\mathsf{C},\mathsf{D}\}) \qquad \text{and} \qquad C_2 = Th_{\{\mathsf{A}\}\cup\{\mathsf{C},\mathsf{D}\}\cup\{\mathsf{E}\}}(\{\mathsf{B},\neg\mathsf{B}\}).$$

Obviously, E_i and C_i correspond to the final contextual extension given in (11) and (12) for $i \geq 2$.

Finally, it is worth mentioning that the specification of Δ_i in Theorem 5.2 leaves open even more application conditions than given in (23). First of all, Δ_i could be modified in order to refer to the final extension and its context. This amounts to the conditions

$$\phi \in E \qquad \text{and} \qquad \phi \in C.$$

These conditions enable "ungrounded" beliefs which are in accord with reasoning patterns found in autoepistemic logic [15]. Second, we could modify the statement for Δ_i in Theorem 5.2 in order to check consistency wrt to partial extensions and their contexts:

$$\neg\phi \notin E_i \qquad \text{and} \qquad \neg\phi \notin C_i$$

These conditions have a procedural flavor since they refer to a certain stage of the construction. However, they could turn out to be useful for approximate reasoning.

Of course, we could continue by defining "autoepistemic" and "procedural" versions of the application conditions referring to the kernel given in (22). For clarity, however, we do not pursue this inquiry any further and rather close this section by summarizing the aforementioned conditions along with their strength.

We use the notation given in Theorem 5.2 for referring to the conditions given in (23). Accordingly, we rewrite the conditions given in (22) in the flavor of Theorem 5.2 by $\alpha_{C_K} \in C_{K_i}$ and $\neg\beta_{C_K} \notin C_K$, where $C_{K_i} = Th(W \cup Conseq(\Delta_i) \cup Justif_C(\Delta_i))$. Then, we can relate the strength of the preceding conditions to our former ones as follows:

$$\begin{aligned}\alpha_W \in W > \alpha_E \in E_i > \left\{ \begin{array}{c} \alpha_{C_K} \in C_{K_i} > \alpha_C \in C_i \\ \phi \in E \end{array} \right\} > \phi \in C > \\ > \neg\beta_C \notin C > \left\{ \begin{array}{c} \neg\beta_{C_K} \notin C_K > \neg\beta_E \notin E \\ \neg\phi \notin C_i \end{array} \right\} > \neg\phi \notin E_i > \neg\beta_W \notin W\end{aligned}$$

The braces indicate that the respective conditions are not comparable.

6 A possible worlds semantics

In contextual default logic, an extension is supplied with a context which may be composed of several incompatible subcontexts. These subcontexts share a common kernel given by the deductive closure of the extension and all formulas relative to joint consistency requirements. However, they differ regarding formulas relative to individual consistency requirements.

In analogy to [3], we employ Kripke structures [4] in order to characterize contextual extensions. A Kripke structure has a distinguished world, the "actual" world, and a set of worlds accessible from it (each world is associated with a first-order interpretation).

Then, the idea is roughly as follows. In a class of Kripke structures, the actual worlds characterize an extension, whereas the accessible worlds characterize its context consisting of a number of subcontexts. In concrete terms, given a contextual extension (E,C) and a Kripke structure m, we require that the actual world ω_0 of m be a model of the extension, E, and demand that each world in m accessible from ω_0 be a model of some subcontext of C. Thus, each world of m accessible from the actual world ω_0 is to be a model of the kernel of C.

We follow the definitions in [4] of a Kripke structure (called K-model in the sequel) as a quadruple $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, where Ω is a non-empty set (also called a set of worlds), $\omega_0 \in \Omega$ a distinguished world, $\mathcal{R}$ a binary relation on Ω (also called the accessibility relation) and $\mathcal{I}$ is a function that defines a first order interpretation $\mathcal{I}_\omega$ for each $\omega \in \Omega$. As usual, a K-model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$ is such that the domain of $\mathcal{I}_\omega$ is a subset of the domain of $\mathcal{I}_{\omega'}$ whenever $(\omega, \omega') \in \mathcal{R}$.

Formulas in K-models are interpreted using a language enriched in the following way: In a K-model $\langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, for each $\omega \in \Omega$, the first order interpretation $\mathcal{I}_\omega$ is extended so that for each $e \in D_\omega$ (the domain of $\mathcal{I}_\omega$), a constant $\overline{e}$ is introduced, letting $\mathcal{I}_\omega(\overline{e}) = e$. In every world ω, each term is mapped into an element of D_ω as follows:

$$\mathcal{I}_\omega(f(t_1, \ldots, t_n)) = (\mathcal{I}_\omega(f))\,(\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n))\,,\ n \geq 0.$$

Given a K-model $m = \langle \omega_0, \Omega, \mathcal{R}, \mathcal{I} \rangle$, the modal entailment relation $\omega \models \alpha$ (in m) is defined by recursion on the structure of α:

$$\begin{array}{lll} \omega \models P(t_1, \ldots, t_n) & \text{iff} & (\mathcal{I}_\omega(t_1), \ldots, \mathcal{I}_\omega(t_n)) \in \mathcal{I}_\omega(P) \\ \omega \models \neg\alpha & \text{iff} & \omega \not\models \alpha \\ \omega \models \alpha \vee \beta & \text{iff} & \omega \models \alpha \text{ or } \omega \models \beta \\ \omega \models \forall x\ \alpha[x] & \text{iff} & \omega \models \alpha[\overline{e}] \text{ for all } e \in D_\omega \\ \omega \models \Box\alpha & \text{iff} & \omega' \models \alpha \text{ whenever } (\omega, \omega') \in \mathcal{R} \end{array}$$

We write $m \models \alpha$ if $\omega_0 \models \alpha$ (in m). This means that m is a model of α. We denote classes of K-models by M. We extend the modal entailment relation $\models$ to classes of K-models M and write $M \models \alpha$ to mean that each element in M (that is, a K-model) entails α.

First, we define the class of K-models associated with W as $M_W = \{m \mid m \models \gamma \wedge \Box\gamma, \gamma \in W\}$. We will semantically characterize contextual extensions by maximal elements of a strict partial order on classes of K-models. Given a contextual default rule δ, its application conditions and the result of applying it are captured by an order $>_\delta$ as follows.

Definition 6.1 *Let* $\delta = \frac{\alpha_W | \alpha_E | \alpha_C : \beta_C | \beta_E | \beta_W}{\gamma}$. *Let* M *and* M' *be distinct classes of* K*-models. We define* $M >_\delta M'$ *iff*

$$M = \{m \in M' \mid m \models \gamma \wedge \Box\gamma \wedge \Box\beta_C \wedge \Diamond\beta_E\}$$

and

1. $M_W \models \alpha_W$
2. $M' \models \alpha_E$
3. $M' \models \Diamond\alpha_C$
4. $M' \not\models \Diamond\neg\beta_C$
5. $M' \not\models \neg\beta_E$
6. $M_W \not\models \neg\beta_W$

Given a set of contextual default rules D, the strict partial order $>_D$ is defined as the transitive closure of the union of all orders $>_\delta$ such that $\delta \in D$.

Conditions *1-6* in the preceding definition constitute the semantical counterparts of Conditions *1-6* in Definition 3.2. Thus, they semantically capture the application conditions of a contextual default rule. The correspondence between the first two and last two conditions in Definition 6.1 is obvious, so that we focus on the remaining context-sensitive conditions. Condition *3* accounts for the applicability condition expressing "membership in a context of reasoning". Now, in terms of possible worlds, this boils down to the requirement that all considered K-models possess an accessible world satisfying α_C. According to the aforementioned intuition, this amounts to stipulating that α_C belongs to some subcontext of the context captured by M'. Condition *4* provides (one half of the) semantical underpinnings for the joint consistency requirement of C-justifications. Notice that this condition is equivalent to

$$\exists m \in M'.\ m \models \Box\beta_C. \tag{24}$$

That is, the consistency condition for β_C corresponds semantically to the requirement that there is a K-model in which all accessible worlds satisfy β_C. Hence, following the above intuition, β_C belongs to the kernel of the context induced by all such K-models m.

The other half of the semantical characterization of joint (and individual) consistency is expressed in the specification of M. The K-models in M capture the result of applying a contextual default rule by gathering the default's conclusion along with its underlying consistency assumptions. Hence they provide the semantical counterpart to Conditions *7-9* in Definition 3.2. This is accomplished by enforcing the satisfiability of the consequent γ in all actual as well as in all worlds accessible from the actual worlds by stipulating $m \models \gamma \wedge \Box\gamma$. The joint consistency of β_C is preserved by requiring that all accessible worlds satisfy β_C, ie. $m \models \Box\beta_C$. In this way, M forms a subset of all models m in (24). The individual consistency of β_E is preserved by enforcing accessible worlds satisfying β_E along with γ and β_C, ie. $m \models \Diamond(\gamma \wedge \beta_C \wedge \beta_E)$. Roughly speaking, such accessible worlds capture the subcontexts "spanned" by E-justifications like β_E (cf. Theorem 6.1 below).

Now, let us give a detailed example illustrating the main idea. Consider contextual default theory (10)

$$\left(\left\{\frac{\mathsf{A}\,||\,:\,|\mathsf{B}|}{\mathsf{C}},\frac{|\mathsf{C}|\,:\,\mathsf{E}|\neg\mathsf{B}|}{\mathsf{D}}\right\}, Th(\{\mathsf{A}\})\right)$$

along with its contextual extensions described in (11) and (12). In what follows, let δ_1 and δ_2 stand for the first and the second contextual default rule.

In this example, M_W is the class of all K-models satisfying A and $\Box\mathsf{A}$. Starting from M_W, we obtain a $>_{\delta_1}$-greater class M such that

$$M \models (\mathsf{A}\wedge\mathsf{C})\wedge\Box(\mathsf{A}\wedge\mathsf{C})\wedge\Diamond\mathsf{B}.$$

That is, $M >_{\delta_1} M_W$. This is so because $M_W \models \mathsf{A}$ according to Condition *1* in Definition 6.1 and $M_W \not\models \neg\mathsf{B}$ according to Condition *5*. On the other hand, $M_W \not\models \mathsf{C}$ thus violating Condition *2* in the case of the second contextual default rule. Therefore, there is no $>_{\delta_2}$-greater class of M_W.

However, $M \models \mathsf{C}$; thus establishing Condition *2* for the second contextual default rule relative to M. Since moreover $M \not\models \Diamond\neg\mathsf{E}$ and $M \not\models \mathsf{B}$ confirm Condition *4* and *5*, we obtain a $>_{\delta_2}$-greater class of M, say M'. That is, $M' >_{\delta_2} M$, where

$$M' \models (\mathsf{A}\wedge\mathsf{C}\wedge\mathsf{D})\wedge\Box(\mathsf{A}\wedge\mathsf{C}\wedge\mathsf{D}\wedge\mathsf{E})\wedge\Diamond\mathsf{B}\wedge\Diamond\neg\mathsf{B}.$$

In all, this amounts to a maximal class of K-models, M' above M_W, ie.

$$M' >_{\{\delta_1,\delta_2\}} M_W.$$

According to the intuition given above, M' characterizes the contextual extension of contextual default theory (10) described in (11) and (12). A canonical K-model out of M' is given in Figure 9. This K-model consists merely of three worlds: An actual world (at the bottom) and two worlds accessible from it. Each world is labeled with the least set of formulas commonly entailed by all other actual and accessible worlds of K-models in M'. Compare this figure with Figure 5 illustrating the contextual extension

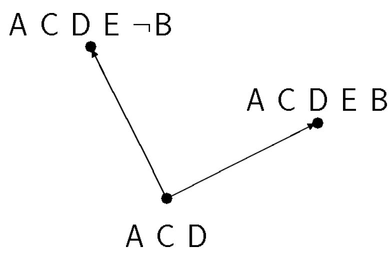

Figure 9: A canonical K-model characterizing a contextual extension.

of contextual default theory (10).

In fact, the non-modal formulas entailed by M', given at the actual world, correspond to the extension given in (11). Notice that

$$M' \models \Diamond(\mathsf{A} \wedge \mathsf{C} \wedge \mathsf{D} \wedge \mathsf{E} \wedge \mathsf{B}) \wedge \Diamond(\mathsf{A} \wedge \mathsf{C} \wedge \mathsf{D} \wedge \mathsf{E} \wedge \neg\mathsf{B}).$$

These two modal formulas capture the two subcontexts forming a wider common context given in (12). Pictorially, they correspond to the two accessible worlds in Figure 9.

In general, we obtain the following soundness and completeness result that make precise the intuition given at the start of this section.[10]

Theorem 6.1 *Let (D, W) be a contextual default theory. Let M be a class of K-models, E a deductively closed set of formulas, C a pointwisely closed set of formulas such that*

$$M = \{m \mid m \models E \wedge \Box C_K \wedge \Diamond C_J\}$$

for

$$C_K = Th\Big(E \cup Justif_C\Big(GD^{(E,C)}_{(D,W)}\Big)\Big) \qquad and \qquad C_J = Justif_E\Big(GD^{(E,C)}_{(D,W)}\Big).$$

Then, (E,C) is a consistent contextual extension of (D, W) iff M is a $>_D$-maximal non-empty class above M_W.

Observe that the requirements on a maximal class of K-models correspond to the aforementioned intuitions. Clearly, E is the extension, C the context, C_K the kernel and C_J consists of E-justifications distinguishing the subcontexts from each other.

Let us illustrate this by means of the first contextual extension obtained in the "holidays with broken arms" example, which is presented in (16) and (17). In this case, M_W is given by the class of all K-models satisfying $\neg\mathsf{W} \vee \neg\mathsf{C}$, $\mathsf{Bl} \vee \mathsf{Br}$ and $\Box(\neg\mathsf{W} \vee \neg\mathsf{C})$, $\Box(\mathsf{Bl} \vee \mathsf{Br})$. By means of the generating contextual default rules listed in (18), say δ_1, δ_2 and δ_3, we obtain a maximal class of K-models M such that $M >_{\{\delta_1,\delta_2,\delta_3\}} M_W$ and

$$\begin{array}{lll} M & \models & ((\neg\mathsf{W} \vee \neg\mathsf{C}) \wedge (\mathsf{Bl} \vee \mathsf{Br}) \wedge \mathsf{T} \wedge \mathsf{S} \wedge \mathsf{Ul}) \\ & & \Box((\neg\mathsf{W} \vee \neg\mathsf{C}) \wedge (\mathsf{Bl} \vee \mathsf{Br}) \wedge \mathsf{T} \wedge \mathsf{S} \wedge \mathsf{Ul} \wedge \neg\mathsf{Bl}) \\ & & \Diamond\mathsf{W} \wedge \Diamond\mathsf{C} \end{array}$$

As above, we give a canonical K-model out of M in Figure 10. For readability, however, we omit the underlying facts $(\neg\mathsf{W} \vee \neg\mathsf{C}) \wedge (\mathsf{Bl} \vee \mathsf{Br})$ when labeling the three possible worlds. This figure should be compared with the first contextual extension given in Figure 8.

Now, let us decompose the preceding formula in order to isolate the extension E presented in (16), the context C presented in (17), the kernel C_K and the E-justifications C_J. Clearly, E is given by the non-modal formulas entailed by M. These non-modal formulas hold in the actual world in Figure 10. The kernel C_K is given by all formulas that necessarily hold: [11]

[10] Given a set of formulas T let $\Box T$ stand for $\bigwedge_{\alpha\in T}\Box\alpha$ and $\Diamond T$ stand for $\bigwedge_{\alpha\in T}\Diamond\alpha$.

[11] For simplicity, we still omit the facts $(\neg\mathsf{W} \vee \neg\mathsf{C}) \wedge (\mathsf{Bl} \vee \mathsf{Br})$.

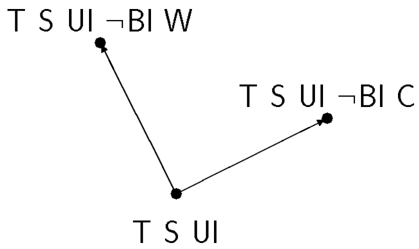

Figure 10: A canonical K-model characterizing a contextual extension.

$$M \models \Box(\mathsf{T} \wedge \mathsf{S} \wedge \mathsf{UI} \wedge \neg\mathsf{BI})$$

They are given by the formulas collectively entailed by the two accessible worlds in Figure 10. Interestingly, the accessible worlds differ exactly in the E-justifications given in C_J; they are possibly entailed by M. To be more precise, we even have

$$M \models \Diamond(\mathsf{T} \wedge \mathsf{S} \wedge \mathsf{UI} \wedge \neg\mathsf{BI} \wedge \mathsf{W}) \wedge \Diamond(\mathsf{T} \wedge \mathsf{S} \wedge \mathsf{UI} \wedge \neg\mathsf{BI} \wedge \mathsf{C}).$$

Now, each of the two modal conjuncts captures one of the two subcontexts forming the wider common context given in (17). In this way, a context is completely described by the set of accessible worlds given in a $>_D$-maximal class of K-models, whereas an extension is captured by the set of actual worlds in the same class of K-models.

Finally, let us return to the application conditions given in Section 5 for exploiting the kernel C_K of a context, namely $\varphi \in C_K$ and $\neg\varphi \notin C_K$. In fact, these conditions have the following semantical counterparts:

$$M \models \Box\varphi \qquad \text{and} \qquad M \not\models \Box\neg\varphi$$

The correspondence between the first conditions is obvious, so that we focus on the consistency condition. The latter is equivalent to

$$\exists m \in M.\ m \models \Diamond\varphi.$$

Recall that the consistency condition $\neg\varphi \notin C_K$ ignores all individual subcontexts by focusing on their common kernel. Suppose that the context induced by M contains a subcontext "spanned" by an E-justifications $\neg\varphi$, ie. $M \models \Diamond\neg\varphi$. Given no other knowledge, this does not falsify the condition $M \not\models \Box\neg\varphi$, so that the original consistency condition $\neg\varphi \notin C_K$ is satisfied and the aforementioned E-justifications is ignored. Observe that this does not apply to the consistency condition for C-justifications which takes into account the entire context. That is, if M satisfies $\Diamond\neg\varphi$ for some E-justifications $\neg\varphi$ there is no way to satisfy Condition *3* of Definition 6.1.

7 Embedding default logics

Since its introduction in [18], several variants of Reiter's original default logic have been proposed, eg. [11, 5, 6, 20, 7]. Each such variant rectified purportedly counterintuitive features of the original approach. However, the evolution of default logic is diverging. Although it has resulted in diverse variants sharing many interesting properties, it has altered the notion of a default rule. In particular, most of the aforementioned variants deal with a different notion of consistency. As we have seen in Section 2, Reiter's default logic employs some sort of local consistency, whereas others, like constrained default logic, employ some sort of global consistency.

Up to now, we were compelled to choose one among the respective variants whenever we want to represent default knowledge. At first sight, this seems to be a good solution, since we may select one of the variants depending on its properties. However, our choice fixes the notion of a default rule. More freedom would be desirable: We should not be forced to commit ourselves to just a single variant of default logic, because all facets of default logic are worth considering.

Fortunately, contextual default logic provides an integrated approach, which allows for embedding existing variants of default logic. In this section, we show that classical [18], justified [11] and constrained default logic [20, 7] are embedded in contextual default logic. Since cumulative default logic [5] is closely connected to constrained default logic (cf. [21]), we may obtain that variant as well.

Thus, as a result of our context-based approach to default logic, we obtained a framework for default logics which allows for integrating different variants of default logic in a more general but uniform system; thereby, combining the expressiveness of various default logics along with more traditional approaches, like the closed world assumption.

7.1 Classical default logic

As mentioned in the introductory section, *classical default logic* employs a sort of local consistency (which we also called individual consistency), as can be seen from the following definition of *classical extensions.*

Definition 7.1 *Let (D, W) be a default theory. For any set of formulas T let $\Gamma(T)$ be the smallest set of formulas T' such that*

1. $W \subseteq T'$,
2. $Th(T') = T'$,
3. *For any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in T'$ *and* $\neg\beta \notin T$ *then* $\gamma \in T'$.

A set of formulas E is a classical extension of (D, W) iff $\Gamma(E) = E$.

In order to have a comprehensive example throughout this section, let us consider the following default theory taken from [3]:

$$\left(\left\{\frac{:\mathsf{B}}{\mathsf{C}}, \frac{:\neg\mathsf{B}}{\mathsf{D}}, \frac{:\neg\mathsf{C}\wedge\neg\mathsf{D}}{\mathsf{E}}\right\}, \emptyset\right) \tag{25}$$

This default theory has one classical extension $Th(\{\mathsf{C},\mathsf{D}\})$. The first two default rules apply, although they have contradictory justifications, and then block the third default rule. Each justification of the applying default rules is separately consistent with $Th(\{\mathsf{C},\mathsf{D}\})$. In this way, the extension is somehow embedded into two (implicit) contradictory subcontexts: One containing the extension and the justification of the first default rule, $Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\})$, and another one containing the justification of the second default rule, $Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})$.

In order to relate classical with contextual default logic, let us agree on identifying default theories in classical default logic with contextual default theories as follows.

Definition 7.2 (Classical default logic) *Let (D,W) be a default theory. We define*

$$\Phi_{\mathsf{DL}}(D,W) = \left(\left\{\tfrac{|\alpha|:|\beta|}{\gamma} \;\middle|\; \tfrac{\alpha:\beta}{\gamma} \in D\right\}, Th(W)\right).$$

Then, classical default logic corresponds to this fragment of contextual default logic.

Theorem 7.1 *Let (D,W) be a default theory and let E be a set of formulas and $C_E = \left\{\beta \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}$. Then, E is a classical extension of (D,W) and $C = Th_E(C_E)$ iff (E,C) is a contextual extension of $\Phi_{\mathsf{DL}}(D,W)$.*

Given a classical extension E, the context C is the pointwise closure of the justifications of the generating[12] default rules under E.

Consider the contextual counterpart of default theory (25):

$$\left(\left\{\frac{||:|\mathsf{B}|}{\mathsf{C}}, \frac{||:|\neg\mathsf{B}|}{\mathsf{D}}, \frac{||:|\neg\mathsf{C}\wedge\neg\mathsf{D}|}{\mathsf{E}}\right\}, Th(\emptyset)\right)$$

We obtain one contextual extension

$$(Th(\{\mathsf{C},\mathsf{D}\}), Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\}) \cup Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\}))$$

whose extension corresponds to the classical extension of default theory (25). The common kernel of the two subcontexts of the context is given by the extension. In addition, the first subcontext, $Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\})$), contains the E-justification of the first contextual default rule, whereas the second one, $Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})$), contains additionally the E-justification of the second contextual default rule. As with classical default logic, the third contextual default rule is blocked by the other ones.

7.2 Justified default logic

Further evidence for the generality of our approach is that it can easily capture a variant of default logic due to [11], which we refer to as *justified default logic*. In this approach, the justifications of the applying default rules are attached to extensions in order to strengthen the applicability condition of default rules. A *justified extension*[13] is defined

[12] Informally, the generating default rules are those which apply in view of E.

[13] Originally, Łukaszewicz called his extensions *modified* extensions.

as follows.

Definition 7.3 *Let (D, W) be a default theory. For any pair of sets of formulas (T, S) let $\Psi(T, S)$ be the pair of smallest sets of formulas T', S' such that*

1. $W \subseteq T'$,
2. $Th(T') = T'$,
3. *For any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in T'$ *and* $\forall \eta \in S \cup \{\beta\}.\ T \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ *then* $\gamma \in T'$ *and* $\beta \in S'$.

A pair of sets of formulas (E, J) is a justified extension of (D, W) iff $\Psi(E, J) = (E, J)$.

First of all, let us return to default theory (25) in order to illustrate Lukaszewicz' approach. This default theory has two justified extensions:

$$(Th(\{\mathsf{C}, \mathsf{D}\}), \{\mathsf{B}, \neg\mathsf{B}\}) \quad \text{and} \quad (Th(\{\mathsf{E}\}), \{\neg\mathsf{C} \wedge \neg\mathsf{D}\}).$$

The first one corresponds to the extension obtained in classical default logic. However, it is supplied with a set of justifications, $\{\mathsf{B}, \neg\mathsf{B}\}$ (which, incidentally, is inconsistent). The second extension stems from applying the third default rule whose justification blocks the two other default rules by contradicting their consequents.

Now, let us identify default theories in justified default logic with contextual default theories in the following way.

Definition 7.4 (Justified default logic) *Let (D, W) be a default theory. We define*

$$\Phi_{\mathsf{JDL}}(D, W) = \left(\left\{ \frac{\lfloor\alpha\rfloor : \gamma \,|\, \beta\wedge\gamma \rfloor}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D\right\}, Th(W)\right).$$

This leads to the following correspondence.

Theorem 7.2 *Let (D, W) be a default theory and let E be a set of formulas. Then, (E, J) is a justified extension of (D, W) and $C = Th_E(J)$ iff (E, C) is a contextual extension of $\Phi_{\mathsf{JDL}}(D, W)$.*

Notice that J consists of the justifications of the generating[14] default rules for E, whereas C is given by the pointwise closure of the same set of justifications under E.

It is interesting to observe how the relatively complicated consistency check in justified default logic is accomplishable in contextual default logic. For a justified extension (E, J) and a default rule $\frac{\alpha:\beta}{\gamma}$ the condition is $\forall \eta \in J \cup \{\beta\}.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$. In fact, it is two-fold: It consists of a joint and an individual consistency check, ie. $\forall \eta \in J.\ E \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ and $E \cup \{\gamma\} \cup \{\beta\} \not\vdash \bot$. Transposed to the case of a contextual extension (E,C) the two subconditions are $\neg\gamma \notin C$ and $\neg(\beta \wedge \gamma) \notin E$. The first check cares about the joint consistency of the consequent γ, whereas the second one checks whether the conjunction of the justification and consequent of the default rule is individually consistent.

Now, let us see what happens to default theory (25) if we apply translation Φ_{JDL}:

[14] In the sense of justified default logic.

$$\left(\left\{\frac{\|: \mathsf{C} | \mathsf{B}\wedge\mathsf{C}|}{\mathsf{C}}, \frac{\|: \mathsf{D} | \neg\mathsf{B}\wedge\mathsf{D}|}{\mathsf{D}}, \frac{\|: \mathsf{E} | \neg\mathsf{C}\wedge\neg\mathsf{D}\wedge\mathsf{E}|}{\mathsf{E}}\right\}, Th(\emptyset)\right)$$

As with justified default logic, we obtain two contextual extensions:

$$(Th(\{\mathsf{C},\mathsf{D}\}), Th(\{\mathsf{C},\mathsf{D},\mathsf{B}\}) \cup Th(\{\mathsf{C},\mathsf{D},\neg\mathsf{B}\})) \quad \text{and} \quad (Th(\{\mathsf{E}\}), Th(\{\mathsf{E},\neg\mathsf{C},\neg\mathsf{D}\})),$$

whose extensions correspond to the extensions obtained in justified default logic. It is interesting to observe that the respective subcontexts differ exactly in the justifications attached to the extensions in justified default logic.

7.3 Constrained default logic

Finally, we turn to a default logic which employs a sort of joint consistency, that is, we consider *constrained default logic* as introduced in [20, 7]. In constrained default logic, an extension comes with a set of constraints. A *constrained extension* is defined as follows.

Definition 7.5 *Let (D, W) be a default theory. For any set of formulas S let $\Upsilon(S)$ be the pair of smallest sets of formulas (T', S') such that*

1. $W \subseteq T' \subseteq S'$,
2. $T' = Th(T')$ *and* $S' = Th(S')$,
3. *For any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in T'$ *and* $S \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\gamma \in T'$ *and* $\beta, \gamma \in S'$.

A pair of sets of formulas (E, C) is a constrained extension of (D, W) iff $\Upsilon(C) = (E, C)$.

Constrained default logic detects inconsistencies among the justifications of default rules. Thus, we obtain three constrained extensions,

$$(Th(\{\mathsf{C}\}), Th(\{\mathsf{C},\mathsf{B}\})),\ (Th(\{\mathsf{D}\}), Th(\{\mathsf{D},\neg\mathsf{B}\})),\ \text{and}\ (Th(\{\mathsf{E}\}), Th(\{\mathsf{E},\neg\mathsf{C},\neg\mathsf{D}\})),$$

of default theory (25). The first extension $Th(\{\mathsf{C}\})$ comes with the set of constraints $Th(\{\mathsf{C},\mathsf{B}\})$ consisting of the justification B and the consequent C of the first default rule. The constraints $Th(\{\mathsf{D},\neg\mathsf{B}\})$ of the second extension $Th(\{\mathsf{D}\})$ contain the justification $\neg\mathsf{B}$ and the consequent D of the second default rule. Finally, we obtain a third extension $Th(\{\mathsf{E}\})$ with constraints $Th(\{\mathsf{E},\neg\mathsf{C},\neg\mathsf{D}\})$ generated by the last default rule.

A default theory in constrained default logic will be identified with a contextual default theory in the following way.

Definition 7.6 (Constrained default logic) *Let (D, W) be a default theory. We define*

$$\Phi_{\mathsf{CDL}}(D, W) = \left(\left\{\frac{|\alpha|: \beta\wedge\gamma\|}{\gamma} \;\middle|\; \frac{\alpha:\beta}{\gamma} \in D\right\}, Th(W)\right).$$

This yields the following correspondence.

Theorem 7.3 *Let (D, W) be a default theory and let E and C be sets of formulas. Then, (E,C) is a constrained extension of (D, W) iff (E,C) is a contextual extension of $\Phi_{\mathsf{CDL}}(D, W)$.*

Notice that C is always deductively closed whenever (E,C) is an extension in either sense.

Finally, let us consider the contextual counterpart of default theory (25) from the perspective of constrained default logic:

$$\left(\left\{\frac{||:\ \mathsf{B} \wedge \mathsf{C}\,||}{\mathsf{C}}, \frac{||:\ \neg\mathsf{B} \wedge \mathsf{D}\,||}{\mathsf{D}}, \frac{||:\ \neg\mathsf{C} \wedge \neg\mathsf{D} \wedge \mathsf{E}\,||}{\mathsf{E}}\right\}, \mathit{Th}(\emptyset)\right)$$

As a result, we obtain three contextual extensions:

$$(\mathit{Th}(\{\mathsf{C}\}), \mathit{Th}(\{\mathsf{C}, \mathsf{B}\})),\ (\mathit{Th}(\{\mathsf{D}\}), \mathit{Th}(\{\mathsf{D}, \neg\mathsf{B}\})),\ \text{and}\ (\mathit{Th}(\{\mathsf{E}\}), \mathit{Th}(\{\mathsf{E}, \neg\mathsf{C}, \neg\mathsf{D}\})),$$

These are identical to the respective constrained extensions.

8 Properties of contextual default logic

In the previous section, we have established correspondences between general default theories in existing default logics and fragments of contextual default logic. In what follows, we take advantage of these relationships for giving the formal properties of contextual default logic. In turn, we discuss the formal properties of existence of extensions, semi-monotonicity, orthogonality, and cumulativity.

First of all, we notice that none of these properties is present in Reiter's original default logic in its full generality. Rather we encounter several restricted subclasses enjoying one or another of the aforementioned property. Hence we cannot expect that contextual default logic improves this situation since it is more expressive than any variant of default logic.

8.1 Existence of extensions

Even though extensions play a central role in default logic, there are default theories that lack classical extensions (even though they possess justified and constrained extensions). For instance, the default theory

$$\left(\left\{\frac{:\ \neg\mathsf{A}}{\mathsf{A}}\right\}, \emptyset\right) \tag{26}$$

has no classical extension, whereas it has the justified extension $(\mathit{Th}(\emptyset), \emptyset)$ and the constrained extension $(\mathit{Th}(\emptyset), \mathit{Th}(\emptyset))$. Now, applying the transformation given in Definition 7.2 to default theory (26), we obtain the following contextual default theory.

$$\Phi_{\mathsf{DL}}\left(\left\{\frac{:\neg \mathsf{A}}{\mathsf{A}}\right\},\emptyset\right) = \left(\frac{||:|\neg \mathsf{A}|}{\mathsf{A}}, Th(\emptyset)\right) \tag{27}$$

Clearly, the contextual default theory in (27) has no contextual extension. Also,

$$\left(\frac{||:\neg \mathsf{A}||}{\mathsf{A}}, Th(\emptyset)\right)$$

has no contextual extension. Hence contextual default logic does not guaranteeing the existence of extensions in general.

In classical default logic, the existence of extensions is guaranteed in the case of so-called *normal default theories* [18], whose default rules are of the form $\frac{\alpha:\beta}{\beta}$. An even more pleasant situation is encountered in justified and constrained default logic, where the existence of extensions is enjoyed in the general case [11, 7]. These observations lead us to the following fragments of contextual default logic guarantee the existence of contextual extensions:

$$\begin{aligned} D_N &= \left\{\frac{|\alpha|:|\beta|}{\beta} \;\middle|\; \alpha,\beta \textit{ formulas}\right\} \\ D_J &= \left\{\frac{|\alpha|:\gamma|\beta\wedge\gamma|}{\gamma} \;\middle|\; \alpha,\beta,\gamma \textit{ formulas}\right\} \\ D_C &= \left\{\frac{|\alpha|:\beta\wedge\gamma||}{\gamma} \;\middle|\; \alpha,\beta,\gamma \textit{ formulas}\right\} \end{aligned}$$

In general, we obtain the following result.

Theorem 8.1 (Existence) *Let* (D,W) *be a contextual default theory. Then,* (D,W) *has a consistent contextual extension if*

$$D \subseteq \left\{\frac{\alpha_W\,|\,\alpha_E\,|\,\alpha_C \;:\; \alpha\wedge\beta\wedge\gamma\,|\,\gamma\,|}{\beta\wedge\gamma} \;\middle|\; \alpha_W,\alpha_E,\alpha_C,\alpha,\beta,\gamma \textit{ formulas}\right\}.$$

As a corollary, we obtain that we can merge contextual default rules belonging to the aforementioned classes without loosing the existence of extensions property. For instance, any contextual default theory whose contextual default rules belong to D_N and D_C (the fragments of contextual default logic corresponding to classical and constrained default logic) has a consistent contextual extension.

8.2 Semi-monotonicity

Another property which holds for normal default theories in classical default logic is *semi-monotonicity*, which stands for monotonicity wrt default rules and stipulates that adding a set of default rules to a default theory can only preserve or enlarge existing extensions. As above, a more pleasant situation is encountered in justified and constrained default logic, where semi-monotonicity is enjoyed in the general case [11, 7].

As an example, consider the default theory

$$\left(\left\{\frac{:\mathsf{B}}{\mathsf{C}}\right\},\emptyset\right) \tag{28}$$

which has in classical, justified, and constrained default logic one extension containing C. Now, adding the default rule $\frac{:\mathsf{D}}{\neg\mathsf{B}}$ yields the default theory

$$\left(\left\{\frac{:\mathsf{B}}{\mathsf{C}},\frac{:\mathsf{D}}{\neg\mathsf{B}}\right\},\emptyset\right) \tag{29}$$

whose only classical extension is $\mathit{Th}(\{\neg\mathsf{B}\})$. This extension does not contain C anymore, which violates semi-monotonicity. In contrast, we obtain two justified extensions, $(\mathit{Th}(\{\mathsf{C}\}),\{\mathsf{B}\})$ and $(\mathit{Th}(\{\neg\mathsf{B}\}),\{\mathsf{D}\})$, as well as two constrained extensions, $(\mathit{Th}(\{\mathsf{C}\}),\mathit{Th}(\{\mathsf{B},\mathsf{C}\}))$ and $(\mathit{Th}(\{\neg\mathsf{B}\}),\mathit{Th}(\{\neg\mathsf{B},\mathsf{D}\}))$, from the last default theory. That is, in both cases, we have one extension containing C and another one containing $\neg\mathsf{B}$.

As above, we can transpose the situation encountered in the variants of default logic onto contextual default logic. This amounts to the following theorem.

Theorem 8.2 (Semi-monotonicity) *Let (D,W) be a contextual default theory and D' a set of contextual default rules such that*

$$D \subseteq D' \subseteq \left\{\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \;:\; \alpha\wedge\beta\wedge\gamma \,|\, \gamma \,|}{\beta\wedge\gamma}\;\middle|\; \alpha_W,\alpha_E,\alpha_C,\alpha,\beta,\gamma \text{ formulas}\right\}.$$

If (E,C) is a contextual extension of (D,W), then there is a contextual extension (E',C') of (D',W) such that $E\subseteq E'$ and $C\subseteq C'$.

8.3 Orthogonality

Default theories may have alternative classical extensions which are consistent, ie. not *orthogonal* to each other. For instance, the default theory

$$\left(\left\{\frac{:\neg\mathsf{B}}{\mathsf{C}},\frac{:\neg\mathsf{C}}{\mathsf{B}}\right\},\emptyset\right)$$

has two extensions in all of the above mentioned variants of default logic, one containing C and another one containing B. Hence none of these variants enjoys orthogonality. Of course, this carries over to contextual default logic. The situation changes in the case of normal default theories, in which orthogonality is enjoyed in all variants of default logic. Clearly, this carries over to the corresponding fragments of contextual default logic, too.

However, a more appropriate notion seems to be that of *weak orthogonality*, as suggested in [19, 7] for cumulative and constrained default logic. This notion takes into account the extension along with its underlying consistency assumptions. For instance, weak orthogonality holds for constrained default logic. That is, given two different constrained extensions the constraints of the extensions are always mutually contradictory.

For contextual default logic, we obtain the following result.

Theorem 8.3 *Let (E,C) and (E',C') be distinct contextual extensions of the contextual default theory (D,W). Let C_K and C'_K be the kernels of C and C'. Then, we have that either $C_K\cup C'_{\beta'}$ is inconsistent for $C'_{\beta'}=\mathit{Th}(E'\cup\{\beta'\})$ and some $\beta'\in \mathit{Prereq}_E\Big(GD^{(E',C')}_{(D,W)}\Big)$ or $C'_K\cup C_\beta$ is inconsistent for $C_\beta=\mathit{Th}(E\cup\{\beta\})$ and some $\beta\in \mathit{Prereq}_E\Big(GD^{(E,C)}_{(D,W)}\Big)$.*

The theorem shows that multiple extensions stem from incompatibilities between kernels, like C_K or C'_K, and subcontexts, $C'_{\beta'}$ or C_β, of different contextual extensions.

As a corollary, we obtain that contexts of distinct contextual extensions are always mutually contradictory. That is, we obtain the property of weak orthogonality as formulated in [7].

Corollary 8.4 (Weak orthogonality) *Let (D, W) be a contextual default theory. If (E,C) and (E', C') are distinct contextual extensions of (D, W), then $C \cup C'$ is inconsistent.*

8.4 Cumulativity

Intuitively, cumulativity stipulates that the addition of a theorem to the set of premises does not change the theory under consideration. Classical, justified and constrained default logic lack this formal property. Since contextual default logic generalizes these variants, it does not enjoy cumulativity either.

As above, we can take advantage of the results found in the literature and transfer them to contextual default logic. The cumulativity of prerequisite-free normal default theories in classical default logic was shown in [8]. In [7], it was shown that constrained default logic is cumulative on the larger fragment of prerequisite-free default theories. Consequently, we obtain cumulativity in contextual default logic, if the default rules are either of the form $\frac{||:|\beta|}{\beta}$ or the form $\frac{||:\beta\wedge\gamma||}{\gamma}$.

9 Conclusion

We have argued that contexts provide an important and meaningful notion in default reasoning. This has been accomplished by thoroughly investigating the notions of contexts found in various variants of Reiter's default logic. This study has led to a new context-based approach to default logic, called contextual default logic.

Contextual default logic is not yet another default logic. Rather it provides a unified framework for default logics by extending the notion of a default rule and supplying each extension with a context. Such contexts are formed by pointwisely closing certain consistency assumptions under a given extension. We have isolated six different application conditions for default rules. We have shown that only three of them are employed in existing default logics, even though two of the three remaining ones correspond to well-known notions, namely first-order derivability and the closed world assumption. The remaining condition expresses "membership in a context" and allows for accessing the consistency assumptions underlying an extension.

From a synthetic point of view, contextual default logic integrates existing default logics along with other concepts like the closed world assumption. But apart from the separate integration of these approaches, we moreover gain expressiveness by combining them. From an analytical point of view, the key advantage of contextual default logic is that it provides a syntactical instrument for comparing existing default logics in a unified setting. In particular, contextual default logic has explicated the context-dependency of default logics and thus revealed that existing default logics differ mainly in the way they deal with an explicit or implicit underlying context. As a result, we have seen that justified default logic compromises individual and joint consistency, whereas other variants strictly employ either of them.

From the perspective of existing default logics, there has been no system simultaneously accounting for different application conditions of default rules yet. Contextual default logic allows for a uniform representation of different notions of default rules. Moreover, this approach allows for combining different application conditions and, therefore, offers a greater expressiveness than is obtainable in any existing default logic.

A Proofs of Theorems

A.1 Proof of alternative characterizations

Proof 5.1 Let (E,C) be a contextual extension of a contextual default theory (D,W). For the sake of readability, let us abbreviate $\mathit{Conseq}\Big(GD^{(E,C)}_{(D,W)}\Big)$ by Γ, $\mathit{Justif}_C\Big(GD^{(E,C)}_{(D,W)}\Big)$ by Λ_C, and $\mathit{Justif}_E\Big(GD^{(E,C)}_{(D,W)}\Big)$ by Λ_E.

Accordingly, we have to show that

$$\mathit{Th}(W \cup \Gamma) = \mathit{Th}(E) = E \qquad \text{and} \qquad \mathit{Th}_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E) = \mathit{Th}_\top(C) = C.$$

"$\subseteq$" If $\frac{\alpha_W \mid \alpha_E \mid \alpha_C \,:\, \beta_C \mid \beta_E \mid \beta_W}{\gamma} \in GD^{(E,C)}_{(D,W)}$ then $\alpha_W \in W$, $\alpha_E \in E$, $\alpha_C \in C$, and $\neg\beta_C \notin C$, $\neg\beta_E \notin E$, $\neg\beta_W \notin W$.

Then, by Definition 3.2, $\mathit{Th}_\gamma(E) \subseteq E$ and $\mathit{Th}_{\beta_E}(E) \subseteq C, \mathit{Th}_{\beta_C}(C) \subseteq C$.

First, we prove $\mathit{Th}(W \cup \Gamma) \subseteq E$

Consider $\varphi \in \mathit{Th}(W \cup \Gamma)$. Assume $\varphi \notin E$.

Consider the following three cases.

1. $\varphi \in W$. By definition, $W \subseteq E$. Hence, $\varphi \in E$. A contradiction.
2. $\varphi \in \Gamma$. Since $\mathit{Th}_\gamma(E) \subseteq E$ for all $\gamma \in \Gamma$, we have $\Gamma \subseteq E$ and thus $\varphi \in E$. A contradiction.
3. Otherwise, we have $W \cup \Gamma \models \varphi$. By compactness, there are finite sets $W' \subseteq W$ and $\Gamma' \subseteq \Gamma$ such that $W' \cup \Gamma' \models \varphi$.
   Since W is deductively closed, we have $(\bigwedge_{(\omega\in W')}\omega) \in W$ and thus $(\bigwedge_{(\omega\in W')}\omega) \in E$.
   Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Since $\mathit{Th}_{\gamma_1}(E) \subseteq E$, we have
   $$\mathit{Th}\Big(\{\gamma_1\} \cup \{\textstyle\bigwedge_{(\omega\in W')}\omega\}\Big) = \mathit{Th}\Big(\{\textstyle\bigwedge_{(\omega\in W'\cup\{\gamma_1\})}\omega\}\Big) \subseteq E;$$
   thus $(\bigwedge_{(\omega\in W'\cup\{\gamma_1\})}\omega) \in E$.
   Proceeding in this way with $\gamma_2, \ldots, \gamma_n$, yields
   $$\mathit{Th}\Big(\{\textstyle\bigwedge_{(\omega\in W'\cup\Gamma')}\omega\}\Big) = \mathit{Th}(W' \cup \Gamma') \subseteq E$$
   Since $W' \cup \Gamma' \models \varphi$, this implies $\varphi \in E$. A contradiction.

Second, we prove $Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E) \subseteq C$ by showing that $Th(W \cup \Gamma \cup \Lambda_C \cup \{\beta_E\})$ for $\beta_E \in \Lambda_E$.

Consider $\varphi \in Th(W \cup \Gamma \cup \Lambda_C \cup \{\beta_E\})$. Assume $\varphi \notin C$.

Consider the following four cases.

1. $\varphi \in W \cup \Gamma$. Since $Th(W \cup \Gamma) \subseteq E$ according to what we have just proven and the fact $E \subseteq C$, we obtain $\varphi \in C$. A contradiction.
2. $\varphi \in \Lambda_C$. Since $Th_{\beta_C}(C) \subseteq C$ for all $\beta_C \in \Lambda_C$, we have $\Lambda_C \subseteq C$ and thus $\varphi \in C$. A contradiction.
3. $\varphi \in \{\beta_E\}$. Since $Th_{\beta_E}(E) \subseteq C$, we have $\varphi \in C$. A contradiction.
4. Otherwise, an analogous argumentation as in part 3. of the first half of this proof yields $\varphi \in C$. Again, a contradiction.

"$\supseteq$" Clearly, $W \subseteq Th(W \cup \Gamma) \subseteq Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)$.

Consider $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D$.

If $\alpha_E \in Th(W \cup \Gamma)$ then $\alpha_E \in E$ according to what we have just proved.

If $\alpha_C \in Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)$ then $\alpha_C \in C$ according to what we have just proved.

If additionally, $\neg\beta_C \notin C$, $\neg\beta_E \notin E$, $\neg\beta_W \notin W$ then $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in GD^{(E,C)}_{(D,W)}$, whence $\gamma \in \Gamma$ and $\beta_C \in \Lambda_C$, $\beta_E \in \Lambda_E$.

Clearly, the last three conditions imply

$$Th_\gamma(Th(W \cup \Gamma)) \subseteq Th(W \cup \Gamma),$$

$$Th_{\beta_E}(Th(W \cup \Gamma)) \subseteq Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E), \text{ and}$$

$$Th_{\beta_C}(Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)) \subseteq Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E).$$

Accordingly, by the minimality of $\nabla(E, C)$, we have $\nabla_1(E, C) \subseteq Th(W \cup \Gamma)$ and $\nabla_2(E, C) \subseteq Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)$. Since (E, C) is a contextual extension, we obtain $E \subseteq Th(W \cup \Gamma)$ and $C \subseteq Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)$.

Clearly, $E = Th(W \cup \Gamma)$ implies $Th(E) = E$. That is, E is deductively closed.

Moreover, $Th_\top(C) = Th_\top(Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E)) = Th_{W\cup\Gamma\cup\Lambda_C}(\Lambda_E) = C$. Hence, C is pointwisely closed.

■

Proof 5.2 First, observe that we have the following properties

- $W \subseteq \bigcup_{i=0}^{\infty} E_i \subseteq \bigcup_{i=0}^{\infty} C_i$.
- For any $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D$,

  if $\alpha_W \in W$, $\alpha_E \in \bigcup_{i=0}^{\infty} E_i$, $\alpha_C \in \bigcup_{i=0}^{\infty} C_i$,

  and $\neg\beta_C \notin \bigcup_{i=0}^{\infty} C_i$, $\neg\beta_E \notin \bigcup_{i=0}^{\infty} E_i$, $\neg\beta_W \notin W$.

  then $Th_\gamma(\bigcup_{i=0}^{\infty} E_i) \subseteq \bigcup_{i=0}^{\infty} E_i$, $Th_{\beta_E}(\bigcup_{i=0}^{\infty} E_i) \subseteq \bigcup_{i=0}^{\infty} C_i$, $Th_{\beta_C}(\bigcup_{i=0}^{\infty} C_i) \subseteq \bigcup_{i=0}^{\infty} C_i$.

By the minimality of $\nabla(E,C)$, we have[15]

$$\nabla_1(E,C) \subseteq \bigcup_{i=0}^{\infty} E_i \qquad \text{and} \qquad \nabla_2(E,C) \subseteq \bigcup_{i=0}^{\infty} C_i, \tag{30}$$

regardless of whether (E,C) is a contextual extension or not.

only-if part Assume (E,C) is a contextual extension.

"$\supseteq$" We have to show that $E_i \subseteq E$ and $C_i \subseteq C$ for $i \geq 0$

Base Clearly, $E_0 = W \subseteq E$ and $C_0 = W \subseteq C$.

Step Assume $E_i \subseteq E$ and $C_i \subseteq C$ and consider $\eta \in E_{i+1} \cup C_{i+1}$.

1. $\eta \in W$. Since $W \subseteq E$, we obtain $\eta \in E$.
2. $\eta \in \{\beta_C, \beta_E, \gamma\}$ for $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D$ such that $\alpha_W \in W, \alpha_E \in E_i, \alpha_C \in C_i$, and $\neg\beta_C \notin C, \neg\beta_E \notin E, \neg\beta_W \notin W$. That is, η is either γ or β_C or β_E for any contextual default rule satisfying the preceding requirements.
   Since $E_i \subseteq E$ and $C_i \subseteq C$ we have $\alpha_E \in E$ and $\alpha_C \in C$.
   Altogether, $\alpha_W \in W$, $\alpha_E \in E$, $\alpha_C \in C$, and $\neg\beta_C \notin C$, $\neg\beta_E \notin E$, $\neg\beta_W \notin W$ imply $Th_\gamma(E) \subseteq E$ and $Th_{\beta_E}(E) \subseteq C$, $Th_{\beta_C}(C) \subseteq C$, by Definition 3.2 and all cases for η are covered.

Thus, we have $E_{i+1} \subseteq E$ and $C_{i+1} \subseteq C$, respectively.

"$\subseteq$" From (30) and the fact that $(E,C) = \nabla(E,C)$ we obtain $E \subseteq \bigcup_{i=0}^{\infty} E_i$ and $C \subseteq \bigcup_{i=0}^{\infty} C_i$, respectively.

We obtain $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

if part Assume $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

"$\supseteq$" Now, we have to show that $E_i \subseteq \nabla_1(E,C)$ and $C_i \subseteq \nabla_2(E,C)$ for $i \geq 0$.

Base Clearly, $E_0 = W \subseteq \nabla_1(E,C)$ and $C_0 = W \subseteq \nabla_2(E,C)$.

Step Assume $E_i \subseteq \nabla_1(E,C)$ and $C_i \subseteq \nabla_2(E,C)$ and consider $\eta \in E_{i+1} \cup C_{i+1}$.

1. $\eta \in W$. Since $W \subseteq \nabla_1(E,C)$ we obtain $\eta \in \nabla_1(E,C)$.
2. $\eta \in \{\beta_C, \beta_E, \gamma\}$ for $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma} \in D$ such that $\alpha_W \in W, \alpha_E \in E_i, \alpha_C \in C_i$, and $\neg\beta_C \notin C, \neg\beta_E \notin E, \neg\beta_W \notin W$. That is, η is either γ or β_C or β_E for any contextual default rule satisfying the preceding requirements.
   Since $E_i \subseteq \nabla_1(E,C)$ and $C_i \subseteq \nabla_2(E,C)$ we have $\alpha_W \in \nabla_1(E,C)$ and $\alpha_E \in \nabla_2(E,C)$. Altogether, $\alpha_W \in W$, $\alpha_W \in \nabla_1(E,C)$ $\alpha_C \in \nabla_2(E,C)$ and $\neg\beta_C \notin C$, $\neg\beta_E \notin E$, $\neg\beta_W \notin W$ imply $Th_\gamma(\nabla_1(E,C)) \subseteq \nabla_1(E,C)$ and $Th_{\beta_E}(\nabla_1(E,C)) \subseteq \nabla_2(E,C)$, $Th_{\beta_C}(\nabla_2(E,C)) \subseteq \nabla_2(E,C)$, and all cases for η are covered.

Accordingly, we have $E_{i+1} \subseteq \nabla_1(E,C)$ and $C_{i+1} \subseteq \nabla_2(E,C)$, respectively.

[15] We refer to the components of $\nabla(E,C)$ as $\nabla_1(E,C)$ and $\nabla_2(E,C)$, respectively.

"$\subseteq$" Follows from (30).

We have shown that $(\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i) = \nabla(E,C)$. Together with the assumption $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$, we obtain that (E,C) is a contextual extension of (D,W). ■

A.2 Proof of correctness and completeness

In the sequel, we frequently employ the following definition.

Definition A.1 *Let (D,W) be a contextual default theory. Given a possibly infinite sequence of contextual default rules $\Delta = \langle \delta_0, \delta_1, \delta_2, \ldots \rangle$ in D, also denoted $\langle \delta_i \rangle_{i \in I}$ where I is the index set for Δ, we define a sequence of classes of K-models $\langle M_i \rangle_{i \in I}$ as follows:*

$$\begin{array}{lcl} M_0 & = & M_W \\ M_{i+1} & = & \{ m \in M_i \mid m \models \gamma_i \wedge \Box\gamma_i \wedge \Box\beta_{C_i} \wedge \Diamond\beta_{E_i} \}, \\ & & \qquad\qquad \textit{where} \quad \delta_i = \frac{\alpha_{W_i} | \alpha_{E_i} | \alpha_{C_i} : \beta_{C_i} | \beta_{E_i} | \beta_{W_i}}{\gamma_i}. \end{array}$$

We will be more liberal here about the orders $>_\delta$ by relaxing the condition that $M >_\delta M'$ holds only if M and M' are distinct. That is, there will be cases where $M >_\delta M$ be true. Clearly, this does not affect the issues under consideration.

Proof 6.1 (Correctness) Assume (E,C) is a consistent contextual extension of (D,W). The set of generating default rules for (E,C) wrt D is defined in Definition 5.1. As an easy adaptation of Theorem 4.3.6. in [21] shows, then there exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of $GD_{(D,W)}^{(E,C)}$ such that for $i \in I$

$$\begin{array}{rcl} \mathit{Prereq}_W(\delta_i) & \in & W, \\ \mathit{Prereq}_E(\delta_i) & \in & \mathit{Th}(W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})), \\ \mathit{Prereq}_C(\delta_i) & \in & \mathit{Th}_{W \cup \mathit{Conseq}(\{\delta_0, \ldots, \delta_{i-1}\})}(\mathit{Justif}(\{\delta_0, \ldots, \delta_{i-1}\})). \end{array} \qquad (31)$$

Let $\langle M_i \rangle_{i \in I}$ be a sequence of classes of K-models obtained from the enumeration $\langle \delta_i \rangle_{i \in I}$ according to Definition A.1. We will show that M coincides with $\bigcap_{i \in I} M_i$ and is $>_D$-maximal above M_W.

Since (E,C) is a contextual extension, it has been proven in Theorem 5.1 that

$$E = \mathit{Th}\Big(W \cup \mathit{Conseq}\Big(GD_{(D,W)}^{(E,C)}\Big)\Big).$$

Then, since $M = \{ m \mid m \models E \wedge \Box C_K \wedge \Diamond C_J \}$ and

$$C_K = \mathit{Th}\Big(E \cup \mathit{Justif}_C\Big(GD_{(D,W)}^{(E,C)}\Big)\Big) \qquad \text{and} \qquad C_J = \mathit{Justif}_E\Big(GD_{(D,W)}^{(E,C)}\Big),$$

we have obviously that $M = \bigcap_{i \in I} M_i$.

Firstly, let us show that $M_{i+1} >_{\delta_i} M_i$ for $i \in I$.

- First, we prove that $M_i \models Prereq_W(\delta_i), Prereq_E(\delta_i), \Diamond Prereq_C(\delta_i)$ for $i \in I$. This leads to the following three cases:

  1. Since $M_i \subseteq M_W$ and $M_W \models W$, then by definition of M_i we have $M_i \models W$ for $i \in I$. By definition, $W \models Prereq_W(\delta_i)$. Hence, $M_i \models Prereq_W(\delta_i)$ for $i \in I$.
  2. Since $M_i \subseteq M_W$ and $M_W \models W$, then by definition of M_i we have $M_i \models W \cup Conseq(\delta_{i-1})$ for $i \in I$. Now, $M_{i+1} \subseteq M_i$ for $i \in I$ implies that $M_i \models W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\})$. By (31), it follows that $M_i \models Prereq_E(\delta_i)$ for $i \in I$.
  3. Since $M_i \subseteq M_W$ and $M_W \models W$, then by definition of M_i we have

     $$M_i \models Conseq(\delta_{i-1}) \wedge \Box(Conseq(\delta_{i-1}) \wedge Justif_C(\delta_{i-1})) \wedge Justif_E(\delta_{i-1})$$

     for $i \in I$. Now, $M_{i+1} \subseteq M_i$ for $i \in I$ implies that

     $$\begin{aligned} M_i \quad \models \quad & \Box(W \cup Conseq(\{\delta_0, \ldots, \delta_{i-1}\}) \cup Justif_C(\{\delta_0, \ldots, \delta_{i-1}\})) \\ & \wedge \Diamond Justif_E(\{\delta_0, \ldots, \delta_{i-1}\}). \end{aligned}$$

     Applying Proposition A.6 to (31) and M_i, yields $M_i \models \Diamond Prereq_C(\delta_i)$ for $i \in I$.

  It all, it follows that $M_i \models Prereq_W(\delta_i), Prereq_E(\delta_i), \Diamond Prereq_C(\delta_i)$ for $i \in I$.

- Let us assume that $M_{i+1} >_{\delta_i} M_i$ fails for some $k \in I$. By definition of $\langle M_i \rangle_{i \in I}$ and the fact that we have just proven that $M_i \models Prereq_W(\delta_i), Prereq_E(\delta_i), \Diamond Prereq_C(\delta_i)$ for $i \in I$, this means that either $M_k \models \Diamond\neg\beta_{C\,k}$ or $M_k \models \neg\beta_{E\,k}$ or $M_W \models \neg\beta_{W\,k}$ for $\delta_k = \frac{\alpha_{W\,k} | \alpha_{E\,k} | \alpha_{C\,k} \,:\, \beta_{C\,k} | \beta_{E\,k} | \beta_{W\,k}}{\gamma_k}$.

  Let us abbreviate $W \cup Conseq(\{\delta_0, \ldots, \delta_{k-1}\})$ by E^k, $W \cup Conseq(\{\delta_0, \ldots, \delta_{k-1}\}) \cup Justif_C(\{\delta_0, \ldots, \delta_{k-1}\})$ by C^k, and $Justif_E(\{\delta_0, \ldots, \delta_{k-1}\})$ by J^k. By definition, $M_k = \{m \mid m \models E^k \wedge \Box C^k \wedge \Diamond J^k\}$.

  Now, we have to consider the following three cases.

  1. Since $C^k \subseteq C_K$ and $J^k \subseteq C_J$, we have that $C^k \wedge \eta$ is satisfiable for each $\eta \in C_J$ and we can apply Corollary A.5 to the definition of M_k and $M_k \models \Diamond\neg\beta_{C\,k}$ we obtain that $\neg\beta_{C\,k} \in Th_{C^k}\left(J^k\right)$. By monotonicity, $\neg\beta_{W\,k} \in C$, contradictory to the fact that $\delta_k \in GD^{(E,C)}_{(D,W)}$.
  2. Since $E^k \subseteq E$ and $J^k \subseteq C_J$, we have that $C^k \wedge \eta$ is satisfiable for each $\eta \in C_J$ and we can apply Corollary A.2 to the definition of M_k and $M_k \models \neg\beta_{E\,k}$. We obtain that $E^k \models \neg\beta_{E\,k}$. By monotonicity, $E \models \neg\beta_{E\,k}$. Since E is deductively closed we have $\neg\beta_{E\,k} \in E$, contradictory to the fact that $\delta_k \in GD^{(E,C)}_{(D,W)}$.
  3. If $M_W \models \neg\beta_{W\,k}$, we have $W \models \neg\beta_{W\,k}$, contradictory to the fact that $\delta_k \in GD^{(E,C)}_{(D,W)}$.

Therefore, $M_{i+1} >_{\delta_i} M_i$ for $i \in I$. As a consequence, $\bigcap_{i \in I} M_i >_{GD^{(E,C)}_{(D,W)}} M_W$. That is, $M >_D M_W$.

Secondly, assume M is not $>_D$-maximal. Then, there exists a contextual default rule $\frac{\alpha_W | \alpha_E | \alpha_C \,:\, \beta_C | \beta_E | \beta_W}{\gamma} \in D \setminus GD^{(E,C)}_{(D,W)}$ such that $M_W \models \alpha_W, M \models \alpha_E, M \models \Diamond\alpha_C$, and $M \not\models \Diamond\neg\beta_C, M \not\models \neg\beta_E, M_W \not\models \neg\beta_W$. This leads us to the following six cases.

1. Trivially, $M_W \models \alpha_W$ implies $W \models \alpha_W$. Since W is deductively closed we have $\alpha_W \in W$.

2. Applying Corollary A.2 to the definition of M and $M \models \alpha_E$ yields $E \models \alpha_E$. Since E is deductively closed we have $\alpha_E \in E$.

3. Applying Corollary A.5 to the definition of M and $M \models \Diamond\alpha_C$ yields $\Diamond\alpha_C \in C$.

4. Since $M \models E \wedge \Box C_K \wedge \Diamond C_J$, we get by monotonicity $\Box C_K \wedge \Diamond C_J \not\models \Diamond\neg\beta_C$. By modal logic K, it follows that $C_K \wedge \eta \not\models \neg\beta_C$ whenever $\eta \in C_J$. Since $C = Th_{C_K}(C_J)$ we obtain $C \not\models \neg\beta_C$. Thus, $\neg\beta_C \notin C$.

5. Since $M \models E \wedge \Box C_K \wedge \Diamond C_J$, we get by monotonicity $E \not\models \neg\beta_E$. Since E is deductively closed, we get $\neg\beta_E \notin E$.

6. Clearly, $M_W \not\models \neg\beta_W$ implies $W \not\models \neg\beta_W$. Thus, $\neg\beta_W \notin W$.

Of course, $\alpha_W \in W$, $\alpha_E \in E$, $\alpha_C \in C$, and $\neg\beta_C \notin C$, $\neg\beta_E \notin E$, $\neg\beta_W \notin W$ imply $\frac{\alpha_W | \alpha_E | \alpha_C : \beta_C | \beta_E | \beta_W}{\gamma} \in GD^{(E,C)}_{(D,W)}$, a contradiction. ■

Proof 6.1 (Completeness) Assume $M = \{m \mid m \models E \wedge \Box C_K \wedge \Diamond C_J\}$ is a $>_D$-maximal class of K-models above M_W.

We define for $i \geq 0$

$$\Delta_i = \left\{ \frac{\alpha_W | \alpha_E | \alpha_C \ : \ \beta_C | \beta_E | \beta_W}{\gamma} \in D \;\middle|\; \begin{array}{lll} \alpha_W \in W, & \alpha_E \in E_i, & \alpha_C \in C_i, \\ \neg\beta_C \notin C, & \neg\beta_E \notin E, & \neg\beta_W \notin W \end{array} \right\}$$

where $E_0 = W$, $C_{K_0} = W$, and $C_{J_0} = \emptyset$ and for $i \geq 0$

$$\begin{array}{rcl} E_{i+1} & = & Th(E_i) \cup Conseq(\Delta_i) \\ C_{K_{i+1}} & = & E_i \cup Justif_C(\Delta_i) \\ C_{J_{i+1}} & = & Justif_E(\Delta_i) \end{array}$$

Furthermore, let $C_i = Th_{C_{K_i}}(C_{J_i})$ for $i \geq 0$.

Then, although the definitions of E_i and C_i are slightly modified, we have that (E, C) is a contextual extension iff $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ according to Theorem 5.2.

Let us abbreviate $\{m \mid m \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} C_{K_i} \wedge \Diamond\bigcup_{i=0}^{\infty} C_{J_i}\}$ by N. We will show that $M = N$, in order to show that $(E, C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$.

Firstly, let us show by induction that $M \subseteq \{m \mid m \models E_i \wedge \Box C_{K_i} \wedge \Diamond C_{J_i}\}$ for $i \geq 0$.

Base By definition, $M_W \models E_0 \wedge \Box C_{K_0} \wedge \Diamond C_{J_0}$. Since $M >_D M_W$, we get $M \subseteq \{m \mid m \models E_0 \wedge \Box C_{K_0} \wedge \Diamond C_{J_0}\}$.

Step The induction hypothesis is: $M \models E_i \wedge \Box C_{K_i} \wedge \Diamond C_{J_i}$

Consider $\eta \in E_{i+1} \cup C_{K_{i+1}} \cup C_{J_{i+1}}$. Then, one of the two following cases holds.

1. $\eta \in Th(E_i)$. By the induction hypothesis, $M \models \eta$ (hence covering the case $\eta \in E_i$).

2. $\eta \in \{\beta_C, \beta_E, \gamma\}$ for $\frac{\alpha_W \mid \alpha_E \mid \alpha_C : \beta_C \mid \beta_E \mid \beta_W}{\gamma} \in D$ such that $\alpha_W \in W, \alpha_E \in E_i, \alpha_C \in C_i$, and $\neg\beta_C \notin C, \neg\beta_E \notin E, \neg\beta_W \notin W$. That is, η is either γ or β_C or β_E for any contextual default rule satisfying the preceding requirements. This leads us to the following six cases.

   (a) Clearly, $\alpha_W \in W$ implies $M_W \models \alpha_W$.
   (b) By the induction hypothesis, $M \models \alpha_E$.
   (c) By the induction hypothesis (in view of $C_i = Th_{C_{K_i}}(C_{J_i})$), $M \models \Diamond\alpha_C$.
   (d) Assume $M \models \Diamond\neg\beta_C$. Since M is non-empty, we obtain that $C_K \wedge \eta$ is satisfiable for each $\eta \in C_J$. So, Corollary A.5 applies to M and $M \models \Diamond\neg\beta_C$. We obtain $\neg\beta_C \in Th_{C_K}(C_J) = C$, a contradiction. So, $M \not\models \Diamond\neg\beta_C$
   (e) Assume $M \models \neg\beta_E$. As in the previous case, we obtain that $C_K \wedge \eta$ is satisfiable for each $\eta \in C_J$. So, Corollary A.2 applies to M and $M \models \neg\beta_E$. As a result, $E \models \neg\beta_E$. Since E is deductively closed, it follows that $\neg\beta_E \in E$, a contradiction. So, $M \not\models \neg\beta_E$.
   (f) Clearly, $\neg\beta_W \notin W$ implies $M_W \not\models \neg\beta_W$.

   Since M is $>_D$-maximal, then $M \models \gamma \wedge \Box\gamma \wedge \Box\beta_C \wedge \Diamond\beta_E$ must hold and all cases for η are covered.

From the two cases, we obtain $M \models E_{i+1} \wedge \Box C_{K_{i+1}} \wedge \Diamond C_{J_{i+1}}$.

Therefore, we have shown that $M \subseteq \{m \mid m \models E_i \wedge \Box C_{K_i} \wedge \Diamond C_{J_i}\}$ for $i \geq 0$. So, $M \subseteq N$.

Secondly, since M is a $>_D$-maximal class above M_W for (D, W), then $M = \bigcap_{i \in I} M_i$ where $\langle M_i \rangle_{i \in I}$ is a sequence of classes of K-models defined for some $\langle \delta_i \rangle_{i \in I}$ according to Definition A.1 such that $M_{i+1} >_{\delta_i} M_i$ for $i \in I$.

Let us show by induction that $N \subseteq M_i$ for $i \in I$.

Base Since $M_0 = M_W$ and $E_0 = C_0 = W$, the result is obvious.

Step The induction hypothesis is: $N \subseteq M_i$

Since $M_{i+1} >_{\delta_i} M_i$ for $i \in I$ we have $M_{i+1} = \{m \in M_i \mid m \models \gamma_i \wedge \Box\gamma_i \wedge \Box\beta_{C_i} \wedge \Diamond\beta_{E_i}\}$ and $M_W \models \alpha_{W_i}$, $M_i \models \alpha_{E_i}$, $M_i \models \Diamond\alpha_{C_i}$, and $M_i \not\models \Diamond\neg\beta_{C_i}$, $M_i \not\models \neg\beta_{E_i}$, $M_W \not\models \neg\beta_{W_i}$ where $\delta_i = \frac{\alpha_{W_i} \mid \alpha_{E_i} \mid \alpha_{C_i} : \beta_{C_i} \mid \beta_{E_i} \mid \beta_{W_i}}{\gamma_i}$. This leads us to the following six cases.

1. Clearly, $M_W \models \alpha_{W_i}$ implies $W \models \alpha_{W_i}$. Since W is deductively closed, we have $\alpha_{W_i} \in W$.
2. By the induction hypothesis, we have $N \models \alpha_{E_i}$. Suppose that $\bigcup_{i=0}^{\infty} C_{K_i} \wedge \eta$ is unsatisfiable for some $\eta \in \bigcup_{i=0}^{\infty} C_{J_i}$. Then, there is some k such that $\eta \in C_k$ and $C_{K_k} \models \neg\eta$. We have shown above that $M \subseteq \{m \mid m \models E_i \wedge \Box E_i \wedge \Diamond C_i\}$ for $i \geq 0$. Then, $M \models \Box C_{K_k} \wedge \Diamond\eta$. From $C_{K_k} \models \neg\eta$, modal logic K yields $\Box C_{K_k} \models \Box\neg\eta$. Therefore, $M \models \Box\neg\eta \wedge \Diamond\eta$. Then, M is empty, a contradiction. So, $\bigcup_{i=0}^{\infty} C_{K_i} \wedge \eta$ is satisfiable for each $\eta \in \bigcup_{i=0}^{\infty} C_{J_i}$. Since $N \models \alpha_{E_i}$, we can now apply Corollary A.2 to obtain that $\bigcup_{i=0}^{\infty} E_i \models \alpha_{E_i}$. Since $\bigcup_{i=0}^{\infty} E_i$ is deductively closed, we have $\alpha_{E_i} \in \bigcup_{i=0}^{\infty} E_i$.

3. By the induction hypothesis, we have $N \models \Diamond\alpha_{C\,i}$. By Corollary A.5, $\alpha_{C\,i} \in Th_{\bigcup_{i=0}^{\infty} C_{K_i}}(\bigcup_{i=0}^{\infty} C_{J_i}) = \bigcup_{i=0}^{\infty} C_i$.

By compactness and monotonicity, there exist k' and k'' such that $\alpha_{E\,i} \in E_{k'}$ and $\alpha_{C\,i} \in C_{k''}$. Let k be the maximum of k' and k''. Thus, $\alpha_{E\,i} \in E_k$ and $\alpha_{C\,i} \in C_k$.

By definition, $M_{i+1} \models \Box\beta_{C\,i} \wedge \Diamond\beta_{E\,i}$ hence $M \models \Box\beta_{C\,i} \wedge \Diamond\beta_{E\,i}$ because $M = \bigcap_{i \in I} M_i$.

4. From $M \models \Box\beta_{C\,i}$ and $M \models \Box C_K \wedge \Diamond C_J$, it follows by modal logic K that $M \models \Diamond(C_K \wedge \beta_{C\,i} \wedge \eta)$ for $\eta \in C_J$. That is, since M is non-empty, $C_K \cup \{\eta\} \not\vdash \neg\beta_{C\,i}$ for $\eta \in C_J$. Since $C = Th_{C_K}(C_J)$, we obtain $\neg\beta_{C\,i} \notin C$.
5. From $M \models \Diamond\beta_{E\,i}$ and $M \models \Box E$, it follows by modal logic K that $E \not\models \neg\beta_{E\,i}$. That is, since M is non-empty, $\neg\beta_{E\,i} \notin E$.
6. Clearly, $M_W \not\models \neg\beta_{W\,i}$ implies $\neg\beta_{W\,i} \notin W$.

From $\alpha_{W\,i} \in W$, $\alpha_{E\,i} \in E_k$, $\alpha_{C\,i} \in C_k$, and $\neg\beta_{C\,i} \notin C$, $\neg\beta_{E\,i} \notin E$, $\neg\beta_{W\,i} \notin W$, we conclude that $\gamma_i \in E_{k+1}$ and $\gamma_i \wedge \beta_{C\,i} \in C_{K_{k+1}}$ and $\beta_{E\,i} \in C_{J_{k+1}}$. Hence, $N \models \gamma_i \wedge \Box\gamma_i \wedge \Box\beta_{C\,i} \wedge \Diamond\beta_{E\,i}$. By the induction hypothesis and the definition of M_{i+1} we obtain $N \subseteq M_{i+1}$.

Therefore, we have shown that $N \subseteq M_i$ for $i \in I$. That is, $N \subseteq M$.

In all, $M = N$. That is, $\{m \mid m \models E \wedge \Box C_K \wedge \Diamond C_J\} = \{m \mid m \models \bigcup_{i=0}^{\infty} E_i \wedge \Box\bigcup_{i=0}^{\infty} C_{K_i} \wedge \Diamond\bigcup_{i=0}^{\infty} C_{K_i}\}$. Since M hence N is non-empty, $\Box\bigcup_{i=0}^{\infty} C_{K_i} \wedge \Diamond\beta_E$ is satisfiable for each $\beta_E \in \bigcup_{i=0}^{\infty} C_{J_i}$ (as $\Box p \wedge \Diamond q \rightarrow \Diamond(p \wedge q)$ and $\Diamond\bot \rightarrow \bot$ are valid in modal logic K). By Corollary A.2, $\bigcup_{i=0}^{\infty} E_i \models E$. The converse is proved in a similar way, it is just simpler. Therefore, $\bigcup_{i=0}^{\infty} E_i = E$ because $\bigcup_{i=0}^{\infty} E_i$ and E are both deductively closed sets of formulas.

Moreover, $Th(\bigcup_{i=0}^{\infty} C_{K_i}) = C_K$, by an analogous argumentation. The rest of $\bigcup_{i=0}^{\infty} C_i = C$ is straightforward.

Then, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ and according to Theorem 5.2 this means (E,C) is a contextual extension of (D,W). ■

Some modal propositions

modal This subsection summarizes some modal propositions taken from [3] along with some others needed in the preceding proofs.

Proposition A.1 *Let p, q, r, $s_1, \ldots, s_n$ be non-modal formulas such that $q \wedge s_i$ is satisfiable for $i = 1, \ldots, n$.*

If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow r$ then $\models p \rightarrow r$.

Corollary A.2 *Let S, T, U and V be sets of non-modal formulas and $T \wedge u$ is satisfiable for each $u \in U$.*

If $M = \{m \mid m \models S \wedge \Box T \wedge \Diamond U\}$ and $M \models V$ then $S \models V$.

Proposition A.3 *Let p, q, r, $s_1, \ldots, s_n$, t be non-modal formulas, with p and $q \wedge s_i \wedge \neg t$ satisfiable for $i = 1, \ldots, n$.*

If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Box r \vee \Diamond t$ *then* $\models q \rightarrow r \vee t$.

In addition, the following propositions are needed.

Proposition A.4 *Let p, q, r, $s_1, \ldots, s_n$ be non-modal formulas such that p and $q \wedge s_i$ are satisfiable for $i = 1, \ldots, n$.*

If $\models p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Diamond r$ *then* $\models q \wedge s_i \rightarrow r$ *for some* $i \in \{1, \ldots, n\}$

Proof A.4 Assume the contrary. Thus, $q \wedge s_i \wedge \neg r$ is satisfiable for each $i = 1, \ldots, n$. We now construct the K-model $m = \langle \omega_0, \{\omega_i \mid i = 0, \ldots, n\}, \{(\omega_0, \omega_i) \mid i = 1, \ldots, n\}, \mathcal{I} \rangle$ such that $\omega_0 \models p$ and $\omega_i \models q \wedge s_i \wedge \neg r$ for each $i = 1, \ldots, n$. Clearly, m contradicts the validity of $p \wedge \Box q \wedge \Diamond s_1 \wedge \ldots \wedge \Diamond s_n \rightarrow \Diamond r$. ■

Corollary A.5 *Let S, T, and U be sets of non-modal formulas and v a non-modal formula such that S is satisfiable and $T \wedge u$ is satisfiable for each $u \in U$.*

If $M = \{m \mid m \models S \wedge \Box T \wedge \Diamond U\}$ *and* $M \models \Diamond v$ *then* $v \in Th_T(U)$.

Proof A.5 $M \models \Diamond v$ means $S \wedge \Box T \wedge \Diamond U \models \Diamond v$. By compactness, $S' \wedge \Box T' \wedge \Diamond U' \models \Diamond v$ for some finite subsets S', T' and U' of S, T and U, respectively. Since the deduction theorem for material implication holds in modal logic K, we get $\models S' \wedge \Box T' \wedge \Diamond U' \rightarrow \Diamond v$. Applying Proposition A.4, $\models T' \wedge u \rightarrow v$ for some $u \in U'$. That is, $T' \cup \{u\} \models v$. By monotonicity, $T \cup \{u\} \models v$. So, $v \in Th_T(U)$ since $u \in U' \subseteq U$. ■

Proposition A.6 *Let S and T be sets of non-modal formulas and u a non-modal formula.*

If $u \in Th_S(T)$ *then* $\Box S \wedge \Diamond T \models \Diamond u$.

Proof A.6 By definition, $Th_S(T) = \bigcup_{t \in T} Th(S \cup \{t\})$. Thus, $u \in Th_S(T)$ is equivalent to $u \in \bigcup_{t \in T} Th(T \cup \{t\})$. That is, $u \in Th(S \cup \{t\})$ for some $t \in T$. By modal logic K, if $S \wedge t \models u$ then $\Box S \wedge \Diamond t \models \Diamond u$ for some $t \in T$. By monotonicity of modal logic K, $\Box S \wedge \Diamond T \models \Diamond u$. ■

A.3 Proofs of relationships to existing default logics

Proof 7.1 In what follows, let (D, W) be a default theory and $(D', W') = \Phi_{\mathsf{DL}}(D, W)$. For a set of formulas E, let $C_E = \left\{ \beta \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E \right\}$.

only-if part Let E be a classical extension of (D,W). Let $C = Th_E(C_E)$. Then, E is the smallest set of formulas such that

1. $W \subseteq E$,
2. $Th(E) = E$,
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $\neg\beta \notin E$ then $\gamma \in E$.

Then, E (recall that C is defined in terms of E) is also the smallest set satisfying the following properties:

1. By definition, $W \subseteq E$. Furthermore, $E \subseteq C$, by definition of C. Hence, $W \subseteq E \subseteq C$.
2. For any $\frac{\lfloor\alpha\rfloor:\lfloor\beta\rfloor}{\gamma} \in D'$, if $\alpha \in E$, $\neg\beta \notin E$ then $Th_\gamma(E) \subseteq E$, $Th_\beta(E) \subseteq C$.

   This is so because by Definition 7.2, $\frac{\lfloor\alpha\rfloor:\lfloor\beta\rfloor}{\gamma} \in D'$ iff $\frac{\alpha:\beta}{\gamma} \in D$. Furthermore, $\gamma \in E$ and the fact that E is deductively closed imply $Th_\gamma(E) \subseteq E$. By the definition of C_E, we have $\beta \in C_E$. Therefore, by definition of C, $Th(E \cup \{\beta\}) \subseteq C$, which in turn implies $Th_\beta(E) \subseteq C$.

Finally, a reductio ad absurdum argument shows that also C is the smallest set satisfying the aforementioned properties.

By Definition 3.2, we thus obtain that (E,C) is a contextual extension of $\Phi_{\mathsf{DL}}(D,W)$.

if part Let (E,C) be a contextual extension of $\Phi_{\mathsf{DL}}(D,W)$. Then, E and C are the smallest sets of formulas such that

1. $W \subseteq E \subseteq C$
2. For any $\frac{\lfloor\alpha\rfloor:\lfloor\beta\rfloor}{\gamma} \in D'$, if *2.* $\alpha \in E$, *4.* $\neg\beta \notin E$ then *7.* $Th_\gamma(E) \subseteq E$, *8.* $Th_\beta(E) \subseteq C$.

Then, E is also the smallest set satisfying the following three properties:

1. By definition, $W \subseteq E$.
2. By Theorem 5.1, $E = Th(E)$.
3. For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in E$ and $\neg\beta \notin E$ then $\gamma \in E$.

   This is so because by Definition 7.2, $\frac{\lfloor\alpha\rfloor:\lfloor\beta\rfloor}{\gamma} \in D'$ iff $\frac{\alpha:\beta}{\gamma} \in D$. Furthermore, $Th_\gamma(E) \subseteq E$ implies $\gamma \in E$.

By Definition 7.1, we thus obtain that E is a classical extension of (D,W).

Moreover, we have according to Theorem 5.1 that

$$\begin{aligned}
C &= Th_E\left(Justif_E\left(GD^{(E,C)}_{(D',W')}\right)\right)\\
&= Th_E\left(\left\{\ \beta \ \middle|\ \frac{\lfloor\alpha\rfloor:\lfloor\beta\rfloor}{\gamma} \in D',\ \alpha \in E, \neg\beta \notin E\right\}\right)\\
&= Th_E\left(\left\{\ \beta \ \middle|\ \frac{\alpha:\beta}{\gamma} \in D,\ \alpha \in E, \neg\beta \notin E\right\}\right)\\
&= Th_E(C_E).
\end{aligned}$$

■

Proof 7.2 Analogous to Proof 7.1. ■

Proof 7.3 Analogous to Proof 7.1. ■

A.4 Proof of properties

Proof 8.3 The unsatisfiable case is trivial.

According to Theorem 5.2, $(E,C) = (\bigcup_{i=0}^{\infty} E_i, \bigcup_{i=0}^{\infty} C_i)$ such that $E_0 = W$ and $C_0 = W$, and for $i \geq 0$

$$\Delta_i = \left\{ \frac{\alpha_W \mid \alpha_E \mid \alpha_C \ : \ \beta_C \mid \beta_E \mid \beta_W}{\gamma} \in D \;\middle|\; \begin{array}{lll} \alpha_W \in W, & \alpha_E \in E_i, & \alpha_C \in C_i, \\ \neg\beta_C \notin C, & \neg\beta_E \notin E, & \neg\beta_W \notin W \end{array} \right\}$$

$$\begin{aligned} E_{i+1} &= Th(W \cup Conseq(\Delta_i)) \\ C_{i+1} &= Th_{W \cup Conseq(\Delta_i) \cup Justif_C(\Delta_i)}(Justif_E(\Delta_i)) \end{aligned}$$

Also $(E', C') = (\bigcup_{i=0}^{\infty} E'_i, \bigcup_{i=0}^{\infty} C'_i)$ where Δ'_i, E'_i and C'_i are defined analogously.

Since (E,C) and (E',C') are distinct, there exists a least k such that $\Delta_k \neq \Delta'_k$ and $\Delta_i = \Delta'_i$ for $0 \leq i \leq k$.

Then, there is a contextual default rule such that $\delta \in \Delta_k$ but $\delta \notin \Delta'_k$. As a consequence, $Prereq_W(\delta) \in W$, $Prereq_E(\delta) \in E_k$, $Prereq_C(\delta) \in C_k$, and $\neg Justif_C(\delta) \notin C$, $\neg Justif_E(\delta) \notin E$, $\neg Justif_W(\delta) \notin W$. Since $E_i = E'_i$ and $C_i = C'_i$ for $0 \leq i \leq k$, we also have $Prereq_E(\delta) \in E'_k$, and $Prereq_C(\delta) \in C'_k$. Consequently, we have $\neg Justif_C(\delta) \in C'$ or $\neg Justif_E(\delta) \in E'$.

On the other hand, we have $Justif_C(\delta) \in C_K$ and $Justif_E(\delta) \in C$, since $\delta \in \Delta_k$.

Suppose that $\neg Justif_C(\delta) \in C'$. Since $C' = Th_{E' \cup Justif_C(\Delta')}(Justif_E(\Delta'))$ according to Theorem 5.1 for $\Delta' = GD_{(D,W)}^{(E',C')}$, there is ${\beta_E}' \in Justif_E(\Delta')$ such that $\neg Justif_C(\delta) \in Th(E' \cup \{{\beta_E}'\})$. Since $Justif_C(\delta) \in C_K$, we obtain that $C_K \cup Th(E' \cup \{{\beta_E}'\})$ is inconsistent.

Suppose that $\neg Justif_E(\delta) \in E'$. This implies $\neg Justif_E(\delta) \in C'_K$ since $E' \subseteq C'_K$. With $\delta \in GD_{(D,W)}^{(E,C)}$, we have $Th(E \cup \{Justif_E(\delta)\}) \subseteq C$ and $Justif_E(\delta) \in Th(E \cup \{Justif_E(\delta)\})$. Clearly, we that $C'_K \cup Th(E \cup \{Justif_E(\delta)\})$ is inconsistent. ■

Proof 8.1 Let (D,W) be a contextual default theory. Then, there is a consistent contextual default theory $(\emptyset, W)$ which has a unique contextual extension $(Th(W), Th(W))$. From this and Theorem 8.2 the result follows immediately. ■

Proof 8.2 The inconsistent case is easily dealt with, so that we prove below the theorem for E and C being consistent.

We define a sequence $\langle\Delta_\iota\rangle$ of subsets of D' as follows. For the sake of simplicity, let us abbreviate $Th(W \cup Conseq(\Delta_\kappa))$ by E^κ and $Th_{W \cup Conseq(\Delta_\kappa) \cup Justif_C(\Delta_\kappa)}(Justif_E(\Delta_\kappa))$

by C^κ. Moreover, we say that a contextual default rule $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \beta_C \,|\, \beta_E \,|\, \beta_W}{\gamma}$ applies wrt (E^κ, C^κ) iff $\alpha_W \in W$, $\alpha_E \in E^\kappa$, $\alpha_C \in C^\kappa$, and $\neg\beta_C \notin C^\kappa$, $\neg\beta_E \notin E^\kappa$, $\neg\beta_W \notin W$.

$$\Delta_\iota = \begin{cases} GD^{(E,C)}_{(D,W)} & \text{if } \iota = 0 \\ \bigcup_{\kappa<\iota} \Delta_\kappa & \text{if } \iota \text{ is a limit ordinal} \\ \Delta_\kappa \cup \{\delta\} & \text{if } \iota = \kappa + 1 \text{ is a successor ordinal in the case there exists} \\ & \quad \delta = \frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \alpha\wedge\beta\wedge\gamma \,|\, \gamma \,|}{\beta\wedge\gamma} \in D' \setminus \Delta_\kappa \\ & \text{such that } \delta \text{ applies wrt } (E^\kappa, C^\kappa) \end{cases}$$

Since the sequence Δ is strictly increasing, the process eventually stops. Let χ be the greatest ordinal such that Δ_χ is defined. Define

$$E' = Th(E \cup Conseq(\Delta_\chi)) \text{ and } C' = Th_{E \cup Conseq(\Delta_\chi) \cup Justif_C(\Delta_\chi)}\left(Justif_E(\Delta_\chi)\right).$$

By definition, $E \subseteq E'$ and $C \subseteq C'$. Thus, it remains to be shown that (E', C') is a contextual extension of (D', W). First, observe the following properties.

1. By definition of E' and C', clearly $W \subseteq E'$. We have also $E' \subseteq C'$ because every contextual default rule is of the form $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \alpha\wedge\beta\wedge\gamma \,|\, \gamma \,|}{\beta\wedge\gamma}$.

2. If $\frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \alpha\wedge\beta\wedge\gamma \,|\, \gamma \,|}{\beta\wedge\gamma} \in D'$, and δ applies wrt (E', C') , we obtain $Th_{\beta\wedge\gamma}(E') \subseteq E'$, $Th_\gamma(E') \subseteq C'$, $Th_{\alpha\wedge\beta\wedge\gamma}(C') \subseteq C'$, (otherwise $\Delta_{\chi+1}$ could be defined).

Then, by the minimality of $\nabla(E', C')$, we have[16] $\nabla_1(E', C') \subseteq E'$ and $\nabla_2(E', C') \subseteq C'$.

Now, assume $\nabla_1(E', C') \subset E'$ and $\nabla_2(E', C') \subset C'$, ie. none of the former inclusions are proper. Then (provided that $E \subseteq \nabla_1(E', C')$ and $C \subseteq \nabla_2(E', C')$), there exists a least ordinal κ such that $\Delta_\kappa = \Delta_{\kappa-1} \cup \{\delta\}$ where $\delta = \frac{\alpha_W \,|\, \alpha_E \,|\, \alpha_C \,:\, \alpha\wedge\beta\wedge\gamma \,|\, \gamma \,|}{\beta\wedge\gamma} \in D'$, such that δ applies wrt (E', C'), and $\beta \wedge \gamma \in E'$ and $\alpha \wedge \beta \wedge \gamma \in C'$ and $\gamma \in C'$, but either $\beta \wedge \gamma \notin \nabla_1(E', C')$ or $\alpha \wedge \beta \wedge \gamma \notin \nabla_2(E', C')$ or $\gamma \notin \nabla_2(E', C')$. Clearly, $\alpha_W \in W$. By definition of Δ, we have $\alpha_E \in E^{\kappa-1}$ and $\alpha_C \in C^{\kappa-1}$. Since κ is the least such ordinal, it follows that $\alpha_E \in \nabla_1(E', C')$ and $\alpha_C \in \nabla_1(E', C')$. But by definition, these, $\alpha_W \in W$, and the consistency conditions imply $\beta \wedge \gamma \in \nabla_1(E', C')$ and $\alpha \wedge \beta \wedge \gamma \in \nabla_2(E', C')$ and $\gamma \in \nabla_2(E', C')$. Contradiction.

It remains to be shown that $E \subseteq \nabla_1(E', C')$ and $C \subseteq \nabla_2(E', C')$. This can be proven by applying the same reasoning to a similar sequence of contextual default rules of $GD^{(E,C)}_{(D,W)}$. ■

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[16]We refer to the components of ∇ as ∇_1 and ∇_2, respectively.

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