Reasoning credulously and skeptically within a single extension

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ABSTRACT. *Consistency-based approaches in nonmonotonic reasoning may be expected to yield multiple sets of default conclusions for a given default theory. Reasoning about such extensions is carried out at the meta-level. In this paper, we show how such reasoning may be carried out at the object level for a large class of default theories. Essentially we show how one can translate a (semi-monotonic) default theory Δ, obtaining a second Δ', such that Δ' has a single extension that encodes every extension of Δ. Moreover, our translated theory is only a constant factor larger than the original (with the exception of unique names axioms). We prove that our translation behaves correctly. In the approach we can now encode the notion of* extension *from within the framework of standard default logic. Hence one can encode notions such as skeptical and credulous conclusions, and can reason about such conclusions within a single extension. This result has some theoretical interest, in that it shows how multiple extensions of semi-monotonic default theories are encodable with manageable overhead in a single extension.*

RÉSUMÉ. *A définir par la commande* `\resume{...}`

KEYWORDS: *default logic, skeptical reasoning, credulous reasoning, tagging*

MOTS-CLÉS : *A définir par la commande* `\motscles{...}`

1. Introduction

In nonmonotonic reasoning, in so-called *consistency-based* approaches such as default logic [REI 80] and autoepistemic logic [MOO 85], one typically obtains not just a single set of default conclusions, but rather multiple sets of candidate default conclusions. Consider the by-now hackneyed example wherein Quakers are normally pacifist, Republicans are normally not, along with adults are normally employed. Assume as well that someone is a Quaker, Republican, and an adult. In default logic (see Section 2) this can be encoded by: $(\{\frac{Q:P}{P}, \frac{R:\neg P}{\neg P}, \frac{A:E}{E}\}, \{Q, R, A\})$. This theory has two *extensions* or sets of default conclusions, one containing $\{Q, R, A, E, P\}$ and the other $\{Q, R, A, E, \neg P\}$. In autoepistemic logic the same example appropriately encoded yields two analogous *expansions* or possible belief sets.

Reasoning about these extensions (resp. expansions) is carried out at the meta-level: a default conclusion that appears in some extension (such as P) is called a *credulous* (or *brave*) default conclusion, while one that appears in every extension (such as E) is called a *skeptical* conclusion. Intuitively it might seem that skeptical inference is the more useful notion. However, this is not necessarily the case. In diagnosis from first principles [REI 87] for example, in one encoding there is a 1-1 correspondence between diagnoses and extensions of the (encoding) semi-monotonic default theory. Hence one may want to carry out further reasoning to determine which diagnosis to pursue. More generally there may be reasons to prefer some extensions over others, or to somehow synthesize the information found in several extensions.

In this paper, we show how such reasoning can be carried out at the object level. For a default theory $\Delta = (D, W)$, we translate Δ to obtain a second theory $\Delta' = (D', W')$, such that Δ' has a single extension that encodes every extension of Δ. Given this, one can express in the theory what it means for something to be a skeptical or credulous default conclusion. Our result isn't completely general; however it applies to *semi-monotonic* default theories. The translation has several desirable properties. The translated theory Δ' is only a constant factor larger than the original Δ, with the exception of introduced unique names axioms. As well, we *prove* that our translation behaves correctly.

We first show for a set of defaults D_m how, using an encoding, we can detect the case wherein all defaults in D_m apply. From this, for a default theory $(D \cup D_m, W)$ we show how to obtain a second theory wherein (informally) either all of the defaults in D_m are applied en masse (if possible) or none of them are. This is done by naming each of the defaults in D_m, and then expressing in default logic the applicability conditions for the defaults. We develop this in Section 3. In Section 4 we present our main result, where we show how a default theory can be translated into a second theory whose extension encodes the extensions of the original. Roughly we provide an axiomatisation that "locates" maximal sets of applicable defaults; for such a set, the set of default conclusions is "tagged" with the set name, to distinguish it from other instances. For example, in our original example, let $m_{1,3}$ be the name of the set $\{\frac{Q:P}{P}, \frac{A:E}{E}\}$ and $m_{2,3}$ be the name of $\{\frac{R:\neg P}{\neg P}, \frac{A:E}{E}\}$. These are maximal applicable sets of defaults, and from our translation we would obtain a single extension containing $\{Q(m_{1,3}),Q(m_{2,3}),R(m_{1,3}),R(m_{2,3}),A(m_{1,3}),\ A(m_{2,3}),\ E(m_{1,3}),\ E(m_{2,3}),P(m_{1,3}),\neg P(m_{2,3})\}$. As mentioned, we are able to prove that our translations in fact accomplish what is claimed.

The advantage of this approach is that we can encode the notion of extension within the framework of standard default logic. Hence one can reason about (skeptical and credulous) conclusions within the framework of a single extension of a default theory. Thus for example, in a diagnosis setting one could go on and axiomatise notions of preference among diagnoses having to do with, perhaps, number of faulty components, or based on components expected to fail first. This result has some theoretical interest, in that it shows (for theories that we consider) how multiple extensions are encodable, with no significant overhead in a single extension. The overall approach builds on work in [DEL 00].

2. Default Logic

Default logic [REI 80] augments classical logic by *default rules* of the form $\frac{\alpha:\beta}{\gamma}$. A default rule is *normal* if β is equivalent to γ; it is *semi-normal* if β implies γ. We sometimes denote the *prerequisite* α of a default δ by $\mathit{PRE}(\delta)$, its *justification* β by $\mathit{JUS}(\delta)$, and its *consequent* γ by $\mathit{CON}(\delta)$. Accordingly, $\mathit{PRE}(D)$ is the set of prerequisites of all defaults in D; $\mathit{JUS}(D)$ and $\mathit{CON}(D)$ are defined analogously. Empty components, such as no prerequisite or even no justifications, are assumed to be tautological. Semantically, defaults with unbound variables are taken to stand for all corresponding instances. A set of default rules D and a set of formulas W form a *default theory* (D,W) that may induce a single or multiple *extensions* in the following way [REI 80].

Definition 2.1 *Let (D,W) be a default theory and let E be a set of formulas. Define $E_0 = W$ and for $i \geq 0$:*

$$\begin{aligned} \mathit{GD}_i &= \left\{ \tfrac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D \,\middle|\, \alpha \in E_i, \neg\beta_1 \notin E, \ldots, \neg\beta_n \notin E \right\} \\ E_{i+1} &= \mathit{Th}(E_i) \cup \{\mathit{CON}(\delta) \mid \delta \in \mathit{GD}_i\} \end{aligned}$$

Then E is an extension for (D,W) if $E = \bigcup_{i=0}^{\infty} E_i$.

Any such extension represents a possible set of beliefs about the world at hand. Further, define for a set of formulas S and a set of defaults D, the *set of generating default rules* as $\mathit{GD}(D,S) = \{\delta \in D \mid \mathit{PRE}(\delta) \in S \text{ and } \neg\mathit{JUS}(\delta) \notin S\}$.

An enumeration $\langle\delta_i\rangle_{i\in I}$ of default rules is *grounded* in a set of formulas W, if we have for every $i \in I$ that $W \cup \mathit{CON}(\{\delta_0,\ldots,\delta_{i-1}\}) \vdash \mathit{PRE}(\delta_i)$. A default theory (D,W) is said to be *semi-monotonic* if, for $D' \subseteq D'' \subseteq D$, if E' is an extension of D' then there is an extension E'' of D'' where $E' \subseteq E''$.

3. Applying All, or None, of a Set of Defaults

In this section we consider the problem of how to apply all defaults in some set, or none in the set. We will thus work with default theories (D, W) having some distinguished finite subset $D_m \subseteq D$. For making the set D_m explicit, we denote such theories by $(D \cup D_m, W)$. The idea is that we wish to obtain extensions of $(D \cup D_m, W)$ subject to the constraint that *all* defaults in D_m are applied, or *none* are. For example, in the theory $\left(\left\{\frac{:A}{A}\right\} \cup \left\{\frac{:B}{B}, \frac{C:D}{D}\right\}, \emptyset\right)$ we would want to obtain an extension containing A, but not B (since both defaults in $\left\{\frac{:B}{B}, \frac{C:D}{D}\right\}$ cannot be jointly applied). For $\left(\left\{\frac{:A}{A}\right\} \cup \left\{\frac{:B}{B}, \frac{B:D}{D}\right\}, \{\emptyset\}\right)$ we would want to obtain an extension containing A, B, and D.

We begin by associating a unique name with each default. This is done by extending the original language by a set of constants[1] N such that there is a bijective mapping $n : D \rightarrow N$. We write n_δ instead of $n(\delta)$ (and we often abbreviate n_{δ_i} by n_i to ease notation). Also, for default δ along with its name n, we sometimes write $n : \delta$ to render naming explicit. To encode the fact that we deal with a finite set of distinct default rules, we adopt a unique names assertion (UNA$_N$) and domain closure assertion (DCA$_N$) with respect to N. So, for a name set $N = \{n_1, \ldots, n_k\}$, we add axioms

$$\begin{array}{ll} \text{UNA}_N: & \neg(n_i = n_j) \text{ for all } n_i, n_j \in N \text{ with } i \neq j \\ \text{DCA}_N: & \forall x.\, name(x) \equiv (x = n_1 \vee \cdots \vee x = n_k)\,. \end{array}$$

We write $\forall x \in N.\, P(x)$ for $\forall x.\, name(x) \supset P(x)$.

We introduce a new constant m as the name of the designated rule set D_m. We relate the name of the rule set denoted by m with the names of its members by introducing a binary predicate in where $in(x, y)$ is true just if the default named by x is a member of the set named by y. In this section, instances of in will be of the form $in(\cdot, m)$. While we could get away with not using in (and m) here, this additional machinery is required in Section 4, and it is most straightforward to introduce it here. Note that we do not need a full axiomatization of in, representing set membership, since we use it in a very restricted fashion.

For applying all, or none, of the defaults in D_m, we need to be able to, first, detect when a rule has been applied or is blocked and, second, control the application of a rule based on other prerequisite conditions. There are two cases for a default $\frac{\alpha:\beta}{\gamma}$ to not be applied: the prerequisite is not known to be true (and so its negation $\neg\alpha$ is consistent), or the justification is not consistent (and so its negation $\neg\beta$ is derivable). For detecting this case, we introduce a new, special-purpose predicate $\mathsf{bl}/1$. Similarly we introduce a special-purpose predicate $\mathsf{ap}/1$ to detect when a rule has been applied. For controlling application of a rule we introduce predicates $\mathsf{ok}/1$ and $\mathsf{ko}/1$.

1. [MCC 86] first suggested naming defaults using a set of *aspect* functions. See also [POO 88, BRE 94].

We are given a default theory $(D \cup D_m, W)$ over language $\mathcal{L}$ and its set of associated default names $N\dot{\cup}\{m\}$. [2] Let

$$D_m = \left\{ n_j : \tfrac{\alpha_j : \beta_j}{\gamma_j} \;\middle|\; j = 1..k \right\}.$$

(For simplicity, we reuse the symbols j, k, m, n_j, α_j, etc. below.) We define $\mathcal{S}_m((D \cup D_m, W)) = (D', W')$ over $\mathcal{L}^*$, obtained by extending $\mathcal{L}$ to $\mathcal{L}^*$ with new predicates symbols $\mathsf{ok}/1$, $\mathsf{ko}/1$, $\mathsf{bl}/1$, $\mathsf{ap}/1$, and names $N\dot{\cup}\{m\}$, as follows

$$\begin{aligned} D' &= D \cup D_N \cup D_M \\ W' &= W \cup W_M \cup \{\text{DCA}_N, \text{UNA}_N\} \end{aligned}$$

where

$$\begin{aligned}
D_N &= \left\{ \tfrac{\alpha_j \wedge \mathsf{ok}(n_j) : \beta_j}{\gamma_j \wedge \mathsf{ap}(n_j)} \;\middle|\; j = 1..k \right\} && (1) \\
D_M &= \left\{ \tfrac{: \neg \mathsf{ko}(m)}{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)} \right\} && (2) \\
&\cup \left\{ \tfrac{: \neg \alpha_j}{\mathsf{bl}(m)}, \tfrac{(\gamma_1 \wedge \cdots \wedge \gamma_k) \supset \neg \beta_j :}{\mathsf{bl}(m)} \;\middle|\; j = 1..k \right\} && (3) \\
W_M &= \{\forall x \in N. in(x, m) \equiv (x = n_1 \vee ... \vee x = n_k)\} && (4) \\
&\cup \{\mathsf{bl}(m) \supset \mathsf{ko}(m)\} && (5) \\
&\cup \left\{ \big(\forall x \in N.\, in(x, m) \supset \mathsf{ap}(x)\big) \supset \mathsf{ap}(m) \right\} && (6)
\end{aligned}$$

Clearly, D_N contains the images of the original rules in D_m. Each rule $\delta_j \in D_N$ is applicable, if $\mathsf{ok}(n_j)$ is derivable. In fact, we assert $\mathsf{ok}(n_j)$ for every $\delta_j \in D_m$, *unless* we cannot jointly apply all rules of D_m. That is, before activating the constituent rules, we have to make sure that none of them will be blocked. This is accomplished through the justification $\neg \mathsf{ko}(m)$ in (2) together with Axiom (5). We block rule (2) (and with it the derivability of all $\mathsf{ok}(n_j)$) when we detect that one of $\delta_1, \ldots, \delta_k$ is blocked. That is, $\mathsf{ko}(m)$ will be an immediate consequence of $\mathsf{bl}(m)$.

Now, we have that D_m is blocked ($\mathsf{bl}(m)$) just if some rule in D_m is blocked. However, since we must control a whole set of defaults, we must check for the blockage of one of the constituent default rules in the context of all other rules in the set applying. For detecting the failure of consistency, we verify for D_m and some set of formulas S (cf. Definition 2.1), whether $S \cup \{\gamma_1, \ldots, \gamma_k\} \vdash \neg \beta_j$ rather than $S \vdash \neg \beta_j$. This motivates the prerequisite of the second rule in (3). This context, $(\gamma_1 \wedge \cdots \wedge \gamma_k)$, is not needed for detecting the failure of derivability by means of the first rule in (3), since this test is effectuated with respect to the final extension E via $\neg \alpha_j \notin E$.

Finally, as given in (6), D_m is applied ($\mathsf{ap}(m)$) just if every rule in D_m is applied; it is only in this last case that the consequents of the constituent rules in D_m are asserted.

2. We let $\dot{\cup}$ stand for disjoint union.

Consider theory $(D \cup D_m, W)$, where

$$D = \left\{ \frac{:E}{E} \right\},\ D_m = \left\{ n_1 : \frac{:P}{P},\ n_2 : \frac{:S}{S} \right\}. \tag{7}$$

For D_N and D_M, we obtain (after simplifying and removing redundant defaults):

$$\frac{\mathsf{ok}(n_1) : P}{P \wedge \mathsf{ap}(n_1)},\quad \frac{\mathsf{ok}(n_2) : S}{S \wedge \mathsf{ap}(n_2)},\quad \frac{: \neg\mathsf{ko}(m)}{\mathsf{ok}(n_1) \wedge \mathsf{ok}(n_2)},\quad \frac{\neg P \vee \neg S :}{\mathsf{bl}(m)}.$$

The in predicate has instances: $in(n_1, m)$ and $in(n_2, m)$. From (6) we can deduce $[\mathsf{ap}(n_1) \wedge \mathsf{ap}(n_2)] \supset \mathsf{ap}(m)$.

Let $W = \{\neg(P \wedge E \wedge S)\}$. We obtain two extensions, one containing $P, S, \neg E$ and the other containing $E, \neg(P \wedge S)$. For the first case, we obtain $\mathsf{ok}(n_1)$ and $\mathsf{ok}(n_2)$. If both δ_1 and δ_2 are applicable (which they are) then we conclude $P \wedge \mathsf{ap}(n_1)$ and $S \wedge \mathsf{ap}(n_1)$ as well as $\mathsf{ap}(m)$. From this we get P and S and so $\neg E$. For the other extensions, if the default $\frac{:E}{E}$ is applied, then $\neg P \vee \neg S$ is derivable, and so $\frac{\neg P \vee \neg S :}{\mathsf{bl}(m)}$ is applicable, from which we obtain $\mathsf{bl}(m)$, and so $\mathsf{ko}(m)$, blocking application of $\frac{: \neg\mathsf{ko}(m)}{\mathsf{ok}(n_1) \wedge \mathsf{ok}(n_2)}$. Consequently neither $\frac{\mathsf{ok}(n_1) : P}{P \wedge \mathsf{ap}(n_1)}$ nor $\frac{\mathsf{ok}(n_2) : S}{S \wedge \mathsf{ap}(n_2)}$ can be applied.

In the next example, defaults inside a set depend upon each other. Consider $(\emptyset \cup D_m, \emptyset)$ with

$$D_m = \left\{ n_1 : \frac{:Q}{Q},\ n_2 : \frac{Q : R}{R} \right\}.$$

We get for D_N and D_M the following rules.

$$\frac{\mathsf{ok}(n_1) : Q}{Q \wedge \mathsf{ap}(n_1)},\quad \frac{Q \wedge \mathsf{ok}(n_2) : R}{R \wedge \mathsf{ap}(n_2)},\qquad \frac{: \neg\mathsf{ko}(m)}{\mathsf{ok}(n_1) \wedge \mathsf{ok}(n_2)},\quad \frac{\neg Q \vee \neg R :}{\mathsf{bl}(m)},\quad \frac{: \neg Q}{\mathsf{bl}(m)}.$$

We obtain $\mathsf{ok}(n_1)$, and $\mathsf{ok}(n_2)$, which allow us to apply default δ_1, yielding in turn $Q \wedge \mathsf{ap}(n_1)$. Given Q, we can now apply default δ_2, yielding $R \wedge \mathsf{ap}(n_2)$. From this we deduce $\mathsf{ap}(m)$. We thus get an extension containing Q and R. This example also shows why we cannot avoid the translation by replacing D_m by $\frac{\bigwedge_{\delta \in D_m} \mathit{PRE}(\delta) : \bigwedge_{\delta \in D_m} \mathit{JUS}(\delta)}{\bigwedge_{\delta \in D_m} \mathit{CON}(\delta)}$. As well, in Section 4, this replacement would result in an exponential blowup in the encoding.

The next theorem summarizes properties of our approach, and shows that rules are applied either en masse, or not at all.

Theorem 3.1 *Let E be a consistent extension of $\mathcal{S}_m((D \cup D_m, W))$ for default theory $(D \cup D_m, W)$. We have that:*

1) $\mathsf{ap}(m) \in E$ *iff* $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \cup \mathit{CON}(D_m) \subseteq E$

2) $\mathsf{bl}(m) \in E$ *iff* $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \not\subseteq E$

3) $\mathsf{ok}(n_\delta) \in E$ *iff* $\mathsf{ap}(n_\delta) \in E$

4) $\mathsf{ok}(n_\delta) \in E$ *for all* $\delta \in D_m$ *iff* $\mathsf{ko}(m) \notin E$

5) $\mathsf{ap}(n_\delta) \in E$ *implies* $(\mathsf{ap}(m) \wedge in(n_\delta, m)) \in E$ *for some* $\delta \in D_m$

6) $\mathsf{ap}(n_\delta) \in E$ *for* $\delta \in D_m$ *iff* $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \subseteq E$.

Theorem 3.2 *For default theory* $(\emptyset \cup D, W)$, *we have that* $\mathcal{S}_m((\emptyset \cup D, W))$ *has extension* E *where either* $E \cap \mathcal{L} = \mathit{Th}(W \cup \mathit{CON}(D))$ *or else* $E \cap \mathcal{L} = \mathit{Th}(W)$.

The default theory $\left(\emptyset \cup \left\{\frac{:B}{\neg B}\right\},\ \emptyset\right)$ has an extension E where $E \cap \mathcal{L} = \mathit{Th}(\emptyset)$.

Theorem 3.3 *Let* (D, W) *be a (standard) default theory over* $\mathcal{L}$ *with extension* E *and (respective) set of generating defaults* $\mathit{GD}(D, E)$. *Then* $\mathcal{S}_m((\emptyset \cup \mathit{GD}(D, E), W))$ *has extension* E' *where* $E = E' \cap \mathcal{L}$.

4. Encoding extensions using sets

For encoding extensions of a semi-monotonic default theory (D, W), we use the machinery developed in the previous section to determine maximal (with respect to set inclusion) sets of applicable defaults. Names are introduced for each subset of D, and for each instance of a rule in each subset of D. As well, new predicate symbols are introduced to further control application of sets of rules. We then give a translation that yields a second default theory (D', W'). Viewed algorithmically, this second theory carries out the following: If the original set of defaults D constitutes the set of generating defaults of an extension, then a corresponding "ap"-literal is derived; all default consequences are obtained; and all subsets of the defaults are rendered inapplicable. If this isn't the case (and D isn't a set of generating defaults), we proceed along the partial order induced by set inclusion and consider every set $D \setminus \{\delta\}$ for every $\delta \in D$ to see whether it is a set of generating defaults. Crucially, default conclusions are "tagged" with the name of the set in which they appear so as to eliminate possible side effects.

To name sets of defaults, we take some fixed enumeration $\langle n_1, \ldots, n_k \rangle$ of N, and define m as a k-ary function symbol. Then, for $n_\perp \notin N$, define

$$\mathrm{DCA}_M : \forall x_1, \ldots, x_k.\ \textit{set-name}(m(x_1, \ldots, x_k)) \equiv$$
$$(x_1 = n_1 \vee x_1 = n_\perp) \wedge \cdots \wedge (x_k = n_k \vee x_k = n_\perp).$$

Intuitively, $x_i = n_\perp$ tells us that n_i does not belong to the set at hand.

Accordingly, for $\vec{x} = x_1..x_k$ and $\vec{x}' = x'_1..x'_k$ define

$$\mathrm{UNA}_M : \forall \vec{x}, \vec{x}'.\ \textit{set-name}(m(\vec{x})) =$$
$$\textit{set-name}(m(\vec{x}')) \equiv x_1 = x'_1 \wedge \cdots \wedge x_k = x'_k\ .$$

The advantage of this "vector-oriented" representation over a dynamic one including a binary function symbol (as with lists) is that each set has a unique representation. We write $\forall x \in M.\ P(x)$ instead of $\forall x.\ \textit{set-name}(x) \supset P(x)$. Further, we use M for denoting the set of all valid set-names, that is,

$$M = \{m \mid \mathrm{DCA}_M \models \textit{set-name}(m)\}\ .$$

In order to ease notation, we write $m_{1,3}$ instead of $m(n_1, n_\perp, n_3, n_\perp, \ldots, n_\perp)$ when representing the set $\{\delta_1, \delta_3\}$. Also, we abbreviate $m(n_\perp, \ldots, n_\perp)$ by $m_\emptyset$ and $m(n_1, \ldots, n_k)$ by m_D. Note the difference between names n_i and m_i, induced by our notational convention.

We also rely on the "vector-oriented" representation for capturing set membership, denoted by $in/2$. Consider for instance $N = \{n_1, n_2\}$. Membership is then axiomatized through the formulas

$$\forall x_1, x_2.\ in(n_1, m(x_1, x_2)) \equiv (n_1 = x_1)$$
$$\forall x_1, x_2.\ in(n_2, m(x_1, x_2)) \equiv (n_2 = x_2).$$

While this validates $in(n_1, m_{1,2})$, it falsifies $in(n_1, m_2)$. See (15) for the general case.

We need to be able to refer to separate instances of the same default appearing in different sets. For this we introduce a function-symbol $\cdot/2$. For $\delta_j \in D_i$ we write $n_{\delta_j}{\cdot}m_i$ or $n_j{\cdot}m_i$ to name the instance of δ_j appearing in D_i. This results in name set $N{\cdot}M = \{n{\cdot}m \mid n \in N, m \in M\}$. Corresponding axioms, as $\mathrm{DCA}_{N\cdot M}$ and $\mathrm{UNA}_{N\cdot M}$, are obtained in a straightforward way. In what follows, we refer to the various domain closure and unique names axioms pertaining to N, M, and $N{\cdot}M$ as $Ax(N)$.[3]

Given language $\mathcal{L}$, we define a family of languages $\mathcal{L}(m)$ for $m \in M$ as follows. If P is an i-ary predicate symbol then $P(\cdot)$ is a distinct $(i+1)$-ary predicate symbol. If $\gamma \in \mathcal{L}$ then $\gamma(m) \in \mathcal{L}(m)$ is the formula obtained by replacing all predicate symbols in γ with predicate symbols extended as described, and with term m as the $(i+1)^{st}$ argument. This extra argument is used to index formulas by the (names of) sets in which they are used.

Lastly, we introduce special-purpose predicates for controlling the application of sets of defaults. These are summarised in the following table:

Name	Use/meaning
$m \sqsubset m'$	$D_m \subset D_{m'}$
$\mathsf{ok}(e)$	It is ok to try to apply set/rule e
$\mathsf{ap}(e)$	Set/rule e is applied
$\mathsf{bl}(m)$	Not all rules in set m can be applied
$\mathsf{ovr}(m)$	Some set named m' is applied and $m \sqsubset m'$
$\mathsf{ko}(m)$	For set m, $\mathsf{bl}(m) \vee \mathsf{ovr}(m)$ is true

Taking all this into account, we obtain the following translation, mapping default theories in language $\mathcal{L}$ onto default theories in the language $\mathcal{L}^+$ obtained by unioning all languages $\mathcal{L}(m)$ for $m \in M$ and using the aforementioned names and introduced predicates and functions:

3. Note that names in M and $N{\cdot}M$ are obtained from those in N.

Definition 4.1 *Given a finite default theory* (D, W) *over* $\mathcal{L}$ *and its set of associated default names* N*, define* $\mathcal{E}((D, W)) = (D', W')$ *over* $\mathcal{L}^+$ *by*

$$
\begin{aligned}
D' &= D_N \cup D_M \cup D_\neg \\
W' &= W_D \cup W_W \cup W_M \cup W_\sqsubset \cup Ax(N)
\end{aligned}
$$

where

$$
\begin{aligned}
D_N &= \left\{ \frac{\alpha(x) \wedge in(n,x) \wedge \mathsf{ok}(n \cdot x) : \beta(x)}{\gamma(x) \wedge \mathsf{ap}(n \cdot x)} \;\middle|\; n : \frac{\alpha : \beta}{\gamma} \in D \right\} && (8) \\
D_M &= \left\{ \frac{\mathsf{ok}(x) : \neg \mathsf{ko}(x)}{\forall y \in N.\ in(y,x) \supset \mathsf{ok}(y \cdot x)} \right\} && (9) \\
&\cup \left\{ \frac{in(n,x) \wedge \mathsf{ok}(x) : \neg \alpha(x)}{\mathsf{bl}(x)} \;\middle|\; n : \frac{\alpha : \beta}{\gamma} \in D \right\} && (10) \\
&\cup \left\{ \frac{([\forall y \in N.\ in(y,x) \supset c(y,x)] \supset \neg \beta(x)) \wedge \mathsf{ok}(x) :}{\mathsf{bl}(x)} \;\middle|\; n : \frac{\alpha : \beta}{\gamma} \in D \right\} && (11) \\
D_\neg &= \left\{ \frac{: \neg(x \sqsubset y)}{\neg(x \sqsubset y)}, \frac{: \neg in(x,y)}{\neg in(x,y)} \right\} && (12) \\
W_W &= \{\forall x \in M.\ \alpha(x) \mid \alpha \in W\} && (13) \\
W_D &= \{\forall x \in M.\ c(n_\delta, x) \equiv \mathit{CON}(\delta)(x) \mid \delta \in D\} && (14) \\
W_M &= \{\forall x_1, \ldots, x_k.\ in(n_i, m(x_1, \ldots, x_k)) && (15) \\
&\qquad \equiv (n_i = x_i) \mid n_i \text{ in } \langle n_1, \ldots, n_k \rangle\} \\
&\cup \{\forall x, x' \in M. [\exists y \in N. \neg in(y,x) \wedge in(y,x')] && (16) \\
&\qquad \wedge [\forall y.\ in(y,x) \supset in(y,x')] \supset x \sqsubset x' \} \\
W_\sqsubset &= \{\mathsf{ok}(m_D)\} && (17) \\
&\cup \{\forall x \in M. \big[\forall y \in M.\ x \sqsubset y \supset \mathsf{bl}(y)\big] \supset \mathsf{ok}(x)\} && (18) \\
&\cup \{\forall x \in M.\ \big[\mathsf{bl}(x) \vee \mathsf{ovr}(x)\big] \supset \mathsf{ko}(x)\} && (19) \\
&\cup \{\forall x \in M.\ \big[\forall y \in N.\ in(y,x) \supset \mathsf{ap}(y \cdot x)\big] \supset \mathsf{ap}(x)\} && (20) \\
&\cup \{\forall x, x' \in M.\ \mathsf{ap}(x) \supset (x' \sqsubset x \supset \mathsf{ovr}(x'))\} && (21)
\end{aligned}
$$

The rules in D_N and D_M directly generalise those in (1–3), from treating a single set named m to an arbitrary set referenced by variable x. The specific consequents used in the second rule in (3) are dealt with via the axioms in (W_D/14) that allows us to quantify over default consequents (via predicate c). This trick avoids the exponential blowup that would occur in (11) if we were to explicitly give the consequences of the rules.

The rules in ($D_\neg$/12) provide us with complete knowledge on predicates $\sqsubset$ and in. The axioms in $(W_W/13)$ propagate the information in W to all possible contexts.

W_M takes care of what we need wrt set operations. That is, (15) formalises set membership, while (16) formalises strict set inclusion. $W_\sqsubset$ axiomatises the control flow along the partial order induced by $\sqsubset$. Axioms (17) and (18) tell us when it is ok to consider a certain set: we always consider the maximum set D; otherwise, via (18), we consider a set just when every superset is known to be blocked (and so inapplicable). (19) tells us when the consideration of a set is cancelled. This either happens because a set is inapplicable (given by bl) or because it has been explicitly cancelled (given by ovr). (20) asserts that a set is applied just if all of its member rules are. Once we have found an applicable set of rules (and hence a set of generating defaults) we need not consider any subset; (21) annuls the consideration of all such subsets.

For example, consider the following normal default theory:

$$\Delta_{22} = \left(\left\{ n_1 : \frac{:A}{A}, n_2 : \frac{:B}{B}, n_3 : \frac{:\neg B}{\neg B}, n_4 : \frac{B:D}{D} \right\}, \emptyset \right). \tag{22}$$

From $\mathcal{E}(\Delta_{22})$ we get an extension, where the only "ap-literals" are $\mathsf{ap}(m_{1,2,4})$ and $\mathsf{ap}(m_{1,3})$. That is, Δ_{22} has two extensions with generating defaults, the first with δ_1, δ_2, δ_4, and the second with δ_1, δ_3. Among formulas in the extension of $\mathcal{E}(\Delta_{22})$ are $A(m_{1,2,4})$, $A(m_{1,3})$, $B(m_{1,2,4})$, $\neg B(m_{1,3})$, and $D(m_{1,2,4})$. To see this, let us take a closer look at the image of Δ_{22}, namely $\mathcal{E}(\Delta_{22})$. For D_N, we get

$$\frac{in(n_1,x) \wedge \mathsf{ok}(n_1 \cdot x) : A(x)}{A(x) \wedge \mathsf{ap}(n_1 \cdot x)} \qquad \frac{in(n_2,x) \wedge \mathsf{ok}(n_2 \cdot x) : B(x)}{B(x) \wedge \mathsf{ap}(n_2 \cdot x)} \tag{23}$$

$$\frac{in(n_3,x) \wedge \mathsf{ok}(n_3 \cdot x) : \neg B(x)}{\neg B(x) \wedge \mathsf{ap}(n_3 \cdot x)} \qquad \frac{B(x) \wedge in(n_4,x) \wedge \mathsf{ok}(n_4 \cdot x) : D(x)}{D(x) \wedge \mathsf{ap}(n_4 \cdot x)} \tag{24}$$

We get a single nontrivial rule in (10), namely

$$\frac{in(n_4,x) \wedge \mathsf{ok}(x) : \neg B(x)}{\mathsf{bl}(x)} \tag{25}$$

and four rules in (11)

$$\frac{([\forall y \in N.\ in(y,x) \supset c(y,x)] \supset \neg A(x)) \wedge \mathsf{ok}(x) :}{\mathsf{bl}(x)} \tag{26}$$

$$\frac{([\forall y \in N.\ in(y,x) \supset c(y,x)] \supset \neg B(x)) \wedge \mathsf{ok}(x) :}{\mathsf{bl}(x)} \tag{27}$$

$$\frac{([\forall y \in N.\ in(y,x) \supset c(y,x)] \supset \ \ B(x)) \wedge \mathsf{ok}(x) :}{\mathsf{bl}(x)} \tag{28}$$

$$\frac{([\forall y \in N.\ in(y,x) \supset c(y,x)] \supset \neg D(x)) \wedge \mathsf{ok}(x) :}{\mathsf{bl}(x)} \tag{29}$$

Given $\mathsf{ok}(m_D)$, we may consider any rule in D_M. However, given that $\forall y \in N.\ in(y, m_D)$ is true, we obtain that (14) and $\forall y \in N.\ in(y, m_D) \supset c(y, m_D)$ are inconsistent and thus imply any formula. Consequently, rules (26) to (29) are applicable and provide $\mathsf{bl}(m_D)$, yielding $\mathsf{ko}(m_D)$, which in turn blocks (9) for $x = m_D$. From (16), we obtain (among other relations) $m_{1,2,3} \sqsubset m_D$, $m_{1,2,4} \sqsubset m_D$, $m_{1,3,4} \sqsubset m_D$, and $m_{2,3,4} \sqsubset m_D$. From (18), we then get $\mathsf{ok}(m_{1,2,3})$, $\mathsf{ok}(m_{1,2,4})$, $\mathsf{ok}(m_{1,3,4})$, and $\mathsf{ok}(m_{2,3,4})$.

Now, consider $\mathsf{ok}(m_{1,2,4})$. From (9), we obtain

$$\forall y \in N.\ in(y, m_{1,2,4}) \supset \mathsf{ok}(y \cdot m_{1,2,4})$$

yielding $\mathsf{ok}(n_1{\cdot}m_{1,2,4})$, $\mathsf{ok}(n_2{\cdot}m_{1,2,4})$, and $\mathsf{ok}(n_4{\cdot}m_{1,2,4})$. This allows us to apply three of the four rules in (23/24) and we obtain $A(m_{1,2,4}) \wedge \mathsf{ap}(n_1{\cdot}m_{1,2,4})$, $B(m_{1,2,4}) \wedge \mathsf{ap}(n_2{\cdot}m_{1,2,4})$, and $D(m_{1,2,4}) \wedge \mathsf{ap}(n_4{\cdot}m_{1,2,4})$. From (20), we obtain $\mathsf{ap}(m_{1,2,4})$, from which we deduce with (21) in turn $\mathsf{ovr}(m_{1,2,4})$, $\mathsf{ovr}(m_{2,4})$, $\ldots$, $\mathsf{ovr}(m_4)$, and $\mathsf{ovr}(m_\emptyset)$.

Next, consider $\mathsf{ok}(m_{1,2,3})$. As with $\mathsf{ok}(m_D)$, we obtain an inconsistency among $in(n_1, m_{1,2,3})$, $in(n_2, m_{1,2,3})$, $in(n_3, m_{1,2,3})$, $\forall y \in N.\ in(y, m_{1,2,3}) \supset c(y, m_{1,2,3})$, and (14). This validates the prerequisites of rules (26), (27), and (28), thus yielding $\mathsf{bl}(m_{1,2,3})$. As above, we then get from W_M that $\mathsf{ok}(m_{1,2})$, $\mathsf{ok}(m_{1,3})$, $\mathsf{ok}(m_{2,3})$. Note that we have already obtained $\mathsf{ovr}(m_{1,2})$ from $\mathsf{ap}(m_{1,2,4})$.

Given $\mathsf{ok}(m_{1,3})$, (9) provides us with $\mathsf{ok}(n_1{\cdot}m_{1,3})$ and $\mathsf{ok}(n_3{\cdot}m_{1,3})$. Using the two first rules in (23/24), we get $A(m_{1,3}) \wedge \mathsf{ap}(n_1{\cdot}m_{1,3})$ and $\neg B(m_{1,3}) \wedge \mathsf{ap}(n_3{\cdot}m_{1,3})$. From (20), we then get $\mathsf{ap}(m_{1,3})$, from which we deduce with (21) in turn $\mathsf{ovr}(m_1)$, $\mathsf{ovr}(m_3)$, and $\mathsf{ovr}(m_\emptyset)$ (again).

Given $\mathsf{ok}(m_{2,3})$, along with the fact that $in(n_2, m_{2,3})$, $in(n_3, m_{2,3})$, $\forall y \in N.\ in(y, m_{2,3}) \supset c(y, m_{2,3})$, and (14) imply $B(m_{2,3})$ and $\neg B(m_{2,3})$, Rule (27) and (28) fire and we get $\mathsf{bl}(m_{2,3})$.

The next results show that our default theories resulting from $\mathcal{E}$ have appropriate properties.

Theorem 4.1 *Let E be a consistent extension of $\mathcal{E}((D, W))$ for semi-monotonic default theory (D, W). We have for all $\delta \in D$ and for all $D_m, D_{m'} \subseteq D$ that:*

1) $(m \sqsubset m') \in E$ iff $\neg(m \sqsubset m') \notin E$

2) $in(n_\delta, m) \in E$ iff $\neg in(n_\delta, m) \notin E$

3) $\mathsf{ok}(m) \in E$ if $\mathsf{ovr}(m) \notin E$

4) $\mathsf{ok}(m) \in E$ if ($\mathsf{ap}(m) \in E$ or $\mathsf{bl}(m) \in E$)

5) $\mathsf{ap}(m) \in E$ iff $\mathsf{ko}(m) \notin E$

6) $\mathsf{ko}(m) \in E$ iff ($\mathsf{bl}(m) \in E$ or $\mathsf{ovr}(m) \in E$)

7) $\mathsf{ovr}(m) \in E$ iff $\mathsf{ap}(m') \in E$ and $m \sqsubset m' \in E$ for some $m' \in M$.

8) If $\mathsf{ap}(m) \in E$ then $\mathsf{bl}(m') \in E$ for all $m' \in M$ with $m \sqsubset m' \in E$.

9) If $\mathsf{ap}(m) \in E$ then $\mathsf{ovr}(m') \in E$ for all $m' \in M$ with $m' \sqsubset m \in E$.

10) If $\mathsf{ap}(m), \mathsf{ap}(m') \in E$ then $\neg(m \sqsubset m') \in E$

Theorem 4.2 *If (D, W) is a semi-monotonic default theory then $\mathcal{E}((D, W))$ has a unique extension.*

The next two theorems show that our translation captures an encoding of extensions of a semi-monotonic default theory.

Theorem 4.3 *Let (D, W) be a semi-monotonic default theory and let E be the extension of $\mathcal{E}((D, W))$.*

Then for any $\mathsf{ap}(m) \in E$ with $m \in M$, we have that $Th(\{\gamma \mid \gamma(m) \in E\})$ is an extension of (D, W).

Theorem 4.4 *Let (D, W) be a semi-monotonic default theory with extensions $E_1, ..., E_n$ and let E be the extension of $\mathcal{E}((D, W))$.*

Then, for any $i \in \{1, \ldots, n\}$, there is some $m \in M$ naming GD(D, E_i) such that $\mathsf{ap}(m) \in E$.

Lastly, our claim that a translated theory is a constant factor larger than the original requires a caveat. UNA$_N$ yields a quadratic number of unique names assertions. In practice this is no problem, since any sensible implementation would not explicitly list such axioms. With the exception of unique names assertions, a translated theory is a constant factor larger than the original. To see this, it suffices to examine Definition 4.1. Each of (8, 10, 11, 14, 15) introduce $|D|$ axioms/rules; (13) introduces $|W|$ axioms. All remaining terms introduce a single axiom. Moreover, the size of individual axioms is similarly bounded. (For example, each instance of (8) is a constant factor larger than the original default.)

5. Discussion

We have shown how we can encode a semi-monotonic default theory so that the extension from the encoding represents all extensions of the original theory. The fact that we encode all extensions of a theory within a single extension means that we can now encode phenomena of interest, usually dealt with at the metalevel, at the object level. Specifically we can now encode the notions of skeptical and credulous inference within a theory. In order to do this, we introduce two new constants $skep$ and $cred$, for "skeptical" and "credulous" respectively.

A formula is a skeptical inference if it is a member of every extension. In our approach, this means that it follows in every "ap-set". Hence we define skeptical inference within a theory, for a given formula γ, by

$$(\forall x \in M.\ \mathsf{ap}(x) \supset \gamma(x)) \supset \gamma(skep).$$

For credulous inference, the simplest option is to assert that a formula is a credulous inference if it is a member of some extension:

$$(\exists x \in M.\ \mathsf{ap}(x) \wedge \gamma(x)) \supset \gamma(cred).$$

However, this is overly simplistic, since with this definition, a formula and its negation may be credulous inferences. A more reasonable definition is to assert that a formula is a credulous inference if it is a member of some extension, and its negation is a member of no extension. We can define this notion of credulous inference (indicated by $cred'$) for a formula γ by means of the default:

$$\frac{\exists x \in M.\ \mathsf{ap}(x) \wedge \gamma(x)\ :\ \forall x \in M.\ \mathsf{ap}(x) \supset \gamma(x)}{\gamma(cred')}.$$

Hence in Example (22), we obtain that A is a $skep$tical inference, while D is a $cred'$ulous inference. B and $\neg B$ are $cred$ulous inferences.

We have suggested that the approach may be applicable in diagnosis programs, such as found in [REI 87]. Similarly, the approach can be used to directly encode applications expressible in Theorist [POO 88]. That is, there is a correspondence between so-called *Poole-type* theories and Theorist with constraints [DIX 92]. Since Poole-type theories are semi-monotonic, this means that our approach can encode any application encodable in Theorist.

Our approach relies on a first-order language. Despite this, the image of a theory over a finite language remains finite. As regards implementation, however, it is not advisable to use a bottom-up grounding approach, as done in many implementations of extended logic programming [EIT 97, NIE 97]. Instead, a query-oriented approach seems to be advantageous, because it may rely on unification rather than ground instantiation.

In Definition 4.1, sets of defaults were ordered based on the partial order given by set containment. This order represents one example of a *preference* order on sets of defaults. A natural avenue for future work would be to generalise our approach to address arbitrary preference orders on sets of defaults. In an arbitrary preference order on sets, one could represent desiderata as found in configuration, scheduling, or (generally) decision-theoretic problems. This could also be combined with the present approach yielding an encoding of preferences on extensions. Hence, for our diagnosis example, we might want to prefer extensions (diagnoses) on the basis of an ordering based on reliability of components.

6. Conclusion

We have described an approach for encoding default extensions within a single extension. Using constants and functions for naming, we can refer to default rules, sets of defaults, and instances of a rule in a set. Via these names we can, first, determine whether a set of defaults is its own set of generating defaults and, second, consider the application of sets of defaults ordered by set containment. The translated theory requires a modest increase in space: except for unique names axioms, only a constant-factor increase is needed. The translated theory is a (regular, Reiter) default theory. Hence we essentially axiomatise the notion of “extensions” for the class of semi-monotonic default theories, resulting in a single extension. Further, we are able to prove that our translation behaves correctly.

Using the approach we can now express notions such as skeptical and credulous inference within a theory. Arguably this will prove beneficial in expressing at the object level problems and approaches generally expressed at the metalevel. Areas of application range from specific areas such as diagnosis, to broadly-applicable approaches such as Theorist. Lastly, we suggest that the approach may be easily extended to address arbitrary preferences over sets of defaults.

A. Proofs of Theorems

A.1. *Proofs for Section 3*

The following definition is used in the proofs.

Definition A.1 ([REI 80]) *Let (D, W) be a default theory. For any set of formulas S, let $\Gamma(S)$ be the smallest set of formulas S' such that*

1) $W \subseteq S'$,

2) $Th(S') = S'$,

3) For any $\frac{\alpha : \beta}{\gamma} \in D$, if $\alpha \in S'$ and $\neg\beta \notin S$ then $\gamma \in S'$.

A set of formulas E is an extension of (D, W) if $\Gamma(E) = E$.

With respect to the various translations we adopt the following notation: For a set of defaults with name m and one of its members with name n, let δ_a^m, $\delta_{b_1}^{m,n}$, and $\delta_{b_2}^{m,n}$ be the corresponding default rules in D_M. Let $\delta_a^{m,n}$ denote the transform of the individual default named n with x instantiated to m in D_N.

Proof 3.1

1. **if part** Suppose $\mathsf{ap}(n_\delta) \in E$ for all $\delta \in D_m$. Since E is deductively closed and since E contains Formula (6), we deduce that $\mathsf{ap}(m) \in E$.

only-if part Suppose $\mathsf{ap}(m) \in E$. By construction, this implies $\mathsf{ap}(n_\delta) \in E$ for all δ such that $in(n_\delta, m) \in E$, or $\delta \in D_m$. By the definition of D_N and W_D, however, we have $\mathsf{ap}(n_\delta) \in E$ only if $CON(\delta) \wedge \mathsf{ap}(n_\delta) \in E$. Since this holds for all $\delta \in D_m$, we obtain $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \cup CON(D_m) \subseteq E$.

2. **if part** Suppose $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \not\subseteq E$. Thus, there is some $\delta \in D_m$ such that $\delta_a^{m,n_\delta} \notin GD(D', E)$. Then, one of the following cases must be true.

- If $\neg JUS(\delta) \in E$, then clearly from W_D we get $CON(\delta_1) \wedge \cdots \wedge CON(\delta_k) \supset \neg JUS(\delta) \in E$, where $\{\delta_1, \ldots, \delta_k\} = D_m$. By Theorem 3.1.3 and the fact that E is deductively closed, we get that

$$[CON(\delta_1) \wedge \cdots \wedge CON(\delta_k) \supset \neg JUS(\delta)] \wedge \mathsf{ok}(m) \in E.$$

Hence $\delta_{b_2}^{m,n_\delta} \in GD(D', E)$, that is, $\mathsf{bl}(m) \in E$.

- Suppose $PRE(\delta) \wedge \mathsf{ok}(n_\delta) \notin E$. Since E is deductively closed, we may distinguish the following cases.

- Assume $PRE(\delta) \notin E$. Consequently, $\delta_{b_1}^{m,n_\delta} \in GD(D', E)$, that is, $\mathsf{bl}(m) \in E$.

- If $\mathsf{ok}(n_\delta) \notin E$, then $\delta_a^m \notin GD(D', E)$, since this is the only means by which we can fail to obtain $\mathsf{ok}(n_\delta) \in E$. Hence $\mathsf{ovr}(m) \in E$; hence from (5) we get $\mathsf{bl}(m) \in E$.

Thus, in all cases we obtain that $\mathsf{bl}(m) \in E$.

only-if part Suppose $\mathsf{bl}(m) \in E$. We distinguish the following two cases.

- If $\delta_{b_1}^{m,n_j} \in GD(D', E)$, then we have that $PRE(\delta_j) \notin E$ for some $\delta_j \in D_m$. Therefore, $(\delta_j)_a^{m,n_j} \notin GD(D', E)$ and clearly $\mathsf{ap}(n_j) \notin E$.

- If $\delta_{b_2}^{m,n_j} \in GD(D', E)$, then we have for some $\delta_j \in \{\delta_1, \ldots, \delta_k\} = D_m$ that

$$CON(\delta_1) \wedge \cdots \wedge CON(\delta_k) \supset \neg JUS(\delta_j) \in E. \quad (30)$$

Assume $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \subseteq E$, that is, by definition of D_N that $\{CON(\delta) \wedge \mathsf{ap}(n_\delta) \mid \delta \in D_m\} \subseteq E$. Since E is deductively closed we get from (30) that $\neg JUS(\delta_j) \in E$ and therefore $(\delta_j)_a^{m,n_j} \notin GD(D', E)$ and clearly $\mathsf{ap}(n_j) \notin E$, a contradiction.

In both cases we thus obtain $\{\mathsf{ap}(n_\delta) \mid \delta \in D_m\} \not\subseteq E$.

if part Suppose $\mathsf{ap}(n_\delta) \in E$. Then we have necessarily that $\delta_a^{m,n_\delta} \in GD(D', E)$, and therefore that $PRE(\delta) \wedge \mathsf{ok}(n_\delta) \in E$, and so $\mathsf{ok}(n_\delta) \in E$.

only-if part Suppose $\mathsf{ok}(n_\delta) \in E$. Then, we have by definition of D_M that

$$\frac{: \neg \mathsf{ovr}(m)}{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_\delta) \wedge \cdots \wedge \mathsf{ok}(n_k)} \in GD(D', E).$$

Clearly, we thus have $\mathsf{ovr}(m) \notin E$; this implies $\mathsf{bl}(m) \notin E$. As a consequence, we get $\delta_{b_1}^{m,n_\delta} \notin GD(D', E)$ and $\delta_{b_2}^{m,n_\delta} \notin GD(D', E)$. We obtain for each $\delta \in \{\delta_1, \ldots, \delta_k\} = D_m$ that

$$PRE(\delta) \in E \quad \text{and} \quad CON(\delta_1) \wedge \cdots \wedge CON(\delta_k) \supset \neg JUS(\delta) \notin E.$$

Furthermore, the latter gives $\neg JUS(\delta) \notin E$. With $\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_\delta) \wedge \cdots \wedge \mathsf{ok}(n_k) \in E$ and the fact that E is deductively closed, we get that $\delta_a^{m,n_\delta} \in GD(D', E)$ for all $\delta \in D_m$. That is, since E is deductively closed, $\mathsf{ap}(n_\delta) \in E$ for all $\delta \in D_m$.

if part Suppose $\mathsf{ovr}(m) \notin E$, and so $\mathsf{bl}(m) \notin E$. As a consequence, we get $\delta_{b_1}^{m,n_\delta} \notin GD(D', E)$ and $\delta_{b_2}^{m,n_\delta} \notin GD(D', E)$ for some $\delta \in D_m$. As a corollary of Theorem 3.1.1-2, we obtain that: $\delta_a^m \in GD(D', E)$ iff ($\delta_{b_1}^{m,n_\delta} \notin GD(D', E)$ and $\delta_{b_2}^{m,n_\delta} \notin GD(D', E)$ for all $\delta \in D_m$). This implies that $\delta_a^m \in GD(D', E)$. Therefore $\mathsf{ok}(n_\delta) \in E$.

only-if part If $\mathsf{ok}(n_\delta) \in E$, then $\delta_a^m \in GD(D', E)$, that is, $\mathsf{ovr}(m) \notin E$.

The if-part is trivial.

For the only-if part, assume that $\mathsf{ap}(n_\delta) \in E$ for some $\delta \in D_m$. Then $\delta_a^{m,n_\delta} \in GD(D', E)$ and therefore $\delta_a^m \in GD(D', E)$.

We also have $in(n_\delta, m) \in E$ for all $\delta \in D_m$. Further, $\delta_a^m \in GD(D', E)$ implies $\mathsf{ok}(n_\delta) \in E$ for all $\delta \in D_m$. By Theorem 3.1.3, this implies $\mathsf{ap}(n_\delta) \in E$ for all $\delta \in D_m$.

■

Proof 3.2

1) First, assume that default theory $(\emptyset \cup D, W)$ has an extension E where $GD(D, E) = D$.

From Definition 2.1 we have that $E = \bigcup_{i=0}^{\infty} E_i$ where

$$\begin{aligned} E_0 &= W \\ E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}. \end{aligned}$$

Obviously then $E = \bigcup_{i=0}^{\infty} E_i$ where

$$\begin{aligned} E_0 &= W \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\} \\ E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}. \end{aligned}$$

defines an extension of $(D_N, W \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\})$.

Replacing W with W' in the above defines an extension of $(D_N, W' \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\})$ as well as of $(D_N \cup D_M, W' \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\})$ or $(D', W' \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\})$

From this it follows that

$$\begin{aligned} E'_{-1} &= W' \\ E'_0 &= Th(W') \cup \{\mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\} \\ &= Th(E_{-1}) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D', \alpha \in E_{-1}, \neg\beta \notin E' \right\} \\ E'_{i+1} &= Th(E'_i) \cup \left\{ \gamma \;\middle|\; \tfrac{\alpha : \beta}{\gamma} \in D', \alpha \in E'_i, \neg\beta \notin E' \right\} \qquad \text{for } i > 1. \end{aligned}$$

and $E' = \bigcup_{i=-1}^{\infty} E'_i$ defines an extension of (D', W').

Thus for this case we have that $\mathcal{S}_m((\emptyset \cup D, W))$ has extension E where $E \cap \mathcal{L} = Th(W \cup CON(D))$.

(Note for this case that having $\mathsf{bl}(m)$ in our purported extension E' would contradict the assumption that D is a set of generating defaults for $(\emptyset \cup D, W)$.

2) Assume that D is not a set of generating defaults for default theory $(\emptyset \cup D, W)$. Thus for any set E and for

$$\begin{aligned} E_0 &= W \\ E_{i+1} &= Th(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i, \neg\beta \notin E \right\}. \end{aligned}$$

we have that $E \neq \bigcup_{i=0}^{\infty} E_i$.

In particular this holds for $E = Th(W \cup CON(D))$.

Since $\bigcup_{i=0}^{\infty} E_i = Th(W \cup C)$ for some $C \subset CON(D)$, this means that some default $\delta_j \in D$ fails to apply. There are two possibilities:

a) $\alpha_j \notin E_i$ for every $i \geq 0$, or

b) $\neg\beta_j \in E$.

For the first case, assume that there is an extension E' of $\mathcal{S}_m((\emptyset \cup D, W))$ containing α_j. Since $\alpha_j \notin W' \setminus W$ we have that $W \cup C' \vdash \alpha_j$ for some $C' \subset CON(D)$.

Since $W \vdash \alpha_j$ contradicts $\alpha_j \notin E_i$ above, we have that $C' \neq \emptyset$ and hence $\mathsf{ap}(n) \in E'$ for some default $n : \delta$.

From Theorem 3.1.6 we obtain that $\{\mathsf{ap}(n_\delta) \mid \delta \in D\} \subseteq E'$, hence in particular $\mathsf{ap}(n_j) \in E'$ and so $\alpha_j \in GD_i$, a contradiction.

Hence there is no extension E' of $\mathcal{S}_m((\emptyset \cup D, W))$ containing α_j.

It follows that $Th(\{W', \mathsf{bl}(m), \mathsf{ok}(n_1) \wedge \cdots \wedge \mathsf{ok}(n_k)\})$ is an extension of $\mathcal{S}_m((\emptyset \cup D, W))$: we have shown that $\alpha_j \in E'$ is not possible for any purported extension. Hence $\delta_{b_1}^{m,n_j}$ does apply, yielding $\mathsf{bl}(m)$, and $\mathsf{ovr}(m)$. This then prevents δ_a^m and any of $\delta_a^{m,n}$ from applying.

In the second case, where $\neg\beta_j \in E$, since $E = Th(W \cup CON(D))$ we get that $W \cup CON(D) \vdash \neg\beta_j$ which, by the previous argument, gives an extension $\mathcal{S}_m((\emptyset \cup D, W))$ by virtue of the applicability of $\delta_{b_2}^{m,n_j}$

■

Proof 3.3

This follows immediately from the first part of the proof of Theorem 3.2.

■

A.2. *Proofs for Section 4*

We first show the following results:

Lemma 1 *Let E be a consistent extension of $\mathcal{S}((D, W, <)) = (D', W')$ for set–ordered default theory $(D, W, <)$.*

1) $(m \sqsubset m') \in E$ *iff* $\neg(m \sqsubset m') \notin E$

2) $in(n_\delta, m) \in E$ *iff* $\neg in(n_\delta, m) \notin E$

Proof 1

1. By the consistency of E, we cannot have both $m \sqsubset m' \in E$ and $\neg(m \sqsubset m') \in E$.

Assume that for some $D_m, D_{m'} \subseteq D$, we have neither $m \sqsubset m' \in E$ nor $\neg(m \sqsubset m') \in E$. Then, however, the default rule $\frac{:\neg(m \sqsubset m')}{\neg(m \sqsubset m')}$ in $D_\neg$ is applicable and we obtain $\neg(m \sqsubset m') \in E$, which contradicts our assumption.

We have thus shown that $m \sqsubset m' \in E$ iff $\neg(m \sqsubset m') \notin E$.

2. Analogous to Proof 1.1.

■

Lemma 2 *Let E be a consistent extension of $\mathcal{S}((D, W, <)) = (D', W')$ for set–ordered default theory $(D, W, <)$.*

1) We have for all $D_1, D_2 \subseteq D$ that $(m_1 \sqsubset m_2) \in E$ iff $(m_1 \sqsubset m_2) \in W'$.

2) We have for all $D_m \subseteq D$ and $\delta \in D$ that $in(n_\delta, m) \in E$ iff $in(n_\delta, m) \in W'$.

Proof 2

1) Clearly, we have $(m_1 \sqsubset m_2) \in E$ if $(m_1 \sqsubset m_2) \in W'$.

Assume we have $(m_1 \sqsubset m_2) \in E$ and $(m_1 \sqsubset m_2) \notin W'$. Since $(m_1 \sqsubset m_2) \notin W' = E_0$, there must exist (according to Definition 2.1) some $i \geq 0$ with $(m_1 \sqsubset m_2) \notin E_i$ but $(m_1 \sqsubset m_2) \in E_{i+1}$. Since there are no default rules with consequents containing positive occurrences of $\sqsubset$-literals, we must have $(m_1 \sqsubset m_2) \in \mathit{Th}(E_i)$. For the same reason, all positive occurrences in E_i must stem from $W_\sqsubset$. In fact, all positive occurrences of $\sqsubset$-literals in $W_\sqsubset$ (in clause form) come from (16) or (18) in $W_\sqsubset$. For (16), we obtain $(m_1 \sqsubset m_2) \in W'$, a contradiction. (18) can be written in the form $((m_1 \sqsubset m_2) \wedge \phi) \vee \varphi \vee \mathsf{ok}(m_1)$ for some formulas ϕ, φ. A proof for $E_i \vdash (m_1 \sqsubset m_2)$ must thus contain the negative ok-literal $\mathsf{ok}(m_1)$. There are however no negative occurrences of ok-literals in $\mathcal{S}((D, W, <))$, neither in D' nor in W', a contraction.

2) Analogous to proof of Lemma 2.1.

■

Proof 4.1

4+5+6. We show for all $D_m \in 2^D$ by induction on $\sqsubset$ that $\mathsf{ok}(m) \in E$ iff $\mathsf{bl}(m) \in E$ or $\mathsf{ap}(m) \in E$, and that $\mathsf{ap}(m) \in E$ iff $\mathsf{ko}(m) \notin E$, or $\mathsf{ko}(m) \in E$ iff ($\mathsf{bl}(m) \in E$ or $\mathsf{ovr}(m) \in E$).

Consider $D_m \in 2^D$ and assume that for all $D_{m'}$ with $D_m \subset D_{m'}$ we have $\mathsf{ok}(m') \in E$ iff $\mathsf{bl}(m') \in W$ or $\mathsf{ap}(m') \in E$, and $\mathsf{ap}(m') \in E$ iff $\mathsf{ko}(m') \notin E$, and $\mathsf{ko}(m) \in E$ iff ($\mathsf{bl}(m) \in E$ or $\mathsf{ovr}(m) \in E$).

First, we have the following lemma.

Lemma 3 *Given the induction hypothesis, we have* $\mathsf{ok}(m) \in E$ *iff for every* m' *where* $m \sqsubset m'$ *we have* $\mathsf{bl}(m') \in E$.

Proof 3 The lemma holds trivially for $m = m_D$.

Otherwise, by the induction hypothesis, we have $\mathsf{ap}(m') \in E$ iff $\mathsf{ko}(m') \notin E$ for all $D_{m'}$ with $D_m \subset D_{m'}$. Hence $\mathsf{bl}(m') \notin E$ since the only way $\mathsf{ovr}(m')$ is derivable is via (19).

By definition of $W_{\sqsubset}$ and Lemma 1, we have $m \sqsubset m' \in E$ for all $D_m, D_{m'}$ with $D_m \subset D_{m'}$.

Analogously, we get $(m \sqsubset m') \notin E$ for all $D_m, D_{m'}$ with $D_m \not\subset D_{m'}$. From this, we get by means of $D_\neg$ that $\neg(m \sqsubset m') \in E$ for all $D_m, D_{m'}$ with $D_m \not\subset D_{m'}$.

Consider the following cases.

- There is m' where $m \sqsubset m' \in E$ and $\mathsf{ap}(m') \in E$.

Because E is deductively closed and contains (21) we derive $\mathsf{ovr}(m) \in E$, and $\mathsf{ko}(m)$ from (19).

The only way in which $\mathsf{ok}(m)$ can be derived is via (18). $\mathsf{bl}(m') \notin E$ by the induction hypothesis, and so we deduce that $\mathsf{ok}(m) \notin E$.

- For every m' where $m \sqsubset m' \in E$ we have $\mathsf{ap}(m') \notin E$.

We obtain that for every such m' that $\mathsf{ko}(m') \in E$. As well, $\mathsf{ovr}(m') \notin E$ since $\mathsf{ovr}(m')$ is derivable only via (19). Thus $\mathsf{bl}(m') \in E$. Then $\mathsf{ok}(m) \in E$ by (18).

■

For $D_m = \{\delta_j \mid j = 1..k\} \in 2^D$, we distinguish the following cases.

- If $\frac{\mathsf{ok}(m) : \neg\mathsf{ko}(m)}{\forall y \in N.\ in(y,x) \supset \mathsf{ok}(y \cdot x)} \in GD(D', E)$, then $\mathsf{ok}(m) \in E$ and $\mathsf{ko}(m) \notin E$. The latter implies $\mathsf{bl}(m) \notin E$. As a consequence, we get $\delta_{b_1}^{m,n_j} \notin GD(D', E)$ and $\delta_{b_2}^{m,n_j} \notin GD(D', E)$ for $j = 1..k$. Since $\mathsf{ok}(m) \in E$, we thus have for each $\delta_j \in \{\delta_1, \ldots, \delta_k\} = D_m$ that

$$\begin{array}{ll} & \mathit{PRE}(\delta_j) \in E \\ \text{and} & \mathit{CON}(\delta_1) \wedge \cdots \wedge \mathit{CON}(\delta_k) \supset \neg \mathit{JUS}(\delta_j) \notin E. \end{array} \qquad (31)$$

Furthermore, (31) implies $\neg\mathit{JUS}(\delta_j) \notin E$. With $\forall y \in N.\ in(y, x) \supset \mathsf{ok}(y \cdot x) \in E$ and the fact that E is deductively closed, we get that $(\delta_j)_a^{m,n} \in GD(D', E)$ for $j = 1..k$. That is, since E is deductively closed, $\mathsf{ap}(n_j \cdot m) \in E$ for $j = 1..k$. And from this we conclude by Theorem 3.1.6 that $\mathsf{ap}(m) \in E$.

- If $\frac{\mathsf{ok}(m) : \neg\mathsf{ko}(m)}{\forall y \in N.\ in(y,x) \supset \mathsf{ok}(y \cdot x)} \notin GD(D', E)$, then we have that $\mathsf{ko}(m) \in E$: The only other possibility is that $\mathsf{ok}(m) \notin E$. But then by Lemma 3 we would have that there is m' where $m \sqsubset m'$ and $\mathsf{ap}(m') \in E$. But then, since E is deductively closed, we get $\mathsf{ovr}(m) \in E$ via (21) and so $\mathsf{ko}(m) \in E$.

Consequently we have $\mathsf{ko}(m) \in E$. It follows that $\mathsf{ap}(m) \notin E$: Assume to the contrary that $\mathsf{ap}(m) \in E$. $\mathsf{ap}(m)$ is derivable via (20) only. But this means that $\mathsf{ap}(n_j \cdot m) \in E$ for every $\delta_j \in D_m$; hence $\mathsf{ok}(n_j \cdot m) \in E$ for every $\delta_j \in D_m$, or for $in(n_j, m)$. But $\mathsf{ok}(n_j \cdot m)$ is obtainable only from application of the default $\frac{\mathsf{ok}(m_i) : \neg\mathsf{ovr}(m_i)}{\forall y \in N.\ in(y,x) \supset \mathsf{ok}(y \cdot x)}$, contradiction.

A similar argument established that $\mathsf{bl}(m) \notin E$.

This demonstrates that $\mathsf{ok}(m) \in E$ iff $\mathsf{ap}(m) \in E$ or $\mathsf{bl}(m) \in E$, and that $\mathsf{ap}(m) \in E$ iff $\mathsf{ko}(m) \in E$ and that $\mathsf{ko}(m) \in E$ iff ($\mathsf{bl}(m) \in E$ or $\mathsf{ovr}(m) \in E$) for all $D_m \in 2^D$.

3. This is a corollary of the preceding.

7. The if-part follows immediately from the last line in $W_\sqsubset$ (21). For the only-if part, we observe that $\mathsf{ovr}(m)$ can be derived only from (21).

8. Assume that $\mathsf{ap}(m) \in E$ for some m and that for some m' where $m \sqsubset m' \in E$ we have $\mathsf{bl}(m') \notin E$. This means that $\mathsf{ko}(m') \notin E$ since $\mathsf{ko}(m') \in E$ is derivable only by (19). $\mathsf{ko}(m') \notin E$ implies $\mathsf{ap}(m) \in E$ (Theorem 4.1.5) and (from $W_\sqsubset$) we get $\mathsf{ovr}(m) \in E$. But then $\mathsf{ko}(m) \in E$ and $\mathsf{ap}(m) \in E$, $\mathsf{ko}(m) \in E$ contradicts Theorem 4.1.5.

9. This is obvious from (21).

10. Assume that $\mathsf{ap}(m), \mathsf{ap}(m') \in E$ where $(m \sqsubset m') \in E$. Since $\mathsf{ap}(m') \in E$ we have from (21) that $\mathsf{ovr}(m) \in E$ and $\mathsf{ko}(m) \in E$. But $\mathsf{ap}(m) \in E$, $\mathsf{ko}(m) \in E$ contradicts Theorem 4.1.5.

■

Proof 4.3 Let (D, W) be a semi-monotonic default theory and let E be an extension of $\mathcal{E}(D, W) = (D', W')$.

We make use of the following definition:

Definition A.2 $\downarrow(S, m) = \{\gamma \in \mathcal{L} \mid \gamma(m) \in S\}$.

Assume that $\mathsf{ap}(m) \in E$ where m is the name of D_m. We show that $\downarrow(Th(W \cup \mathit{CON}(D_m)), m)$ is an extension of (D, W).

We have $\mathsf{ok}(m) \in E$ by Theorem 4.1.4. Let i be the least integer such that $\mathsf{ok}(m) \in E_i$, and let j be the least integer such that $\mathsf{ap}(m) \in E_j$. (That is, in the definition of an extension there is some step, i, where $\mathsf{ok}(m)$ is asserted. Following this the defaults corresponding to elements of D_m are applied. At (later) step $j > i$ we are "done" applying the defaults and $\mathsf{ap}(m)$ is asserted.)

Lemma 4 $\downarrow(E_i, m) = \mathit{Th}(W)$ *for i as above.*

Proof 4 Since $\mathsf{ap}(m) \in E_j$ we have $\mathsf{ap}(m) \in E$, and from Theorem 4.1.8 we get that $\mathsf{bl}(m') \in E$ for all m' such that $m \sqsubset m'$. Thus $\mathsf{ko}(m') \in E$.

Thus for every m' where $m \sqsubset m'$ and for default $\delta_l \in D_{m'}$ we have $\mathsf{ok}(n_l \cdot m') \notin E$ (since the only way $\mathsf{ok}(n_l \cdot m')$ can be inferred is from (9)). Hence default δ_l isn't applied in E. Since this holds for arbitrary m' where $m \sqsubset m'$, it follows that $\downarrow(E_i, m) = \downarrow(E_0, m) = \mathit{Th}(W)$. ■

Since $\mathsf{ap}(m) \in E$, $\mathsf{ko}(m) \in E$ via Theorem 4.1.5, so (9) is applicable at step $i+1$.

We have:

$$\begin{aligned}
E_{i+1} &\subseteq \mathit{Th}(E_i) \cup \{\mathsf{ok}(n_j \cdot m) \mid \delta_j \in D_m\} \\
E_{i+k+1} &\subseteq \mathit{Th}(E_{i+k}) \cup \Big\{\gamma(m) \wedge \mathsf{ap}(n \cdot m) \,\Big|\, \tfrac{\alpha(m) \wedge in(n,m) \wedge \mathsf{ok}(n \cdot m) : \beta(m)}{\gamma(m) \wedge \mathsf{ap}(n \cdot m)} \in D_m, \\
&\qquad \alpha(m) \wedge in(n,m) \wedge \mathsf{ok}(n \cdot m) \in E_{i+k}, \neg\beta(m) \notin E\Big\} \qquad \text{for } 0 < k < j - i. \\
E_j &\subseteq \mathit{Th}(E_{j-1})\,.
\end{aligned}$$

Observe that for $k > j$ we have $\downarrow(E_k, m) = \downarrow(E_j, m)$ since the name m appears only in relation to the set D_m.

Define: $E_k^m = \downarrow(E_{i+k+1}, m)$ for $0 \leq k$.

For later use, we have the following small lemma.

Lemma 5

$$\bigcup_{k=0}^{\infty} E_k^m = \downarrow(E, m).$$

Proof 5 Since $E_k^m = \downarrow(E_{i+k+1}, m)$ for $0 \leq k$ we have
$\bigcup_{k=0}^{\infty} E_k^m = \bigcup_{k=i+1}^{\infty} \downarrow(E_k, m) = \downarrow(\bigcup_{k=i+1}^{\infty} E_k, m)$
We also have $\downarrow(E_0, m) = \cdots = \downarrow(E_i, m) = \mathit{Th}(W)$ from Lemma 4.

Hence:

$$\begin{aligned}
\bigcup_{k=0}^{\infty} E_k^m &= \downarrow(E_0, m) \cup \cdots \cup \downarrow(E_i, m) \cup \downarrow(\textstyle\bigcup_{k=i+1}^{\infty} E_k, m) \\
&= \downarrow(\textstyle\bigcup_{k=0}^{\infty} E_k, m) \\
&= \downarrow(E, m).
\end{aligned}$$

■

We show that the sets E_k^m ($0 \leq k$) and $\bigcup_{k=0}^{\infty} E_k^m$ satisfy the conditions for an extension.

To begin with, we have by definition:

$$E_k^m = \downarrow(E_{i+k+1}, m) \qquad \text{for } 0 \leq k < j - i. \tag{32}$$

and in particular for $k = 0$ we have

$$E_0^m = \downarrow(E_{i+1}, m) = Th(W)\,.$$

For $0 \leq k < j - i$ the only applicable defaults with consequents with name m are of the form $\frac{\alpha(m)\wedge in(n,m)\wedge \mathsf{ok}(n\cdot m)\,:\,\beta(m)}{\gamma(m)\wedge \mathsf{ap}(n\cdot m)}$. We expand the right hand side of (32) using Definition 2.1 to obtain:

$$\begin{aligned}
E_{k+1}^m &= \downarrow\!\Big(Th(E_{i+k+2}) \cup \\
&\qquad \Big\{\gamma(m) \wedge \mathsf{ap}(n\cdot m) \,\Big|\, \tfrac{\alpha(m)\wedge in(n,m)\wedge \mathsf{ok}(n\cdot m)\,:\,\beta(m)}{\gamma(m)\wedge \mathsf{ap}(n\cdot m)} \in D', \\
&\qquad\qquad \alpha(m) \wedge in(n,m) \wedge \mathsf{ok}(n\cdot m) \in E_{i+k+2}, \neg\beta(m) \notin E\Big\}, m\Big) \\
&= \downarrow(Th(E_{i+k+2}), m) \cup \\
&\qquad \downarrow\!\Big(\Big\{\gamma(m) \wedge \mathsf{ap}(n\cdot m) \,\Big|\, \tfrac{\alpha(m)\wedge in(n,m)\wedge \mathsf{ok}(n\cdot m)\,:\,\beta(m)}{\gamma(m)\wedge \mathsf{ap}(n\cdot m)} \in D', \\
&\qquad\qquad \alpha(m) \wedge in(n,m) \wedge \mathsf{ok}(n\cdot m) \in E_{i+k+2}, \neg\beta(m) \notin E\Big\}, m\Big)
\end{aligned}$$

so

$$\begin{aligned}
E_{k+1}^m &= Th(E_k^m) \cup \Big\{\gamma \,\Big|\, \tfrac{\alpha(m)\wedge in(n,m)\wedge \mathsf{ok}(n\cdot m)\,:\,\beta(m)}{\gamma(m)\wedge \mathsf{ap}(n\cdot m)} \in D', \alpha \in E_k^m, \neg\beta \notin \downarrow(E,m)\Big\} \\
&= Th(E_k^m) \cup \Big\{\gamma \,\Big|\, \tfrac{\alpha\,:\,\beta}{\gamma} \in D, \alpha \in E_k^m, \neg\beta \notin \downarrow(E,m)\Big\}\,.
\end{aligned}$$

This together with Lemma 5 shows that $\downarrow(E, m)$ satisfies the definition of an extension.

■

Proof 4.4

Define: for $D_1, D_2 \subseteq D$, $D_1 < D_2$ iff $D_1 \subset D_2$.

Let (D, W) be a semi-monotonic default theory with extensions $E_1, \ldots, E_n$. $\mathcal{E}(D, W) = (D', W')$ be given as in Definition 4.1.

For ease of notation, let m_i name $\zeta(GD(D, E_i))$ for $i = 1..n$.

Define in $\mathcal{L}^+$:

$$
\begin{array}{rcl}
E' = Th\big(W' & \cup & \{\mathit{CON}(\delta)(m_i) \wedge \mathsf{ap}(n_\delta{\cdot}m_i), \mathsf{ok}(n_\delta{\cdot}m_i) \mid \delta \in \mathit{GD}(D, E_i) \text{ for } i = 1..n\} \\
 & \cup & \{\neg(m_i \sqsubset m_j) \mid (D_i, D_j) \not\in <\} \\
 & \cup & \{\neg in(n_\delta, m_i) \mid \delta \notin D_i,\ D_i \subseteq D\} \\
 & \cup & \{\mathsf{ok}(m) \mid \mathit{GD}(D, E_i) \subseteq D_m \text{ for some } i \in \{1..n\}\} \\
 & \cup & \{\mathsf{ap}(m) \mid D_m = \mathit{GD}(D, E_i),\ \text{for some } i \in \{1..n\}.\} \\
 & \cup & \{\mathsf{ko}(m) \mid D_m \neq \mathit{GD}(D, E_i) \text{ for every } i \in \{1..n\}\} \\
 & \cup & \{\mathsf{bl}(m) \mid \mathit{GD}(D, E_i) \subset D_m \text{ for some } i \in \{1..n\}\} \\
 & \cup & \{\mathsf{ovr}(m) \mid D_m \subset \mathit{GD}(D, E_i) \text{ for some } i \in \{1..n\}\}\ \big)
\end{array}
$$

To begin with, we show that for m_i naming a set of generating defaults of (D, W) that

$$\downarrow\!(E', m_i) = E_i. \tag{33}$$

– If $\alpha \in W$ then since $W \subseteq E$ and $W \subseteq \downarrow\!(E', m_i)$, we have $\alpha \in E$ iff $\alpha \in \downarrow\!(E', m_i)$.

– If $\alpha \in \mathit{CON}(\mathit{GD}(D, E_i))$ then since $\mathit{CON}(\mathit{GD}(D, E_i)) \subseteq E$ and $\mathit{CON}(\mathit{GD}(D, E_i)) \subseteq \downarrow\!(E', m_i)$ again $\alpha \in \downarrow\!(E', m_i)$ iff $\alpha \in E$.

– Last, we have $W \cup \mathit{CON}(\mathit{GD}(D, E_i)) \subseteq E_i$ and $W \cup \mathit{CON}(\mathit{GD}(D, E_i)) \subseteq \downarrow\!(E', m_i)$. Since E_i and $\downarrow\!(E', m_i)$ are deductively closed, this implies that $\alpha \in Th(W \cup \mathit{CON}(\mathit{GD}(D, E_i))) \subseteq E_i$ iff $\alpha \in Th(W \cup \mathit{CON}(\mathit{GD}(D, E_i))) \subseteq \downarrow\!(E', m_i)$.

Consequently for every $\alpha \in \mathcal{L}$, we have shown that $\alpha \in \downarrow\!(E', m_i)$ iff $\alpha \in E_i$, hence $\downarrow\!(E', m_i) = E_i$.

Second, for m_i not a name of a set of generating defaults, it follows easily that

$$\downarrow\!(E', m_i) = W. \tag{34}$$

We show next that E' is an extension of $\mathcal{E}(D, W) = (D', W')$, and subsequently that for m_i naming $\eta(\mathit{GD}(D, E_i))$ we have $\mathsf{ap}(m_i) \in E'$.

To show that E' is an extension of (D', W'), we first show the following three propositions:

1) $W' \subseteq E'$. This holds by the definition of E'.

2) $Th(E') = E'$. This holds by the definition of E'.

3) For any $\delta \in D'$, if $\mathit{PRE}(\delta) \in E'$ and $\neg \mathit{JUS}(\delta) \notin E'$ then $\mathit{CON}(\delta) \in E'$.

To show this, suppose $\mathit{PRE}(\delta) \in E'$ and $\neg \mathit{JUS}(\delta) \notin E'$. For brevity, we assume without further mention elementary results arising from deductively-closed sets. E.g. (and most frequently) $\alpha, \beta \in E'$ iff $\alpha \wedge \beta \in E'$.

- If $\delta = \frac{:\neg(m_i \sqsubset m_j)}{\neg(m_i \sqsubset m_j)}$ then we have $(m_i \sqsubset m_j) \notin E'$. The definition of E' and the fact that $(m_i \sqsubset m_j) \notin E'$ implies that $(m_i \sqsubset m_j) \notin W'$, specifically

$(m_i \sqsubset m_j) \notin W_M$, and so $(D_i, D_j) \notin <$. But from the definition of E' this means that $\neg(m_i \sqsubset m_j) \in E'$.

- If $\delta = \frac{: \neg in(n_i, m_j)}{\neg in(n_i, m_j)}$ then we have $in(n_i, m_j) \notin E'$. As in the preceding this implies that $in(n_i, m_j) \notin W'$, and in particular that $in(n_i, m_j) \notin W_M$ or $\delta_i \notin D_j$ for $D_j \subseteq D$. Consequently, according to the definition of E' this means that $\neg in(n_i, m_j) \in E'$.

- If $\delta = \frac{in(n_j, m_i) \wedge \mathsf{ok}(m_i) : \neg \alpha_j(m_i)}{\mathsf{bl}(m_i)}$ for $\delta_j \in D_i$ and $D_i = \{\delta_1, \ldots, \delta_k\}$ then $\mathsf{ok}(m_i) \in E'$ and $\alpha_j(m_i) \notin E'$. Since $\downarrow(E', m_i) = E_i$ for each extension E_i we get that D_i is not a set of generating defaults for (D, W). From the definition of E' we obtain that $\mathsf{bl}(m_i) \in E'$.

- If $\delta = \frac{([\forall y \in N.\ in(y, m_i) \supset c(y, m_i)] \supset \neg \beta(m_i)) \wedge \mathsf{ok}(m_i) :}{\mathsf{bl}(m_i)}$ for $\delta_j \in D_i$ and $D_i = \{\delta_1, \ldots, \delta_k\}$ then $((\gamma_1(m_i) \wedge \cdots \wedge \gamma_k(m_i)) \supset \neg \beta_j(m_i)) \wedge \mathsf{ok}(m_i) \in E'$.

For extension E_i, we note that $\{\delta_1, \ldots, \delta_k\} \not\subseteq GD(D, E_i)$, since if this were the case we would have $CON(\delta_1), \ldots, CON(\delta_k) \in E_i$, and this together with $(\gamma_1 \wedge \cdots \wedge \gamma_k) \supset \neg \beta_j$ and the fact that E is deductively closed means that $\neg \beta_j \in E_i$ for some default δ_j. But this means that $\neg JUS(\delta_j) \in E_i$, contradicting the assumption that $\delta_j \in GD(D, E_i)$.

So for $D_i = \{\delta_1, \ldots, \delta_k\}$ we have $D_i \not\subseteq GD(D, E_i)$ for any extension E_i of (D, W), and from the definition of E' we obtain that $\mathsf{bl}(m_i) \in E'$.

- If $\delta = \frac{\mathsf{ok}(m_i) : \neg \mathsf{ko}(m_i)}{\forall y \in N.\ in(y, m_i) \supset \mathsf{ok}(y \cdot m_i)}$ for $D_i = \{\delta_1, \ldots, \delta_k\}$ then $\mathsf{ok}(m_i) \in E'$ and $\mathsf{ko}(m_i) \notin E'$. Consequently (via $W_{\sqsubset}$) $\mathsf{bl}(m_i) \notin E'$. Hence from the definition of E' this means that $D_i = GD(D, E)$. In the definition of E' we have that for every $\delta_j \in GD(D, E_i)$ that $\mathsf{ok}(m_i) \in E'$. Hence for every $\delta_j \in D_i$ we have that $\mathsf{ok}(n_j \cdot m_i) \in E'$.

- If $\delta = \frac{\alpha(m) \wedge in(n, m) \wedge \mathsf{ok}(n_j \cdot m) : \beta(m)}{\gamma(m) \wedge \mathsf{ap}(n_j \cdot m)}$ for $\delta_j \in D$ then we have $\alpha(m) \wedge in(n, m) \wedge \mathsf{ok}(n_j \cdot m) \in E'$ and $\neg \beta(m) \notin E'$. Since $\mathsf{ok}(n_j \cdot m) \in E'$, by construction of E' we have that there is extension E_i such that $D_m = GD(D, E_i)$. Since $\downarrow(E', m) = E_i$ we have $\alpha \in E_i$ and $\neg \beta \notin E_i$. Since E_i is an extension, we have that $\delta_j \in GD(D, E_i)$ and so from the definition of E' we obtain $\gamma(m) \wedge \mathsf{ap}(n_j \cdot m) \in E'$.

This shows that for any $\delta \in D'$, if $PRE(\delta) \in E'$ and $\neg JUS(\delta) \notin E'$ then $CON(\delta) \in E'$.

According to Definition A.1, we get $\Gamma(E') \subseteq E'$ by the minimality of $\Gamma(E')$.

To show the converse, we show that if $\mu \in E'$ then $\mu \in \Gamma(E')$.

We distinguish the following cases.

– If $\mu \in W'$ then since $W' \subseteq \Gamma(E')$ we obtain $\mu \in \Gamma(E')$.

– If $\mu \in \{\neg(m_i \sqsubset m_j) \mid (D_i, D_j) \notin <\}$ then $(m_i \sqsubset m_j) \notin E'$. Since we have the default $\frac{: \neg(m_i \sqsubset m_j)}{\neg(m_i \sqsubset m_j)} \in D'$, Condition 3 of the definition of $\Gamma(E')$ requires that $\neg(m_i \sqsubset m_j) \in \Gamma(E')$.

– If $\mu \in \{\neg in(n_i, m_j) \mid \delta_i \in D_i, D_i \subseteq D\}$ then $in(n_i, m_j) \notin E'$. But again, since we have the default $\frac{:\neg in(n_i,m_j)}{\neg in(n_i,m_j)} \in D'$, Condition 3 of the definition of $\Gamma(E')$ requires that $\neg in(n_i, m_j) \in \Gamma(E')$.

– We claim that for all $m \in M \setminus \{m_D\}$:

1) If $D_m \supset \mathit{GD}(D, E_i)$ for some extension E_i then $\mathsf{ok}(m) \in \Gamma(E')$ and $\mathsf{bl}(m) \in \Gamma(E')$.

2) If $D_m = \mathit{GD}(D, E_i)$ for some extension E_i then $\mathsf{ok}(m) \in \Gamma(E')$ and $\mathsf{ap}(m) \in \Gamma(E')$, and for every $\delta_j \in D_m$ we have $\mathit{CON}(\delta_j)(m) \wedge \mathsf{ap}(n_j{\cdot}m), \mathsf{ok}(n_j{\cdot}m) \in \Gamma(E')$.

3) If $D_m \subset \mathit{GD}(D, E_i)$ for some extension E_i then $\mathsf{ovr}(m) \in \Gamma(E')$.

Since the set of sets of generating defaults of extensions of (D, W) forms a cut of the lattice of subsets of D, 1.–3. above covers all remaining cases.

We show for all $D_m \subseteq D$ that the claim holds by induction on $<$.

Base: By definition, $\mathsf{ok}(m_D) \in W_{\sqsubset} \subseteq W' \subseteq \Gamma(E')$.

We have the default $\frac{\mathsf{ok}(m_D)\,:\,\neg\mathsf{ko}(m_D)}{in(n_\uparrow, m_D)\supset \mathsf{ok}(n_\uparrow \cdot m_D)}$ and since $in(n_\uparrow, m_D) \in \Gamma(E')$ we get $\mathsf{ok}(m_D) \in \Gamma(E')$ and $\mathsf{ko}(m_D) \notin E'$; hence from Condition 3 of Definition A.1 we obtain $\mathsf{ok}(n_\uparrow{\cdot}m_D) \in \Gamma(E')$.

Similarly we have the default $\frac{\top \wedge in(n_\uparrow, m_D) \wedge \mathsf{ok}(n_\uparrow \cdot m_D)\,:\,\top}{\top \wedge \mathsf{ap}(n_\uparrow \cdot m_D)}$. Since $in(n_\uparrow, m_D) \wedge \mathsf{ok}(n_\uparrow{\cdot}m_D) \in \Gamma(E')$, $\neg\top \notin E'$, we obtain $\mathsf{ap}(n_\uparrow{\cdot}m_D) \in \Gamma(E')$. Since $\Gamma(E')$ is deductively closed we get $\mathsf{ap}(m_D) \in \Gamma(E')$.

Step: Consider $D_m \subseteq D$ and assume that for all D_j such that $D_m < D_j$ that 1.-3. in the claim above hold.

Let $D_m = \{\delta_i, \ldots, \delta_k\}$. There are the following cases.

1) There is extension E_i where $\mathit{GD}(D, E_i) \subset D_m$.

By the induction hypothesis, and in particular 1., for every m' such that $m \sqsubset m'$, we have $\mathsf{bl}(m') \in \Gamma(E')$ or $m' = m_D \in \Gamma(E')$. Since $W_{\sqsubset}$ contains the formula $\forall x \in M\, \big[\forall y \in M.\,(x \sqsubset y) \supset (\mathsf{bl}(y) \vee y = m_D)\big] \supset \mathsf{ok}(x)$ and since $\Gamma(E')$ is deductively closed, we have that $\mathsf{ok}(m) \in \Gamma(E')$.

Observe that D_m is not a set of generating defaults of an extension E'' of (D, W), since $\mathit{GD}(D, E_i) \subset D_m$ would contradict the assumption that E_i is an extension of (D, W).

Since D_m is not a set of generating defaults of an extension, one of two cases hold.

2) $W \cup \mathit{CON}(D_m) \vdash \neg\beta_j$ for some $\delta_j \in D_m$.

Hence $W(m) \cup \mathit{CON}(D_m)(m) \vdash \neg\beta_j(m)$ for some $\delta_j \in D_m$.

Since $W(m) \subseteq W'$ and $W' \subseteq \Gamma(E')$, we have that $\mathit{CON}(D_m)(m) \supset \neg\beta_j(m) \in \Gamma(E')$.

Since $\mathsf{ok}(m) \in \Gamma(E')$, and we have $W_D \subseteq \Gamma(E')$, and we have the rule $\frac{([\forall y \in N.\ in(y,m) \supset c(y,m)] \supset \neg\beta(m)) \wedge \mathsf{ok}(m) :}{\mathsf{bl}(m)}$, via Definition A.1 we get that $\mathsf{bl}(m) \in \Gamma(E')$.

3) For some $\delta \in D_m \setminus \mathit{GD}(D, E_i)$ we have $W \cup \mathit{CON}(\mathit{GD}(D, E_i)) \not\vdash \mathit{PRE}(\delta)$.

Thus $W(m) \cup \mathit{CON}(\mathit{GD}(D, E_i))(m) \not\vdash \mathit{PRE}(\delta)(m)$, so $W(m) \not\vdash \mathit{PRE}(\delta)(m)$ or, using (34) $E' \not\vdash \mathit{PRE}(\delta)(m)$ or, since E' is logically closed, $\mathit{PRE}(\delta)(m) \notin E'$.

Since for some $\delta \in D_m$, say $\delta = \delta_i$, we have that $\mathit{PRE}(\delta) \notin E'$ and since $\mathsf{ok}(m), in(n_j, m) \in \Gamma(E')$ and since we have the rule $\frac{in(n_j,m) \wedge \mathsf{ok}(m) : \neg\alpha_j}{\mathsf{bl}(m)}$, via Definition A.1 we obtain that $\mathsf{bl}(m) \in \Gamma(E')$.

$D_m = \mathit{GD}(D, E_i)$.

First, we have $\mathsf{bl}(m) \notin E'$ by definition of E' and similarly $\mathsf{ovr}(m) \notin E'$.

By the induction hypothesis, and in particular 2., for every m' such that $m \sqsubset m'$, we have $\mathsf{bl}(m') \in \Gamma(E')$ or $m' = m_D \in \Gamma(E')$. Since $W_\sqsubset$ contains the formula $\forall x \in M \left[\forall y \in M.\,(x \sqsubset y) \supset (\mathsf{bl}(y) \vee y = m_D)\right] \supset \mathsf{ok}(x)$ and since $\Gamma(E')$ is deductively closed, we have that $\mathsf{ok}(m) \in \Gamma(E')$.

Since $\mathsf{ok}(m) \in \Gamma(E')$ and $\mathsf{ovr}(m) \notin E'$, so $\mathsf{ko}(m) \notin E'$. From $\frac{\mathsf{ok}(m) : \neg\mathsf{ko}(m)}{\forall y \in N.\ in(y,m) \supset \mathsf{ok}(y \cdot m)}$ we get $\forall y \in N.\, in(y, m) \supset \mathsf{ok}(y \cdot m) \in \Gamma(E')$ via Definition A.1.

Claim: First, for every $\delta_j \in D_m = \mathit{GD}(D, E_i)$ we have $\alpha_j(m) \wedge in(n_j, m) \wedge \mathsf{ok}(n_j \cdot m) \in \Gamma(E')$ and $\neg\beta_j(m) \notin E'$. Second, since we have the rule $\frac{\alpha_j(m) \wedge in(n_j,m) \wedge \mathsf{ok}(n_j \cdot m) : \beta_j(m)}{\gamma_j(m) \wedge \mathsf{ap}(n_j \cdot m)}$ we obtain via Definition A.1 that $\gamma_j(m) \wedge \mathsf{ap}(n_j) \in \Gamma(E')$.

Proof of Claim: We have that $\mathit{GD}(D, E_i)$ is a set of generating defaults of (D, E). The proof is by induction on the grounded enumeration $\langle\delta_j\rangle_{j \in I}$ of defaults in $D_m = \mathit{GD}(D, E_i)$.

Base: There is $\delta_l : \frac{\alpha_l : \beta_l}{\gamma_l} \in \mathit{GD}(D, E)$ such that $\alpha_l \in W$ and $\neg\beta_l \notin E$.
So $\alpha_l(m) \in W'$ and since $W'(m) \subseteq \Gamma(E')$ so $\alpha_l(m) \in \Gamma(E')$.
Also, since $\mathsf{ok}(n_l \cdot m), in(n_l, m) \in \Gamma(E')$, so $\alpha_l(m) \wedge in(n_l, m) \wedge \mathsf{ok}(n_l \cdot m) \in \Gamma(E')$.
Also $\neg\beta_l(m) \notin E'$ (since $\downarrow(E', m) = E_i$ and $\beta_l \notin E_i$).
From Definition A.1 we get that $\gamma_l(m) \wedge \mathsf{ap}(n_l \cdot m) \in \Gamma(E')$.

Step: Assume that the claim holds for $0..k$.
We have some $\delta_l \in \mathit{GD}(D, E_i)$ such that $\alpha_l \in E_k$ and $\neg\beta_l \notin E$ (since we have a grounded enumeration of the defaults in $\mathit{GD}(D, E_i)$).
By the induction hypothesis $E_k(m) \subseteq \Gamma(E')$ and so $\alpha_l(m) \in \Gamma(E')$.
We have $\mathsf{ok}(n_l \cdot m), in(n_l, m) \in \Gamma(E')$, hence $\alpha_l(m) \wedge in(n_l, m) \wedge \mathsf{ok}(n_l \cdot m) \in \Gamma(E')$.

Also $\neg\beta_l(m) \notin E'$ (since $\downarrow(E', m) = E_i$ and $\beta_l \notin E_i$).
From Definition A.1 we get that $\gamma_l(m) \wedge \mathsf{ap}(n_l{\cdot}m) \in \Gamma(E')$.

This takes care of the case where $D_m = \mathit{GD}(D, E_i)$.

$D_m \subset \mathit{GD}(D, E_i)$.

By the induction hypothesis we have that $\mathsf{ap}(m_i) \in \Gamma(E')$. As well, $m \sqsubset m_i \in W_{\sqsubset}$. Since $W_{\sqsubset}$ contains the formula

$$\forall x, y \in M.\, \mathsf{ap}(x) \supset (y \sqsubset x \supset \mathsf{ovr}(y))$$

and $W_{\sqsubset} \subseteq \Gamma(E')$, and $\Gamma(E')$ is logically closed we get $\mathsf{ovr}(m) \in \Gamma(E')$.

We have thus shown that $\mu \in E'$ implies $\mu \in \Gamma(E')$. Since both E' and $\Gamma(E')$ are deductively closed, we get that $E' \subseteq \Gamma(E')$.

■

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