

Verifying Tight Logic Programs with ANTHEM and VAMPIRE

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Abstract

This paper continues the line of research aimed at investigating the relationship between logic programs and first-order theories. We extend the definition of program completion to programs with input and output in a subset of the input language of the ASP grounder GRINGO, study the relationship between stable models and completion in this context, and describe preliminary experiments with the use of two software tools, ANTHEM and VAMPIRE, for verifying the correctness of programs with input and output. Proofs of theorems are based on a lemma that relates the semantics of programs studied in this paper to stable models of first-order formulas. This paper is under consideration for acceptance in *Theory and Practice of Logic Programming*.

1 Introduction

This paper continues the line of research aimed at investigating the relationship between logic programs and first-order theories, which goes back to Keith Clark's classical paper on program completion (Clark 1978). We are interested in extending that idea to input languages of answer set programming (ASP) (Brewka et al. 2011) and in using it for the verification of ASP programs.

These programs are different from logic programs studied by Clark in two ways. First, their semantics is based on the concept of a stable model (Gelfond and Lifschitz 1988), which is not equivalent to the completion semantics, unless some assumptions about the program are made, such as tightness (Fages 1994; Erdem and Lifschitz 2003). Second, ASP programs can include symbols for integers and arithmetic operations. Ways to extend completion to these constructs are proposed in two recent publications (Harrison et al. 2017; Lifschitz et al. 2020).

It is essential for our purposes that, in ASP, we often deal with programs that accept an input and employ auxiliary predicates. Consider, for instance, the program

$$\begin{aligned}
 & \{in_cover(1..n)\}, \\
 & \leftarrow I \neq J \wedge in_cover(I) \wedge in_cover(J) \wedge s(X, I) \wedge s(X, J), \\
 & covered(X) \leftarrow in_cover(I) \wedge s(X, I), \\
 & \leftarrow s(X, I) \wedge not\ covered(X),
 \end{aligned} \tag{1}$$

which encodes exact covers¹ of a finite family of finite sets. It accepts an input, represented by a placeholder n and a predicate symbol $s/2$; n is the number of sets, and the formula $s(X, I)$ expresses that X belongs to the I -th set. The predicate symbol $in_cover/1$ represents the output, and $covered/1$ is auxiliary.

In this paper, we extend the definition of program completion to programs with input and output in a subset of the input language of the ASP grounder GRINGO, state a theorem describing the relationship between stable models and completion in this context, and describe preliminary experiments with the use of two software tools, ANTHEM (Lifschitz et al. 2018; Lifschitz et al. 2019) and VAMPIRE (Kovács and Voronkov 2013), for verifying the correctness of programs with input and output. A lemma that may be of independent interest relates the semantics of simple GRINGO programs to stable models of first-order formulas (Ferraris et al. 2011). It shows, in other words, that the approach to the semantics of logic programs advocated by Lee et al. (2008) is applicable to the subset of GRINGO studied here.

2 Review: Programs

The class of programs studied in this paper is the subset of the input language of GRINGO (Gebser et al. 2015) discussed in earlier publications by Lifschitz et al. (2019; 2020).

We assume that three countably infinite sets of symbols are selected: *numerals*, *symbolic constants*, and *program variables*. We assume a 1-to-1 correspondence between numerals and integers; the numeral corresponding to an integer n is denoted by \bar{n} .

Program terms are defined recursively:

- Numerals, symbolic constants, program variables, and the symbols *inf* and *sup* are program terms;
- if t_1, t_2 are program terms and \circ is one of the *operation names*

$$+ \quad - \quad \times \quad / \quad \setminus \quad ..$$

then $(t_1 \circ t_2)$ is a program term.

If t is a term, then $-t$ is shorthand for $\bar{0} - t$.

A program term, or another syntactic expression, is *ground* if it does not contain variables. A ground expression is *precomputed* if it does not contain operation names (in other words, if it is a numeral, a symbolic constant, or one of the symbols *inf* or *sup*).

We assume that a total order on precomputed program terms is chosen such that

- *inf* is its least element and *sup* its greatest element,
- for any integers m and n , $\bar{m} < \bar{n}$ iff $m < n$, and
- for any integer n and any symbolic constant c , $\bar{n} < c$.

An *atom* is an expression of the form $p(\mathbf{t})$, where p is a symbolic constant and \mathbf{t} is a tuple of program terms. A *literal* is an atom possibly preceded by one or two occurrences

¹ An *exact cover* of a collection S of sets is a subcollection S' of S such that each element of the union of all sets in S belongs to exactly one set in S' .

of *not*. A *comparison* is an expression of the form $(t_1 \prec t_2)$, where t_1, t_2 are program terms and \prec is one of the *comparison symbols*

$$= \neq < > \leq \geq \quad (2)$$

A *rule* is an expression of the form

$$Head \leftarrow Body, \quad (3)$$

where

- *Body* is a conjunction (possibly empty) of literals and comparisons and
- *Head* is either an atom (then, we say that (3) is a *basic rule*) or an atom in braces (then, (3) is a *choice rule*), or empty (then, (3) is a *constraint*).

A *program* is a finite set of rules.

For example, (1) is a program in the sense of this definition, except that the expression $\{in_cover(1..n)\}$, strictly speaking, should be written as $\{in_cover(\bar{1}..n)\} \leftarrow$.

The semantics of programs uses a translation τ , which turns a program into a set of propositional combinations of precomputed atoms (Lifschitz et al. 2019, Section 3). For example, the result of applying τ to the program

$$\begin{aligned} & \{p(\bar{1}.. \bar{3})\}, \\ & q(X + \bar{1}) \leftarrow p(X) \end{aligned} \quad (4)$$

consists of the ground formulas

$$\begin{aligned} \top & \rightarrow (p(\bar{1}) \vee \neg p(\bar{1})) \wedge (p(\bar{2}) \vee \neg p(\bar{2})) \wedge (p(\bar{3}) \vee \neg p(\bar{3})), \\ p(\bar{n}) & \rightarrow q(\overline{n+1}) \quad \text{for all integers } n, \text{ and} \\ p(r) & \rightarrow \top \quad \text{for all precomputed terms } r \text{ other than numerals.} \end{aligned}$$

Using the translation τ , we define the semantics of programs as follows: A set of precomputed atoms is a *stable model* of a program Π if it is a stable model (answer set) of $\tau\Pi$ in the sense of Ferraris (2005).

3 Review: Representing Programs by First-Order Theories

The translation τ^* (Lifschitz et al. 2019) converts a program into a finite set of first-order sentences with variables of two sorts. Besides program variables, which are meant to range over precomputed program terms, these sentences contain *integer variables*, which are meant to range over numerals. The second sort is a subsort of the first.

A *predicate symbol* is a pair p/n , where p is a symbolic constant and n is a nonnegative integer. About a program or another syntactic expression, we say that a predicate symbol p/n *occurs* in it if it contains an atom of the form $p(t_1, \dots, t_n)$.

For any program Π , $\tau^*\Pi$ consists of formulas over the two-sorted signature σ_Π including

- all precomputed terms as object constants; a precomputed constant is assigned the sort *integer* iff it is a numeral;
- the symbols $+$, $-$, and \times as binary function constants; their arguments and values have the sort *integer*;

- all predicate symbols occurring in Π and the comparison symbols other than equality as predicate constants.

An atomic formula $(p/n)(t_1, \dots, t_n)$ can be abbreviated as $p(t_1, \dots, t_n)$. An atomic formula $\prec(t_1, t_2)$, where \prec is a comparison symbol, can be written as $t_1 \prec t_2$. Formulas are formed from atomic formulas using the propositional connectives

$$\perp \text{ ("false")} \quad \wedge \quad \vee \quad \rightarrow$$

and the quantifiers \forall and \exists as usual in first-order languages with equality. We use \top as shorthand for $\perp \rightarrow \perp$, $\neg F$ as shorthand for $F \rightarrow \perp$, and $F \leftrightarrow G$ as shorthand for $(F \rightarrow G) \wedge (G \rightarrow F)$.

Prior to defining τ^* , we define, for every program term t , a formula $val_t(Z)$, where Z is a program variable that does not occur in t . That formula expresses, informally speaking, that Z is one of the values of t . The definition is recursive:

- if t is a numeral, a symbolic constant, a program variable, *inf*, or *sup*, then $val_t(Z)$ is $Z = t$;
- if t is $(t_1 \circ t_2)$, where \circ is $+$, $-$, or \times , then $val_t(Z)$ is

$$\exists I J (Z = I \circ J \wedge val_{t_1}(I) \wedge val_{t_2}(J))$$

where I, J are fresh integer variables;

- if t is (t_1 / t_2) , then $val_t(Z)$ is

$$\begin{aligned} \exists I J Q R (I = J \times Q + R \wedge val_{t_1}(I) \wedge val_{t_2}(J) \\ \wedge J \neq 0 \wedge R \geq 0 \wedge R < Q \wedge Z = Q), \end{aligned}$$

where I, J, Q, R are fresh integer variables;

- if t is $(t_1 \setminus t_2)$, then $val_t(Z)$ is

$$\begin{aligned} \exists I J Q R (I = J \times Q + R \wedge val_{t_1}(I) \wedge val_{t_2}(J) \\ \wedge J \neq 0 \wedge R \geq 0 \wedge R < Q \wedge Z = R), \end{aligned}$$

where I, J, Q, R are fresh integer variables;

- if t is $(t_1 .. t_2)$, then $val_t(Z)$ is

$$\exists I J K (val_{t_1}(I) \wedge val_{t_2}(J) \wedge I \leq K \wedge K \leq J \wedge Z = K),$$

where I, J, K are fresh integer variables.

For example, $val_{\bar{1}..3}(Z)$ is

$$\exists I J K (I = \bar{1} \wedge J = \bar{3} \wedge I \leq K \wedge K \leq J \wedge Z = K),$$

and $val_{X+\bar{1}}(Z)$ is

$$\exists I J (Z = I + J \wedge I = X \wedge J = \bar{1}).$$

The other thing to do in preparation for defining τ^* is to define the translation τ^B that will be applied to expressions in the body of the rule:

- $\tau^B(p(t_1, \dots, t_k))$ is

$$\exists Z_1 \dots Z_k (val_{t_1}(Z_1) \wedge \dots \wedge val_{t_k}(Z_k) \wedge p(Z_1, \dots, Z_k));$$

- $\tau^B(\text{not } p(t_1, \dots, t_k))$ is

$$\exists Z_1 \dots Z_k (\text{val}_{t_1}(Z_1) \wedge \dots \wedge \text{val}_{t_k}(Z_k) \wedge \neg p(Z_1, \dots, Z_k));$$
- $\tau^B(\text{not not } p(t_1, \dots, t_k))$ is

$$\exists Z_1 \dots Z_k (\text{val}_{t_1}(Z_1) \wedge \dots \wedge \text{val}_{t_k}(Z_k) \wedge \neg \neg p(Z_1, \dots, Z_k));$$
- $\tau^B(t_1 < t_2)$ is

$$\exists Z_1 Z_2 (\text{val}_{t_1}(Z_1) \wedge \text{val}_{t_2}(Z_2) \wedge Z_1 < Z_2);$$

where each Z_i is a fresh program variable.

Now, we define

$$\tau^*(\text{Head} \leftarrow B_1 \wedge \dots \wedge B_n)$$

as the universal closure of the formula

$$\tau^B(B_1) \wedge \dots \wedge \tau^B(B_n) \rightarrow H,$$

where H is

- $\forall Z_1 \dots Z_k (\text{val}_{t_1}(Z_1) \wedge \dots \wedge \text{val}_{t_k}(Z_k) \rightarrow p(Z_1, \dots, Z_k))$
if Head is $p(t_1, \dots, t_k)$;
- $\forall Z_1 \dots Z_k (\text{val}_{t_1}(Z_1) \wedge \dots \wedge \text{val}_{t_k}(Z_k) \rightarrow p(Z_1, \dots, Z_k) \vee \neg p(Z_1, \dots, Z_k))$
if Head is $\{p(t_1, \dots, t_k)\}$;
- \perp if Head is empty;

where Z_1, \dots, Z_k are fresh program variables.

For any program Π , $\tau^*\Pi$ stands for the set of formulas τ^*R for all rules R of Π . For example, the result of applying τ^* to program (4) is the pair of sentences

$$\begin{aligned} \top &\rightarrow \forall Z (\text{val}_{\bar{1}, \bar{3}}(Z) \rightarrow p(Z) \vee \neg p(Z)), \\ \forall X (\exists Z (\text{val}_{X+\bar{1}}(Z) \wedge p(Z)) &\rightarrow \forall Z (Z = X \rightarrow p(Z))). \end{aligned}$$

4 Main Lemma

The lemma stated in this section is used in the proofs of Theorems 1 and 2, which are stated later in Section 6. It relates the stable models of a program Π to the stable models of the first-order theory $\tau^*\Pi$.

Recall that, for any first-order sentence F and any list \mathbf{p} of predicate symbols (other than equality), $\text{SM}_{\mathbf{p}}[F]$ stands for the second-order sentence

$$F \wedge \neg \exists \mathbf{u} ((\mathbf{u} < \mathbf{p}) \wedge F^*(\mathbf{u}))$$

(Ferraris et al. 2011, Section 2.3). The definition of $\text{SM}_{\mathbf{p}}$ in that paper is given for formulas with one sort of variables, but its extension to many-sorted formulas is straightforward, and we will apply it here to the class of formulas defined in Section 3.

The models of $\text{SM}_{\mathbf{p}}[F]$ are called the \mathbf{p} -stable models of F . For a set Γ of first-order sentences, the \mathbf{p} -stable models of Γ are the \mathbf{p} -stable models of the conjunction of all formulas in Γ .

For any program Π and any set \mathcal{I} of precomputed atoms, \mathcal{I}^\dagger stands for the interpretation of the signature σ_Π such that

- the universe of the first sort of \mathcal{I}^\uparrow is the set of all precomputed terms;
- the universe of the second sort of \mathcal{I}^\uparrow is the set of all numerals;
- \mathcal{I}^\uparrow interprets every precomputed term t as t ;
- \mathcal{I}^\uparrow interprets $\overline{m} + \overline{n}$ as $\overline{m + n}$, and similarly for subtraction and multiplication;
- \mathcal{I}^\uparrow interprets every precomputed atom $p(t_1, \dots, t_n)$ as true iff it belongs to \mathcal{I} ;
- \mathcal{I}^\uparrow interprets every atomic sentence $t_1 < t_2$, where t_1 and t_2 are precomputed terms, as true iff the relation $<$ holds for the pair (t_1, t_2) .

Main Lemma. *Let Π be a program and let \mathbf{p} be the list of all predicate symbols occurring in Π other than the comparison symbols. A set \mathcal{I} of precomputed atoms is a stable model of Π iff \mathcal{I}^\uparrow is a \mathbf{p} -stable model of $\tau^*\Pi$.*

Proofs are relegated to Appendix A.

5 Programs with Input and Output

5.1 Syntax

A program with input and output, or an *io-program*, is a quadruple

$$(\Pi, PH, In, Out), \tag{5}$$

where

- Π is a finite set of rules;
- PH is a finite set of symbolic constants, called the *placeholders* of (5);
- In is a finite set of predicate symbols that do not occur in the heads of rules of Π ; they are called the *input symbols* of (5);
- Out is a finite set of predicate symbols that is disjoint from In ; they are called the *output symbols* of (5).

The input and output symbols of an io-program are collectively called its *public symbols*; the predicate symbols occurring in the rules that are not public are called *private*. We also distinguish between public and private atoms, depending on whether their predicate symbols are public or private. Every set \mathcal{I} of public atoms is the union of two disjoint sets—the set \mathcal{I}^{in} of atoms in \mathcal{I} that begin with an input symbol and the set \mathcal{I}^{out} of atoms that begin with an output symbol.

As an example, consider the io-program Ω_1 with rules (1), the placeholder n , the input symbol $s/2$, and the output symbol $in_cover/1$. The only private symbol of Ω_1 is $covered/1$.

By σ_Ω , we denote the part of the signature σ_Π (Section 3) obtained from it by removing all private symbols. Formulas of that signature will be used in writing specifications.

Programs can be viewed as a special case of io-programs if we agree to identify a program Π with the io-program (5) in which PH and In are empty and Out is the set of all predicate symbols that occur in the rules of Π .

5.2 Semantics

A *valuation* for an io-program Ω is a function defined on the set of placeholders of Ω such that its values are precomputed terms different from placeholders. An *input* for Ω is a

pair (\mathbf{v}, \mathbf{i}) , where \mathbf{v} is a valuation and \mathbf{i} is a set of precomputed atoms that do not contain placeholders, such that the predicate symbol of each atom in \mathbf{i} is an input symbol of Ω .

For an io-program Ω and an input (\mathbf{v}, \mathbf{i}) , we denote by $\Omega(\mathbf{v}, \mathbf{i})$ the program consisting of

- the rules obtained from the rules of Ω by replacing every occurrence of every placeholder t by the term $\mathbf{v}(t)$ and
- the facts \mathbf{i} .

About a set of precomputed atoms, we say that it is an *io-model* of an io-program Ω for an input (\mathbf{v}, \mathbf{i}) if it is the set of all public atoms of some stable model of $\Omega(\mathbf{v}, \mathbf{i})$.

For example, we can define an input $(\mathbf{v}_1, \mathbf{i}_1)$ for Ω_1 by the conditions

$$\mathbf{v}_1(n) = \bar{3}, \quad \mathbf{i}_1 = \{s(a, \bar{1}), s(b, \bar{1}), s(b, \bar{2}), s(c, \bar{2}), s(c, \bar{3})\}.$$

With this input, Ω_1 encodes the exact covers of the collection $\{\{a, b\}, \{b, c\}, \{c\}\}$. The program $\Omega_1(\mathbf{v}_1, \mathbf{i}_1)$ consists of the rules

$$\begin{aligned} & \{in_cover(\bar{1}.. \bar{3})\}, \\ & \leftarrow I \neq J \wedge in_cover(I) \wedge in_cover(J) \wedge s(X, I) \wedge s(X, J), \\ & covered(X) \leftarrow in_cover(I) \wedge s(X, I), \\ & \leftarrow s(X, I) \wedge not_covered(X), \\ & s(a, \bar{1}), \dots, s(c, \bar{3}). \end{aligned}$$

It has one stable model,

$$\{s(a, \bar{1}), \dots, s(c, \bar{3}), in_cover(\bar{1}), in_cover(\bar{3}), covered(a), covered(b), covered(c)\},$$

so that the only io-model of Ω is $\{s(a, \bar{1}), \dots, s(c, \bar{3}), in_cover(\bar{1}), in_cover(\bar{3})\}$.

It is clear that atoms in an io-model do not contain private symbols. Furthermore, if \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) , then $\mathcal{I}^{in} = \mathbf{i}$. This is clear from the fact that the only rules of $\Omega(\mathbf{v}, \mathbf{i})$ containing input symbols in the head are the facts \mathbf{i} .

IO-models of io-programs also have a characterization in terms of first-order formulas similar to the characterization of stable models in the Main Lemma above (see Appendix A.2). The proof of this characterization is based on the aforementioned Main Lemma and results from (Ferraris et al. 2009) and (Cabalar et al. 2020). This is later used in the proofs of Theorems 1 and 2 below.

5.3 Specifications

In this paper, we are interested in io-programs written in response to formal specifications. A specification, as we understand this term, includes a list *PH* of placeholders, a list *In* of input symbols, and a list *Out* of output symbols that a future io-program Ω is required to have. (The choice of private symbols of Ω is left to the discretion of the programmer.) That part of the specification determines the set of valuations and inputs of Ω as well as the signature σ_Ω . A specification also includes two lists of sentences in first-order logic over that signature: a list *A* of *assumptions* and a list *S* of *specs*. The sentences in *A* do not contain output symbols. They are intended to describe the class of inputs that the author of the specification is interested in. The sentences in *S* are meant to describe the io-models that the future io-program is required to have for these inputs.

For example, a specification for the exact cover problem (see Footnote 1) may include the assumptions

$$\begin{aligned} \exists N (n = N \wedge N \geq \bar{0}), \\ \forall XY (s(X, Y) \rightarrow \exists I (Y = I \wedge I \geq \bar{1} \wedge I \leq n)), \end{aligned} \quad (6)$$

where N and I are integer variables. They tell us that a “good” input describes the length and composition of a list of sets of precomputed atoms. Its specs may look like this:

$$\begin{aligned} \forall Y (in_cover(Y) \rightarrow \exists I (Y = I \wedge I \geq \bar{1} \wedge I \leq n)), \\ \forall X (\exists Y s(X, Y) \rightarrow \exists Y (s(X, Y) \wedge in_cover(Y))), \\ \forall YZ (\exists X (s(X, Y) \wedge s(X, Z)) \wedge in_cover(Y) \wedge in_cover(Z) \rightarrow Y = Z). \end{aligned} \quad (7)$$

These formulas describe the concept of an exact cover.

To give a precise definition of the semantics of specifications, we need to generalize the process of transforming a set \mathcal{I} of precomputed atoms into the interpretation \mathcal{I}^\dagger defined in Section 4. This generalization takes into account the fact that placeholders may be “reinterpreted” by a valuation. For any set \mathcal{I} of precomputed public atoms that do not contain placeholders and any valuation \mathbf{v} , we denote by $\mathcal{I}^\mathbf{v}$ the interpretation of the signature σ_Ω such that

- the universe of the first sort of $\mathcal{I}^\mathbf{v}$ is the set of all precomputed terms;
- the universe of the second sort of $\mathcal{I}^\mathbf{v}$ is the set of all numerals;
- $\mathcal{I}^\mathbf{v}$ interprets every placeholder t as $\mathbf{v}(t)$ and every other precomputed term t as t ;
- $\mathcal{I}^\mathbf{v}$ interprets $\overline{m} + \overline{n}$ as $\overline{m + n}$, and similarly for subtraction and multiplication;
- $\mathcal{I}^\mathbf{v}$ interprets every precomputed atom $p(t_1, \dots, t_n)$ that does not contain placeholders as true iff it belongs to \mathcal{I} ;
- $\mathcal{I}^\mathbf{v}$ interprets every atomic sentence $t_1 \prec t_2$, where t_1 and t_2 are precomputed terms different from placeholders, as true iff the relation \prec holds for the pair (t_1, t_2) .

In the exact cover example, $\mathcal{I}^\mathbf{v}$ interprets the placeholder n as $\mathbf{v}(n)$. This interpretation satisfies the first of assumptions (6) iff $\mathbf{v}(n)$ is a nonnegative numeral.

We say that Ω *implements* the specification with assumptions A and specs S if, for every valuation \mathbf{v} and every set \mathcal{I} of precomputed public atoms without placeholders such that $\mathcal{I}^\mathbf{v}$ satisfies A ,

$$\mathcal{I} \text{ is an io-model of } \Omega \text{ for the input } (\mathbf{v}, \mathcal{I}^{in}) \text{ iff } \mathcal{I}^\mathbf{v} \text{ satisfies } S.$$

In the next section, we show that this condition can be sometimes established by proving a sentence in a first-order theory. This is how ANTHEM works.

6 Completion and Tightness

6.1 Completed Definitions

The *definition* of a predicate symbol p/n in an io-program Ω is the set of all rules of Ω that have the forms

$$p(t_1, \dots, t_n) \leftarrow Body \quad (8)$$

and

$$\{p(t_1, \dots, t_n)\} \leftarrow \text{Body}. \quad (9)$$

If the definition of p/n in Ω consists of rules R_1, \dots, R_k , then the *formula representations* F_1, \dots, F_k of these rules are constructed as follows. Take fresh program variables V_1, \dots, V_n . If R_i is (8), then F_i is the formula

$$\tau^B(\text{Body}) \wedge \text{val}_{t_1}(V_1) \wedge \dots \wedge \text{val}_{t_n}(V_n)$$

(see Section 3 for the definitions of τ^B and val). If R_i is (9), then F_i is the formula

$$\tau^B(\text{Body}) \wedge p(V_1, \dots, V_n) \wedge \text{val}_{t_1}(V_1) \wedge \dots \wedge \text{val}_{t_n}(V_n). \quad (10)$$

We are ready to define “completed definitions” of predicate symbols. Completed definitions are generally second-order formulas—the private symbols of Ω are represented in them by predicate variables.

Let Ω be an io-program with private symbols $p_1/n_1, \dots, p_l/n_l$. Choose pairwise distinct predicate variables P_1, \dots, P_l and take a predicate symbol p/n that occurs in Ω and is different from its input symbols. The *completed definition* of p/n in Ω is obtained from the formula

$$\forall V_1 \dots V_n \left(p(V_1, \dots, V_n) \leftrightarrow \bigvee_{i=1}^k \exists \mathbf{U}_i F_i \right), \quad (11)$$

where \mathbf{U}_i is the list of all variables occurring in rule R_i after substituting P_1, \dots, P_l for $p_1/n_1, \dots, p_l/n_l$.

For example, the completed definition of $\text{in_cover}/1$ in Ω_1 is

$$\forall V (\text{in_cover}(V) \leftrightarrow \top \wedge \text{in_cover}(V) \wedge \exists IJK (I = \bar{1} \wedge J = n \wedge I \leq K \wedge K \leq J \wedge V = K)). \quad (12)$$

It can be equivalently rewritten as

$$\forall V (\text{in_cover}(V) \rightarrow \exists IJK (I = \bar{1} \wedge J = n \wedge I \leq K \wedge K \leq J \wedge V = K)).$$

6.2 Completion

For a constraint $\leftarrow \text{Body}$, its *formula representation* is defined as the formula obtained from the universal closure of $\neg \tau^B(\text{Body})$ by substituting P_1, \dots, P_l for $p_1/n_1, \dots, p_l/n_l$. For example, the formula representation of the second constraint of Ω_1 is obtained from the formula

$$\forall XI \neg \tau^B(s(X, I) \wedge \text{not covered}(X))$$

by substituting a predicate variable Covered for the predicate constant covered . It can be equivalently rewritten as

$$\forall XI \neg (s(X, I) \wedge \neg \text{Covered}(X)). \quad (13)$$

The *completion* $\text{COMP}(\Omega)$ of an io-program Ω is the sentence $\exists P_1 \dots P_l F$, where P_1, \dots, P_l are the predicate variables corresponding to the private symbols of Ω , as described above, and F is the conjunction of

- the completed definitions of all predicate symbols occurring in Ω other than its input symbols and
- the formula representations of all constraints in Ω .

Note that this sentence does not contain any of the private symbols of Ω . In other words, it is a second-order sentence over the signature σ_Ω (Section 5.1).

For example, the predicate completion of Ω_1 is

$$\exists \text{Covered} (F_1 \wedge F_2 \wedge F_3 \wedge F_4),$$

where F_1 is the completed definition (12) of *in_cover*, F_2 is the completed definition of *covered*, and F_3, F_4 are the formula representations of the two constraints.

Theorem 1. *If \mathcal{I} is an io-model of an io-program Ω for an input (\mathbf{v}, \mathbf{i}) , then the interpretation $\mathcal{I}^\mathbf{v}$ satisfies $\text{COMP}(\Omega)$.*

6.3 Tight Programs

The *predicate dependency graph* of an io-program Ω is the directed graph that

- has the predicate symbols occurring in rules of Ω as its vertices and
- has an edge from p/n to q/m if there is a rule R in Ω such that p/n occurs in the head of R and q/m occurs in the body of R .

The edge from p/n to q/m in the predicate dependency graph is *positive* if there is a rule R in Ω such that p/n occurs in the head of R and q/m occurs in an atom in the body of R that is not preceded by *not*.

An io-program is *tight* if its predicate dependency graph has no cycles consisting of positive edges.

Theorem 2. *For any tight io-program Ω and any set \mathcal{I} of precomputed public atoms without placeholders, \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) iff $\mathcal{I}^\mathbf{v}$ satisfies $\text{COMP}(\Omega)$ and $\mathcal{I}^{\mathbf{in}} = \mathbf{i}$.*

This theorem shows that, in application to tight io-programs, the semantics of specifications (Section 5.3) can be characterized in terms of completion: A tight io-program Ω implements the specification with assumptions A and specs S iff, for every valuation \mathbf{v} and every set \mathcal{I} of precomputed public atoms without placeholders, the interpretation $\mathcal{I}^\mathbf{v}$ satisfies the sentence

$$A \rightarrow (\text{COMP}(\Omega) \leftrightarrow S). \quad (14)$$

(We identify a list of sentences with the conjunction of its members.)

An interpretation of the signature σ_Ω is *standard* if it can be represented in the form $\mathcal{I}^\mathbf{v}$ for some set \mathcal{I} of precomputed public atoms without placeholders and some valuation \mathbf{v} . We can then say that a tight io-program Ω implements the specification with assumptions A and specs S iff sentence (14) is satisfied by all standard interpretations.

The automation of reasoning about formulas of form (14) is difficult because $\text{COMP}(\Omega)$ has, generally, bound predicate variables. We will now discuss a special case when reasoning about such formulas can be reduced to first-order reasoning.

6.4 Private Recursion

Recall that the completion of an io-program Ω is a sentence of the form $\exists \mathbf{P} F$, where \mathbf{P} is a list P_1, \dots, P_l of predicate variables (Section 6.2). That list has the same length l as the list \mathbf{p} of the private symbols $p_1/n_1, \dots, p_l/n_l$ of Ω . Formula F is a conjunction that includes, among its conjunctive terms, the completed definitions $F_1(\mathbf{P}), \dots, F_l(\mathbf{P})$ of $p_1/n_1, \dots, p_l/n_l$. Denote the conjunction of all other conjunctive terms of F by $F'(\mathbf{P})$. Then, the completion of Ω can be written as

$$\exists \mathbf{P} (F_1(\mathbf{P}) \wedge \dots \wedge F_l(\mathbf{P}) \wedge F'(\mathbf{P})). \quad (15)$$

About an io-program Ω , we say that it *uses private recursion* if

- its predicate dependency graph has a cycle such that every vertex in it is a private symbol or
- it includes a choice rule with a private symbol in the head.

Theorem 3. *If an io-program Ω does not use private recursion, then its completion is equivalent to*

$$\forall \mathbf{P} (F_1(\mathbf{P}) \wedge \dots \wedge F_l(\mathbf{P}) \rightarrow F'(\mathbf{P})). \quad (16)$$

From this theorem, we can conclude that, if Ω does not use private recursion, sentence (14) is equivalent to

$$\begin{aligned} & (A \rightarrow (\exists \mathbf{P} (F_1(\mathbf{P}) \wedge \dots \wedge F_l(\mathbf{P}) \wedge F'(\mathbf{P})) \rightarrow S)) \\ & \wedge (A \rightarrow (S \rightarrow \forall \mathbf{P} (F_1(\mathbf{P}) \wedge \dots \wedge F_l(\mathbf{P}) \rightarrow F'(\mathbf{P})))) \end{aligned}$$

and consequently to

$$\forall \mathbf{P} (A \wedge F_1(\mathbf{P}) \wedge \dots \wedge F_l(\mathbf{P}) \rightarrow (F'(\mathbf{P}) \leftrightarrow S)). \quad (17)$$

The procedure used by ANTHEM for program verification is applicable to an io-program if it is tight and does not use private recursion. The procedure is based on the following idea. We choose a set of *axioms*—sentences over the signature Ω that are satisfied by all standard interpretations. This set may include, for instance, the formulas $\forall N (N + \bar{0} = N)$ and $\bar{2} \times \bar{2} = \bar{4}$. It may include also the formula $a \neq b$, where a and b are distinct symbolic constants, unless at least one of them is a placeholder. To establish that Ω implements the given specification, ANTHEM verifies that the axioms entail sentence (17). This can be accomplished by first-order reasoning—by verifying that the axioms entail the sentence obtained from (17) by replacing the bound predicate variables with distinct predicate constants that do not occur in the axioms. The predicate constants used by ANTHEM are actually the private predicate symbols from the rules of Ω , so that the formula derived by ANTHEM from the axioms is

$$A \wedge F_1(\mathbf{p}) \wedge \dots \wedge F_l(\mathbf{p}) \rightarrow (F'(\mathbf{p}) \leftrightarrow S).$$

Proof search is performed by the resolution theorem prover VAMPIRE. The task is divided into two parts: deriving the specs S from the axioms, the assumptions A , and the formulas $F_1(\mathbf{p}), \dots, F_l(\mathbf{p}), F'$ (“forward direction”) and deriving F' from the axioms, the assumptions A , and the formulas $F_1(\mathbf{p}), \dots, F_l(\mathbf{p}), S$ (“backward direction”).

```

input: n -> integer, s/2.
output: in_cover/1.
assume: n >= 0.
assume: forall Y (exists X s(X, Y) -> exists I (Y = I and I >= 1 and I <= n)).
spec: forall Y (in_cover(Y) -> exists I (Y = I and I >= 1 and I <= n)).
spec: forall X (exists Y s(X, Y) -> exists Y (s(X, Y) and in_cover(Y))).
spec: forall Y, Z (exists X (s(X, Y) and s(X, Z)) and in_cover(Y) and in_cover(Z)
-> Y = Z).

```

Fig. 1. A specification file to verify that a program encoding the exact cover problem fulfills specification (7) under assumptions (6). First, n is declared a placeholder using an `input` statement. The `-> integer` notation instructs ANTHEM to represent n by an integer variable internally. Next, the input and output predicates of Ω_1 ($s/2$ and $in/1$, respectively) are declared. The `assume` and `spec` statements encode assumptions (6) and specs (7).

7 Verifying Tight Logic Programs with ANTHEM

Let us return to the exact cover problem, which we encoded in rules (1) and represented as io-program Ω_1 earlier. We would like to show that this program implements specification (7) under the assumptions stated in (6). This verification can be automated with ANTHEM 1.0.² To this effect, ANTHEM reads a logic program and a specification, checks that the program is tight and does not contain private recursion, generates the program’s completion, and then performs automated proof searches trying to establish that the program actually implements the given specification.

As the input program, we can simply pass rules (1) to ANTHEM in the input language of GRINGO. For the specification, ANTHEM accepts files in a custom *specification file* format. Specification files may contain multiple types of statements. `input` and `output` statements specify input predicates and placeholders as opposed to output predicates. Axioms, assumptions and specs are expressed by `axiom`, `assume`, and `spec` statements, respectively. ANTHEM’s specification file format has a simple syntax for such formulas. This format is a concise form of the segment of the TPTP language (Sutcliffe 2017) that is required for expressing specifications.

Similar to the input language of GRINGO, identifiers starting with a lowercase letter are interpreted as predicate and object constants, while variables are denoted by identifiers starting with an uppercase letter. An important distinction is that variables starting with letters I, J, K, L, M, or N are implicitly assumed to be integer variables—as opposed to program variables, which must start with U, V, W, X, Y, or Z. ANTHEM shows an error message if variables are used that start with letters not listed above. Terms may contain integer arithmetics with a syntax close to GRINGO. In addition to that, the keywords `not`, `and`, `or`, `->`, `<->`, `exists`, and `forall` denote the common operators of first-order logic.

We can encode specification (7) and assumptions (6) in this format, as shown in Figure 1. If we give this specification to ANTHEM along with the exact cover program encoded in (1), ANTHEM first generates the program’s completion and shows it to the user (see Figure 2). This is done as described in Section 6.2. Afterward, ANTHEM verifies very quickly

² <https://github.com/potassco/anthem/releases>

```

forall X (covered(X) <-> exists N (in_cover(N) and s(X, N)))
forall X (in_cover(X) <-> in_cover(X) and exists N (1 <= N and N <= n and X = N))
forall N1, N2, X not (N1 != N2 and in_cover(N1) and in_cover(N2)
and s(X, N1) and s(X, N2))
forall X, N not (s(X, N) and not covered(X))

```

Fig. 2. ANTHEM generates the completion of the input program—here, Ω_1 , the exact cover problem encoded by rules (1). The first two lines contain the completed definitions of *covered*/1 and *in_cover*/1. They are followed by the two translated constraints. Note that ANTHEM applies basic equivalent transformations to simplify the completion formulas. For example, the completed definition of *in_cover* shown here corresponds to formula (12), but the integer variables I and J were eliminated by ANTHEM. The last line matches formula (13), the formula representation of the second constraint of Ω_1 .

that the program indeed implements the specification in Figure 1 using the theorem prover VAMPIRE and informs the user accordingly.³

To accomplish this, ANTHEM splits the equivalence verification task into two parts, as mentioned at the end of Section 6.4. In either direction, ANTHEM starts with a set of presupposed formulas and performs one proof step for each formula to verify. For each proof step, ANTHEM devises a program in the TFF (“typed first-order formula”) segment of the TPTP language. In this TPTP program, all presupposed formulas are represented as TPTP axioms, and the formula to verify in this proof step is used as the conjecture. ANTHEM then passes this TPTP program to VAMPIRE. If a proof step is successful within a predefined time limit, the formula is added to the list of presupposed formulas and the next proof step is conducted. The fact that ANTHEM uses the TPTP format to communicate with VAMPIRE is completely hidden from the user.

In some cases, VAMPIRE is incapable of verifying the equivalence of a specification to a program directly and within a reasonable time frame, for which there are two possible explanations. First, the set of axioms known to VAMPIRE might not be sufficient to perform the proof, and second, it might be necessary to further guide VAMPIRE’s proof search by providing helper lemmas. To that end, ANTHEM allows users to supply additional axioms and helper lemmas with the `axiom` and `lemma` directives, respectively. ANTHEM verifies helper lemmas in the given order before the specs or the completion of the program. The usage of axioms and helper lemmas can be seen in a second example, a specification for programs that compute the floor of the square root of an integer n shown in Figure 3 and the implementation in Figure 4. ANTHEM and VAMPIRE verify the equivalence between this specification and program only when supplying an axiom unknown to VAMPIRE and a list of helper lemmas (shown in Figure 5).

While ANTHEM 1.0 introduces the mode of operation described in the present paper, earlier versions of ANTHEM allowed users to try to prove that the stable models of a (not necessarily tight) program satisfy a specification (Lifschitz et al. 2020). This functionality is retained as a second mode of operation in ANTHEM 1.0.

³ When running VAMPIRE 4.4 with the options `--mode casc` and `--cores 8` on a computer with an Intel Core i7-7700K CPU and 16 GB RAM, the equivalence proof completes within a total of 0.37 seconds.

```

input: n -> integer.
assume: n >= 0.
output: q/1.
spec: exists N (forall X (q(X) <-> X = N) and N >= 0
                and N * N <= n and (N + 1) * (N + 1) > n).

```

Fig. 3. A specification for programs calculating the floor of the square root of an integer n .

```

p(X) :- X = 0..n, X * X <= n.
q(X) :- p(X), not p(X + 1).

```

Fig. 4. A program meant to implement the specification in Figure 3.

8 Conclusions

This paper has four main contributions. First, we presented an extended definition of completion for programs with input and output in a subset of the input language of GRINGO. Such programs are similar to lp-functions, defined by Gelfond and Przymusinska (1996), except that we allow an input to include the values of placeholders. Also, they are closely related to SMOBELS program modules (Oikarinen and Janhunen 2009). Second, we stated a result describing the relationship between stable models and completion in this context. It uses the theory of stable models of infinitary formulas by (Truszczyński 2012) and (Harrison et al. 2017). Third, we proposed a new theory of formal specifications for programs in the input language of GRINGO. Fourth, we presented preliminary experiments with the new version of ANTHEM, a software tool used in combination with a theorem prover to verify that programs with input and output implement a specification. Only few theorem provers support both the typed first-order form of TPTP and integer arithmetics, and VAMPIRE was chosen because its performance outclassed other provers in our early experiments. In later experiments, CVC4 emerged as a possible alternative with similar performance to VAMPIRE, and we consider using CVC4 in the future. As another direction of future work, we would like to extend the subset of the input language of GRINGO supported by ANTHEM, in particular, by simple forms of aggregates.

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```

axiom: p(0) and forall N (N >= 0 and p(N) -> p(N + 1))
        -> forall N (N >= 0 -> p(N)).

lemma(forward): forall X (p(X) <-> exists N (X = N and N >= 0 and N * N <= n)).
lemma(forward): forall X (q(X) <-> exists N2 (X = N2 and N2 >= 0 and N2 * N2 <= n
        and (N2 + 1) * (N2 + 1) > n)).
lemma(forward): forall N1, N2 (N1 >= 0 and N2 >= 0 and N1 < N2
        -> N1 * N1 < N2 * N2).
lemma(forward): forall N (N >= 0 and p(N + 1) -> p(N)).
lemma(forward): not p(n + 1).
lemma(forward): forall N1, N2 (q(N1) and N2 > N1 -> not q(N2)).
lemma(forward): forall N (N >= 0 and not p(N + 1) -> (N + 1) * (N + 1) > n).

lemma(backward): forall N1, N2 (N1 >= 0 and N2 >= 0 and N1 * N1 <= N2 * N2
        -> N1 <= N2).
lemma(backward): forall N (p(N) <-> 0 <= N and N <= n and N * N <= n).
lemma(backward): forall N (not p(N) and N >= 0 -> N * N > n).
lemma(backward): forall N (N >= 0 -> N * N < (N + 1) * (N + 1)).
lemma(backward): forall N1, N2 (p(N1) and not p(N2) and N2 >= 0 -> N1 < N2).
lemma(backward): forall N1, N2 (p(N1) and not p(N1 + 1) and p(N2)
        and not p(N2 + 1) -> N1 = N2).

```

Fig. 5. VAMPIRE is unable to verify the equivalence of the specification in Figure 3 to the program in Figure 4 in a reasonable time frame. This can be addressed by supplying helper lemmas using the `lemma` keyword, where `forward` and `backward` instruct ANTHEM only to apply lemmas in the respective part of the equivalence proof. With these helper lemmas, VAMPIRE can perform the backward direction of the equivalence proof task but is still unable to verify the forward direction. To address this, the set of axioms is manually extended by a fact that is not known to VAMPIRE, an instance of induction. Such axioms can be added with `axiom` statements.

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Appendix A Proofs

A.1 Main Lemma

Let us now extend the correspondence between stable models defined in terms of the SM operator and infinitary logic (Truszczyński 2012) to the above two-sorted case. We also allow formulas to contain extensional predicate symbols, which are not considered in (Truszczyński 2012).

We use the following notation. By \mathcal{Z} , we denote the set of all numerals, and by \mathcal{T} , we denote the set of all precomputed terms. For an interpretation I and a list \mathbf{p} of predicate symbols, by $I^{\mathbf{p}}$, we denote the set of precomputed atoms $p(t_1, \dots, t_k)$ satisfied by I where $p \in \mathbf{p}$.

Let \mathbf{p}, \mathbf{q} be a partition of the predicate symbols in the signature. Then, the *grounding of a sentence F with respect to an interpretation I and a set of intensional predicate symbols \mathbf{p}* (and extensional predicate symbols \mathbf{q}) is defined as follows:

- $gr_I^{\mathbf{p}}(\perp) = \perp$;
- for $p \in \mathbf{p}$, $gr_I^{\mathbf{p}}(p(t_1, \dots, t_k)) = p((t_1^I)^*, \dots, (t_k^I)^*)$;
- for $p \in \mathbf{q}$, $gr_I^{\mathbf{p}}(p(t_1, \dots, t_k)) = \top$ if $p((t_1^I)^*, \dots, (t_k^I)^*) \in I^{\mathbf{q}}$ and $gr_I^{\mathbf{p}}(p(t_1, \dots, t_k)) = \perp$ otherwise;
- $gr_I^{\mathbf{p}}(t_1 = t_2) = \top$ if $t_1^I = t_2^I$ and \perp otherwise;
- $gr_I^{\mathbf{p}}(F \otimes G) = gr_I^{\mathbf{p}}(F) \otimes gr_I^{\mathbf{p}}(G)$ if \otimes is \wedge , \vee , or \rightarrow ;
- $gr_I^{\mathbf{p}}(\exists X F(X)) = \{gr_I^{\mathbf{p}}(F(u)) \mid u \in \mathcal{T}\}^{\vee}$ if X is a program variable;
- $gr_I^{\mathbf{p}}(\forall X F(X)) = \{gr_I^{\mathbf{p}}(F(u)) \mid u \in \mathcal{T}\}^{\wedge}$ if X is a program variable;
- $gr_I^{\mathbf{p}}(\exists X F(X)) = \{gr_I^{\mathbf{p}}(F(u)) \mid u \in \mathcal{Z}\}^{\vee}$ if X is an integer variable;
- $gr_I^{\mathbf{p}}(\forall X F(X)) = \{gr_I^{\mathbf{p}}(F(u)) \mid u \in \mathcal{Z}\}^{\wedge}$ if X is an integer variable.

For a theory Γ , we define $gr_I^{\mathbf{p}}(\Gamma) = \{gr_I^{\mathbf{p}}(F) \mid F \in \Gamma\}$.

Definition 1. *Let Γ be a theory and \mathbf{p} be a list of predicate symbols. Then, an interpretation I is called an INF- \mathbf{p} -stable model of Γ if $I^{\mathbf{p}}$ is a stable model of $gr_I^{\mathbf{p}}(\Gamma)$ in the sense of Definition 1 in (Truszczyński 2012).*

Any term, formula, or theory over the two-sorted signature σ_{Π} can be seen as one-sorted if we do not assign sorts to variables. On the other hand, some one-sorted terms and formulas cannot be viewed as terms or formulas over σ_{Π} ; for instance, the one-sorted term $X + Y$, where X and Y are program variables, is not a term over σ_{Π} . We will refer to terms, formulas, and theories over σ_{Π} as two-sorted. *One-sorted interpretations* are defined as usual in first-order logic, with integer and program variables ranging over the same domain. *One-sorted \mathbf{p} -stable models* and *one-sorted INF- \mathbf{p} -stable models* are defined as \mathbf{p} -stable models and INF- \mathbf{p} -stable models (see Section 4), respectively, but using one-sorted interpretations rather than two-sorted ones. We also say that two theories are *one-sorted-equivalent* if both theories have exactly the same one-sorted models. The following is a special case of Theorem 5 in (Truszczyński 2012) restricted to our one-sorted language.

Proposition 1. *Let Γ be a finite theory and \mathbf{p} be the list of all predicate symbols in some signature σ_{Π} . Then, the one-sorted \mathbf{p} -stable models of Γ and its one-sorted INF- \mathbf{p} -stable models coincide.*

In the following, we extend this result to two-sorted stable models with extensional predicate symbols (see Proposition 2 below). This requires the following notation and auxiliary results. The expression $is_int(t)$, where t is a one-sorted term, stands for the one-sorted formula $t + \bar{1} \neq t + \bar{2}$. Given a two-sorted sentence F , we write F^{us} to denote the one-sorted sentence resulting from restricting all quantifiers that bind integer variables in F to $is_int(t)$. Formally, formula F^{us} is recursively defined as follows:

- $F^{us} = F$ for any atomic formula F ;
- $(F \otimes G)^{us} = F^{us} \otimes G^{us}$ with $\otimes \in \{\wedge, \vee, \rightarrow\}$;
- $(\forall X F(X))^{us} = \forall X F(X)^{us}$;
- $(\exists X F(X))^{us} = \exists X F(X)^{us}$;
- $(\forall N F(N))^{us} = \forall N (is_int(N) \rightarrow F(N)^{us})$;
- $(\exists N F(N))^{us} = \exists N (is_int(N) \wedge F(N)^{us})$;

where X and N are program variables and integer variables, respectively. We also define $\Gamma^{us} = \{F^{us} \mid F \in \Gamma\}$.

Intuitively, the one-sorted models of Γ^{us} are in a one-to-one correspondence with the two-sorted models of Γ . This one-to-one correspondence is formalized as follows. The *generalized value* of a ground term is its value if it exists and a fixed (arbitrarily chosen) symbolic constant u otherwise. Given a two-sorted interpretation I , by I^{us} , we denote the one-sorted interpretation such that

- the universe of I^{us} is the set of all precomputed terms;
- I^{us} interprets each ground term as its generalized value;
- I^{us} interprets every predicate symbol in the same way as I .

Lemma 1. *Let F be a two-sorted sentence and I be a two-sorted interpretation. Then, $I \models F$ iff $I^{us} \models F^{us}$.*

Proof. By structural induction. In case that F is an atomic formula of the form $p(t_1, \dots, t_n)$, it follows that $I^{us} \models F^{us}$ iff $(s_1^*, \dots, s_n^*) \in p^I$, where each s_i is the generalized value of t_i , iff $((t_1^I)^*, \dots, (t_n^I)^*) \in p^I$ iff $I \models F$. Note that, since F is a two-sorted sentence, the generalized value of t_i coincides with its value.

The only remaining relevant cases are quantifiers over integer variables. We show here the case of a universal quantifier. Let N be an integer variable. Then,

$$\begin{aligned}
I^{us} &\models (\forall N G(N))^{us} \\
&\text{iff } I^{us} \models \forall N (is_int(N) \rightarrow G(N)^{us}) \\
&\text{iff } I^{us} \models is_int(u) \rightarrow G(u)^{us} \text{ for all } u \in \mathcal{T} \\
&\text{iff } I^{us} \models G(u)^{us} \text{ for all } u \in \mathcal{Z} \\
&\text{iff } I \models G(u) \text{ for all } u \in \mathcal{Z} && \text{(induction hypothesis)} \\
&\text{iff } I \models \forall N G(N).
\end{aligned}$$

The case for the existential quantifier is analogous. □

We extend this result also to the stable models of a first-order formula. The following auxiliary result is useful for that purpose.

Lemma 2 (Lemma 5 in Ferraris et al. 2011). *The formula $(\mathbf{u} \leq \mathbf{p}) \wedge (F^*(\mathbf{u}) \rightarrow F)$ is satisfied by all one-sorted interpretations and for any one-sorted formula F .*

Lemma 3. *Let F be a two-sorted sentence, I be a two-sorted interpretation, and \mathbf{p} be a list of predicate symbols. Then, $I \models \text{SM}_{\mathbf{p}}[F]$ iff $I^{\text{us}} \models \text{SM}_{\mathbf{p}}[F^{\text{us}}]$.*

Proof. From Lemma 1, we get that $I \models \text{SM}[F]$ iff $I^{\text{us}} \models (\text{SM}[F])^{\text{us}}$. We show below that formula $(\mathbf{u} \leq \mathbf{p}) \wedge (F^*(\mathbf{u}))^{\text{us}}$ is equivalent to $(\mathbf{u} \leq \mathbf{p}) \wedge (F^{\text{us}}(\mathbf{u}))^*$. This immediately implies that $(\text{SM}[F])^{\text{us}}$ and $\text{SM}[F^{\text{us}}]$ are also equivalent. The proof follows by structural induction, and the only relevant cases are, again, quantifiers over integer variables. We show the case of a universal quantifier here. Let N be an integer variable and assume $(\mathbf{u} \leq \mathbf{p})$. Then,

$$\begin{aligned}
& ((\forall N G(N, \mathbf{u}))^{\text{us}})^* \\
&= (\forall N (is_int(N) \rightarrow G(N, \mathbf{u})^{\text{us}}))^* \\
&= \forall N (is_int(N) \rightarrow G(N, \mathbf{u})^{\text{us}})^* \\
&= \forall N ((is_int(N) \rightarrow G(N)^{\text{us}}) \wedge (is_int(N)^* \rightarrow (G(N, \mathbf{u})^{\text{us}})^*)) \\
&= \forall N ((is_int(N) \rightarrow G(N)^{\text{us}}) \wedge (is_int(N) \rightarrow (G(N, \mathbf{u})^{\text{us}})^*)) \\
&\Leftrightarrow \forall N (is_int(N) \rightarrow (G(N)^{\text{us}} \wedge (G(N, \mathbf{u})^{\text{us}})^*)) \\
&\Leftrightarrow \forall N (is_int(N) \rightarrow (G(N, \mathbf{u})^{\text{us}})^*) \tag{Lemma 2} \\
&\Leftrightarrow \forall N (is_int(N) \rightarrow (G(N, \mathbf{u})^*)^{\text{us}}) \tag{induction hypothesis} \\
&= (\forall N G(N, \mathbf{u})^*)^{\text{us}}. \quad \square
\end{aligned}$$

This correspondence can also be established in terms of groundings as follows. The expression $\Gamma \equiv_s \Gamma'$, where Γ and Γ' are two infinitary propositional theories, stands for *strong equivalence* in the sense of (Harrison et al. 2017, Section 3.1).

Lemma 4. *Let Γ be a finite two-sorted theory, I be a two-sorted interpretation, and \mathbf{p} be a list of predicate symbols. Then, $gr_{\mathbf{I}}^{\mathbf{p}}(\Gamma) \equiv_s gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}(\Gamma^{\text{us}})$.*

Proof. We show $gr_{\mathbf{I}}^{\mathbf{p}}(F) \equiv_s gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}(F^{\text{us}})$ for a formula F , which implies $gr_{\mathbf{I}}^{\mathbf{p}}(\Gamma) \equiv_s gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}(\Gamma^{\text{us}})$. We proceed by structural induction. The only relevant cases are quantifiers over integer variables. We show the case of a universal quantifier here. Let N be an integer variable and $G'(u)$ stand for $gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}(G^{\text{us}}(u))$. Then,

$$\begin{aligned}
gr_{\mathbf{I}}^{\mathbf{p}}(\forall N G(N)) &= \{gr_{\mathbf{I}}^{\mathbf{p}}(G(u)) \mid u \in \mathcal{Z}\}^{\wedge} \\
&\equiv_s \{G'(u) \mid u \in \mathcal{Z}\}^{\wedge} \tag{induction hypothesis} \\
&\equiv_s (\{\top \rightarrow G'(u) \mid u \in \mathcal{Z}\} \cup \{\perp \rightarrow G'(u) \mid u \in \mathcal{T} \setminus \mathcal{Z}\})^{\wedge} \\
&= gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}(\forall N (is_int(N) \rightarrow G^{\text{us}}(N))) \\
&= gr_{\mathbf{I}^{\text{us}}}^{\mathbf{p}}((\forall N G(N))^{\text{us}}).
\end{aligned}$$

The case for the existential quantifier is analogous. \square

Next, we combine these results to establish a correspondence between the stable models of a first-order theory and the stable models of its infinitary grounding.

Lemma 5. *Let Γ be a finite two-sorted theory, I be a two-sorted interpretation, and \mathbf{p} be the list of all predicate symbols in the signature. Then, the \mathbf{p} -stable and the INF- \mathbf{p} -stable models of Γ coincide.*

Proof. We have

$$\begin{aligned}
& I \text{ is a } \mathbf{p}\text{-stable model of } \Gamma \\
& \text{iff } I^{\text{us}} \text{ is a } \mathbf{p}\text{-stable model of } \Gamma^{\text{us}} && \text{(Lemma 3)} \\
& \text{iff } I^{\text{us}} \text{ is an INF-}\mathbf{p}\text{-stable model of } \Gamma^{\text{us}} && \text{(Proposition 1)} \\
& \text{iff } (I^{\text{us}})^{\mathbf{P}} \text{ is a stable model of } gr_{I^{\text{us}}}^{\mathbf{P}}(\Gamma^{\text{us}}) \\
& \text{iff } (I^{\text{us}})^{\mathbf{P}} \text{ is a stable model of } gr_I^{\mathbf{P}}(\Gamma) && \text{(Lemma 4)} \\
& \text{iff } I^{\mathbf{P}} \text{ is a stable model of } gr_I^{\mathbf{P}}(\Gamma) \\
& \text{iff } I \text{ is an INF-}\mathbf{p}\text{-stable model of } \Gamma.
\end{aligned}$$

For the next-to-last equivalence, just note that $(I^{\text{us}})^{\mathbf{P}} = I^{\mathbf{P}}$. \square

Proposition 2. *For any finite two-sorted theory Γ and list of predicate symbols \mathbf{p} , its \mathbf{p} -stable models and its INF- \mathbf{p} -stable models coincide.*

Proof. Let \mathbf{q} be the list of all extensional predicate symbols in Γ , that is, all predicate symbols in the signature that do not belong to \mathbf{p} , and let $Choice(\mathbf{q})$ be the set containing a choice sentence $\forall \mathbf{U} (p(\mathbf{U}) \vee \neg p(\mathbf{U}))$ for every predicate $p \in \mathbf{q}$ and where \mathbf{U} is a list of distinct program variables. Let Γ' be the theory obtained by replacing each occurrence of $p(\mathbf{t})$ in Γ with $p \in \mathbf{q}$ by $\neg \neg p(\mathbf{t})$. Let $\Gamma_1 = \Gamma \cup Choice(\mathbf{q})$ and $\Gamma'_1 = \Gamma' \cup Choice(\mathbf{q})$. Given the choice sentences in Γ_1 and Γ'_1 for the predicate symbols in \mathbf{q} , the \mathbf{pq} -stable models of Γ_1 and Γ'_1 coincide. Then,

$$\begin{aligned}
& I \text{ is a } \mathbf{p}\text{-stable model of } \Gamma \\
& \text{iff } I \text{ is a } \mathbf{pq}\text{-stable model of } \Gamma_1 && \text{(Theorem 2 in Ferraris et al. 2011)} \\
& \text{iff } I \text{ is a } \mathbf{pq}\text{-stable model of } \Gamma'_1 \\
& \text{iff } I \text{ is an INF-}\mathbf{pq}\text{-stable model of } \Gamma'_1 && \text{(Lemma 5)} \\
& \text{iff } I \text{ is an INF-}\mathbf{p}\text{-stable model of } \Gamma. && \text{(see below)}
\end{aligned}$$

It remains to be shown that the INF- \mathbf{pq} -stable models of Γ'_1 coincide with the INF- \mathbf{p} -stable models of Γ . For this, note that

$$\begin{aligned}
[gr_I^{\mathbf{pq}}(\Gamma'_1)]^{I^{\mathbf{pq}}} &= [gr_I^{\mathbf{pq}}(\Gamma' \cup Choice(\mathbf{q}))]^{I^{\mathbf{pq}}} \\
&= [gr_I^{\mathbf{pq}}(\Gamma')]^{I^{\mathbf{pq}}} \cup [gr_I^{\mathbf{pq}}(Choice(\mathbf{q}))]^{I^{\mathbf{pq}}} \\
&\equiv [gr_I^{\mathbf{P}}(\Gamma')]^{I^{\mathbf{pq}}} \cup [gr_I^{\mathbf{pq}}(Choice(\mathbf{q}))]^{I^{\mathbf{pq}}} \\
&= [gr_I^{\mathbf{P}}(\Gamma')]^{I^{\mathbf{P}}} \cup [gr_I^{\mathbf{pq}}(Choice(\mathbf{q}))]^{I^{\mathbf{q}}} \\
&\equiv [gr_I^{\mathbf{P}}(\Gamma')]^{I^{\mathbf{P}}} \cup I^{\mathbf{q}}.
\end{aligned}$$

The first two equalities hold by definition. The third step holds because all predicate symbols in \mathbf{q} occur in Γ' under the scope of negation. Note that, for $q \in \mathbf{q}$, it follows that

$$\begin{aligned}
[gr_I^{\mathbf{pq}}(\neg q(\mathbf{t}))]^{I^{\mathbf{pq}}} &= [\neg q((\mathbf{t}^I)^*)]^{I^{\mathbf{pq}}} = I^{\mathbf{pq}}(\neg q((\mathbf{t}^I)^*)) \\
&\equiv \neg I(q((\mathbf{t}^I)^*)) \\
&= [\neg I(q((\mathbf{t}^I)^*))]^{I^{\mathbf{pq}}} = [\neg gr_I^{\mathbf{P}}(q(\mathbf{t}))]^{I^{\mathbf{pq}}} = [gr_I^{\mathbf{P}}(\neg q(\mathbf{t}))]^{I^{\mathbf{pq}}},
\end{aligned}$$

where

$$I(q((\mathbf{t}^I)^*)) = \begin{cases} \top & \text{if } q((\mathbf{t}^I)^*) \in I^{\mathbf{q}}; \\ \perp & \text{otherwise} \end{cases}$$

$$I^{\mathbf{p}\mathbf{q}}(\neg q((\mathbf{t}^I)^*)) = \begin{cases} \perp \equiv \neg I(q((\mathbf{t}^I)^*)) & \text{if } q((\mathbf{t}^I)^*) \in I^{\mathbf{q}}; \\ \neg \perp = \neg I(q((\mathbf{t}^I)^*)) & \text{otherwise.} \end{cases}$$

The fourth case is because no predicate in \mathbf{q} occurs in $gr_I^{\mathbf{p}}(\Gamma')$. Recall that extensional predicate symbols are removed by grounding. Similarly, $Choice(\mathbf{q})$ only contains predicate symbols from \mathbf{q} .

We now prove the following equivalence:

$$[gr_I^{\mathbf{p}\mathbf{q}}(\Gamma'_1)]^{I^{\mathbf{p}\mathbf{q}}} \equiv [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}} \cup I^{\mathbf{q}} \quad (\text{A1})$$

For this, note that

$$gr_I^{\mathbf{p}}(\Gamma') \equiv gr_I^{\mathbf{p}}(\Gamma)$$

holds because $gr_I^{\mathbf{p}}(\Gamma')$ is the result of replacing each occurrence of $p(\mathbf{t})$ in $gr_I^{\mathbf{p}}(\Gamma)$ by $\neg\neg p(\mathbf{t})$ with $p \in \mathbf{q}$. As a result, $gr_I^{\mathbf{p}}(\Gamma')$ is the outcome of replacing each occurrence of X in $gr_I^{\mathbf{p}}(\Gamma)$ (with $X \in \{\top, \perp\}$) by $\neg\neg X$. Consequently, equivalence (A1) is proven, and we get that any interpretation J satisfies that $J^{\mathbf{p}\mathbf{q}} \models [gr_I^{\mathbf{p}\mathbf{q}}(\Gamma'_1)]^{I^{\mathbf{p}\mathbf{q}}}$ iff $J^{\mathbf{p}} \models [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}}$ and $J^{\mathbf{q}} \models I^{\mathbf{q}}$. Note that $J^{\mathbf{q}} \models I^{\mathbf{q}}$ iff $J^{\mathbf{q}} \supseteq I^{\mathbf{q}}$. Then,

$$\begin{aligned} & I \text{ is an INF-}\mathbf{p}\mathbf{q}\text{-stable model of } \Gamma'_1 \\ & \text{iff } I^{\mathbf{p}\mathbf{q}} \text{ is a stable model of } gr_I^{\mathbf{p}\mathbf{q}}(\Gamma'_1) \\ & \text{iff } I^{\mathbf{p}\mathbf{q}} \text{ is a model of } [gr_I^{\mathbf{p}\mathbf{q}}(\Gamma'_1)]^{I^{\mathbf{p}\mathbf{q}}} \text{ and there is no model } J \subset I^{\mathbf{p}\mathbf{q}} \text{ of } [gr_I^{\mathbf{p}\mathbf{q}}(\Gamma'_1)]^{I^{\mathbf{p}\mathbf{q}}} \\ & \text{iff } I^{\mathbf{p}} \text{ is a model of } [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}} \text{ and there is no model } J \subset I^{\mathbf{p}\mathbf{q}} \text{ of } [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}} \cup I^{\mathbf{q}} \\ & \text{iff } I^{\mathbf{p}} \text{ is a model of } [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}} \text{ and there is no model } J' \subset I^{\mathbf{p}} \text{ of } [gr_I^{\mathbf{p}}(\Gamma)]^{I^{\mathbf{p}}} \\ & \text{iff } I^{\mathbf{p}} \text{ is a stable model of } [gr_I^{\mathbf{p}}(\Gamma)] \\ & \text{iff } I \text{ is an INF-}\mathbf{p}\text{-stable model of } \Gamma. \quad \square \end{aligned}$$

The following adaptation of Proposition 3 from (Lifschitz et al. 2019) to our notation is useful to prove the Main Lemma.

Proposition 3. *Any rule R and interpretation I satisfy $gr_I^{\mathbf{p}}(\tau^*R) \equiv_s \tau R$.*

Proof. By identifying the precomputed terms in $\tau^*\Pi^{\text{PROP}}$ with their names in I , we get $gr_I^{\mathbf{p}}(\tau^*\Pi) = \tau^*\Pi^{\text{PROP}}$, where $\tau^*\Pi^{\text{PROP}}$ is defined as in (Lifschitz et al. 2019, Section 5). \square

Proof of the Main Lemma. Let Π be a program, let \mathbf{p} be the list of all predicate symbols occurring in Π other than the comparison symbols, and let \mathcal{I} be a set of precomputed atoms. By the choice of \mathbf{p} , we get that all predicate symbols in Π and none of the relations belong to \mathbf{p} and, therefore, $\mathcal{I} = (\mathcal{I}^\dagger)^{\mathbf{p}}$. Then, from Proposition 2, it follows that \mathcal{I}^\dagger is a \mathbf{p} -stable model of $\tau^*\Pi$ iff \mathcal{I}^\dagger is an INF- \mathbf{p} -stable model of $\tau^*\Pi$ iff \mathcal{I} is a \subseteq -minimal model of $[gr_{\mathcal{I}^\dagger}^{\mathbf{p}}(\tau^*\Pi)]^{\mathcal{I}}$ iff \mathcal{I} is a stable model of $[gr_{\mathcal{I}^\dagger}^{\mathbf{p}}(\tau^*\Pi)]$ iff \mathcal{I} is a stable model of $\tau\Pi$ (Proposition 3) iff \mathcal{I} is a stable model of Π (by definition). \square

A.2 Main Lemma for IO-Programs

We need the following terminology to extend the Main Lemma to io-programs. The models of formula $\exists \mathbf{H} (\text{SM}_{\mathbf{p}}[F])_{\mathbf{H}}^{\mathbf{h}}$ are called the *\mathbf{p} -stable models with private symbols \mathbf{h}* of F , where \mathbf{H} is a tuple of predicate variables of the same length as \mathbf{h} and $F_{\mathbf{H}}^{\mathbf{h}}$ is the result of replacing all occurrences of constants from \mathbf{h} by the corresponding variables from \mathbf{H} . For a set Γ of first-order sentences, the *\mathbf{p} -stable models with private symbols \mathbf{h}* of Γ are the \mathbf{p} -stable models with private symbols \mathbf{h} of the conjunction of all formulas in Γ (Cabalar et al. 2020). We usually omit parentheses and write just $\exists \mathbf{H} \text{SM}_{\mathbf{p}}[F]_{\mathbf{H}}^{\mathbf{h}}$ instead of $\exists \mathbf{H} (\text{SM}_{\mathbf{p}}[F])_{\mathbf{H}}^{\mathbf{h}}$.

Main Lemma for IO-Programs. *Let $\Omega = (\Pi, PH, In, Out)$ be an io-program, let \mathbf{p} be the list of all predicate symbols occurring in Π other than the comparison and input symbols, and let \mathbf{h} be the list of all its private symbols. A set \mathcal{I} of precomputed public atoms is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) iff $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^* \Pi$ and $\mathcal{I}^{in} = \mathbf{i}$.*

The following is a reformulation of the Splitting Theorem in (Ferraris et al. 2009) adapted to our notation, and it will be useful in proving the above result. We adopt the following terminology. An occurrence of a predicate symbol in a formula is called *negated* if it belongs to a subformula of the form $F \rightarrow \perp$ and *nonnegated* otherwise. An occurrence of a predicate symbol in a formula is called *positive* if the number of implications containing that occurrence in the antecedent is even. It is called *strictly positive* if that number is 0. A *rule* of a first-order formula F is a strictly positive occurrence of an implication in F . The *dependency graph* of a formula is a directed graph that

- has all intensional predicate symbols as vertices and
- has an edge from p to q if, for some rule $G \rightarrow H$ of F , formula G has a positive nonnegated occurrence of q and H has a strictly positive occurrence of p .

Proposition 4. *Let F and G be one-sorted first-order sentences and let \mathbf{p} and \mathbf{q} be two disjoint tuples of distinct predicate symbols such that*

- *each strongly connected component of the of the dependency graph of $F \wedge G$ is a subset either of \mathbf{p} or \mathbf{q} ,*
- *all occurrences in F of symbols from \mathbf{q} are negated, and*
- *all occurrences in G of symbols from \mathbf{p} are negated.*

Then, $\text{SM}_{\mathbf{p}\mathbf{q}}[F \wedge G]$ is equivalent to $\text{SM}_{\mathbf{p}}[F] \wedge \text{SM}_{\mathbf{q}}[G]$.

This result can be straightforwardly lifted to the two-sorted language as follows.

Proposition 5. *Let F and G be two-sorted first-order sentences and let \mathbf{p} and \mathbf{q} be two disjoint tuples of distinct predicate symbols such that*

- *each strongly connected component of the dependency graph of $F \wedge G$ is a subset either of \mathbf{p} or \mathbf{q} ,*
- *all occurrences in F of symbols from \mathbf{q} are negated, and*
- *all occurrences in G of symbols from \mathbf{p} are negated.*

Then, $\text{SM}_{\mathbf{p}\mathbf{q}}[F \wedge G]$ is equivalent to $\text{SM}_{\mathbf{p}}[F] \wedge \text{SM}_{\mathbf{q}}[G]$.

Proof. Let I be any interpretation. Then,

$$\begin{aligned}
 I &\models \text{SM}_{\mathbf{pq}}[F \wedge G] \\
 \text{iff } I^{\text{us}} &\models \text{SM}_{\mathbf{pq}}[(F \wedge G)^{\text{us}}] && \text{(Lemma 3)} \\
 \text{iff } I^{\text{us}} &\models \text{SM}_{\mathbf{pq}}[F^{\text{us}} \wedge G^{\text{us}}] \\
 \text{iff } I^{\text{us}} &\models \text{SM}_{\mathbf{p}}[F^{\text{us}}] \wedge \text{SM}_{\mathbf{q}}[G^{\text{us}}] && \text{(Proposition 4)} \\
 \text{iff } I &\models \text{SM}_{\mathbf{p}}[F] \wedge \text{SM}_{\mathbf{q}}[G]. && \text{(Lemma 3)}
 \end{aligned}$$

Note that the dependency graphs of $F \wedge G$ and $F^{\text{us}} \wedge G^{\text{us}}$ are the same. \square

Lemma 6. *Let $\Omega = (\Pi, PH, In, Out)$ be an io-program and let \mathbf{p} be the list of all predicate symbols occurring in Π other than the comparison and input symbols. A set \mathcal{I} of precomputed public atoms is a stable model of $\Omega(\mathbf{v}, \mathbf{i})$ iff $\mathcal{I}^{\mathbf{v}}$ is a model of $\text{SM}_{\mathbf{p}}[\tau^*\Pi]$ and $\mathcal{I}^{\text{in}} = \mathbf{i}$.*

Proof. Recall that, from the Main Lemma stated in Section 4, we get that \mathcal{I} is a stable model of $\Omega(\mathbf{v}, \mathbf{i})$ iff \mathcal{I}^\uparrow is a \mathbf{pq} -stable model of $\tau^*(\Omega(\mathbf{v}, \mathbf{i}))$ iff \mathcal{I}^\uparrow is a model of $\text{SM}_{\mathbf{pq}}[\tau^*(\Omega(\mathbf{v}, \mathbf{i}))]$, where \mathbf{q} is the list of all input symbols. Let us denote by $\Omega(\mathbf{v})$ the set of rules obtained from the rules of Ω by substituting the precomputed terms $v(c)$ for all occurrences of all placeholders c . Then, $\tau^*(\Omega(\mathbf{v}, \mathbf{i})) = \tau^*(\Omega(\mathbf{v})) \cup \mathbf{i} = \tau^*(\Omega(\mathbf{v})) \cup \tau^*(\mathbf{i}) \equiv_s \tau^*(\Omega(\mathbf{v})) \cup \mathbf{i}$. Furthermore, since there are no occurrences of predicate symbols in \mathbf{q} in the heads of the rules of $\Omega(\mathbf{v})$ nor of any predicate symbol in \mathbf{p} in the head of the rules in \mathbf{i} , we get that each strongly connected component is a subset either of \mathbf{p} or \mathbf{q} . From Proposition 5, this implies that

$$\begin{aligned}
 \mathcal{I}^\uparrow &\text{ is a model of } \text{SM}_{\mathbf{pq}}[\tau^*(\Omega(\mathbf{v}, \mathbf{i}))] \\
 \text{iff } \mathcal{I}^\uparrow &\text{ is a model of } \text{SM}_{\mathbf{p}}[\tau^*(\Omega(\mathbf{v}))] \text{ and } \mathcal{I}^\uparrow \text{ is a model of } \text{SM}_{\mathbf{q}}[\mathbf{i}] \\
 \text{iff } \mathcal{I}^{\mathbf{v}} &\text{ is a model of } \text{SM}_{\mathbf{p}}[\tau^*\Pi] \text{ and } \mathcal{I}^{\text{in}} = \mathbf{i}.
 \end{aligned}$$

For the second equivalence, note that $\mathcal{I}^{\mathbf{v}}$ is identical to \mathcal{J}^\uparrow , except that it interprets each placeholder c as $\mathbf{v}(c)$ and that $\Omega(\mathbf{v})$ is the result of replacing each placeholder c by $\mathbf{v}(c)$. \square

Proof of the Main Lemma for IO-Programs. From left to right. Assume that \mathcal{I} is an io-model of Ω for input (\mathbf{v}, \mathbf{i}) . Let us show that $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^*\Pi$ and $\mathcal{I}^{\text{in}} = \mathbf{i}$. By definition, the assumption implies that there is some stable model \mathcal{J} of $\Omega(\mathbf{v}, \mathbf{i})$ such that \mathcal{I} is the set of all public atoms of \mathcal{J} . From Lemma 6, this implies that $\mathcal{J}^{\mathbf{v}}$ is a model of $\text{SM}_{\mathbf{p}}[\tau^*\Pi]$ and $\mathcal{J}^{\text{in}} = \mathbf{i}$ and, thus, that $\mathcal{I}^{\mathbf{v}}$ is a model of $\exists \mathbf{H} \text{SM}_{\mathbf{p}}[(\tau^*\Pi)]_{\mathbf{H}}^{\mathbf{h}}$ and $\mathcal{I}^{\text{in}} = \mathbf{i}$. For this last step, recall that \mathcal{I} and \mathcal{J} agree on all public predicates. By definition, this means that $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^*\Pi$ and $\mathcal{I}^{\text{in}} = \mathbf{i}$.

From right to left. Assume that $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^*\Pi$. Let us show that \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) . By definition, the assumption implies that $\mathcal{I}^{\mathbf{v}}$ is a model of $\exists \mathbf{H} \text{SM}_{\mathbf{p}}[(\tau^*\Pi)]_{\mathbf{H}}^{\mathbf{h}}$. This implies that there is some model J of $\text{SM}_{\mathbf{p}}[(\tau^*\Pi)]$ such that $\mathcal{I}^{\mathbf{v}}$ and J agree on the interpretation of all public predicates. Let \mathcal{J} be the set of precomputed atoms satisfied by J . Then, there are no occurrences of placeholders in \mathcal{J} and, thus, we get that $\mathcal{J}^{\mathbf{v}} = \mathcal{J}^\uparrow = J$ and, thus, also that $\mathcal{J}^{\mathbf{v}}$ is a stable model of $\text{SM}_{\mathbf{p}}[(\tau^*\Pi)]$. Recall that we also have $\mathcal{I}^{\text{in}} = \mathbf{i}$ and, since \mathcal{I} and \mathcal{J} contain

the same public atoms, we get that $\mathcal{J}^{in} = \mathbf{i}$. From Lemma 6, these two facts together imply that \mathcal{J} is a stable model of $\Omega(\mathbf{v}, \mathbf{i})$ and, therefore, that \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) . \square

A.3 Theorem 1

In order to prove Theorem 1, we need the notions of Clark normal form, completion, and tight theories (Ferraris et al. 2011, Section 6). We adapt these notions to a two-sorted language here. A theory—one-sorted or two-sorted—is in *Clark normal form* relative to a list \mathbf{p} of intensional predicates if it contains exactly one sentence of the form

$$\forall V_1 \dots V_n (G \rightarrow p(V_1, \dots, V_n)) \quad (\text{A2})$$

for each intensional predicate symbol p/n in \mathbf{p} , where G is a formula and V_1, \dots, V_n are distinct program variables. The *completion* of a theory Γ in Clark normal form, denoted by $\text{COMP}_{\mathbf{p}}[\Gamma]$, is obtained by replacing each implication \rightarrow by an equivalence \leftrightarrow in all sentences of form (A2). The following is a special case of Theorem 10 in (Ferraris et al. 2011) adapted to our notation.

Proposition 6. *For any one-sorted sentence F in Clark normal form and list of predicates \mathbf{p} , the implication*

$$\text{SM}_{\mathbf{p}}[F] \rightarrow \text{COMP}_{\mathbf{p}}[F]$$

is satisfied by all one-sorted interpretations.

Then, we can easily extend this result to two-sorted interpretations as follows.

Proposition 7. *For any two-sorted sentence F in Clark normal form, list of predicates \mathbf{p} , and two-sorted interpretation \mathcal{I} , if \mathcal{I} satisfies $\text{SM}_{\mathbf{p}}[F]$, then it also satisfies $\text{COMP}_{\mathbf{p}}[F]$.*

Proof. Let I be any two-sorted interpretation. From Lemma 1, we get that $I \models \text{COMP}_{\mathbf{p}}[F]$ iff $I^{\text{us}} \models (\text{COMP}_{\mathbf{p}}[F])^{\text{us}}$. Furthermore, we can see that $(\text{COMP}_{\mathbf{p}}[F])^{\text{us}} = \text{COMP}_{\mathbf{p}}[F^{\text{us}}]$, and thus, we get

$$I \models \text{COMP}_{\mathbf{p}}[F] \text{ iff } I^{\text{us}} \models \text{COMP}_{\mathbf{p}}[F^{\text{us}}].$$

Similarly, from Lemma 3, we get

$$I \models \text{SM}_{\mathbf{p}}[\Gamma] \text{ iff } I^{\text{us}} \models \text{SM}_{\mathbf{p}}[\Gamma^{\text{us}}].$$

Finally, from Proposition 6, we get

$$I^{\text{us}} \models \text{SM}_{\mathbf{p}}[\Gamma^{\text{us}}] \text{ implies } I^{\text{us}} \models \text{COMP}_{\mathbf{p}}[\Gamma^{\text{us}}].$$

Consequently, the result holds. \square

Let us introduce the *Clark form* of a program without input and output. The *Clark definition* of p/n in Π is a formula of the form

$$\forall V_1 \dots V_n \left(\bigvee_{i=1}^k \exists \mathbf{U}_i F_i \rightarrow p(V_1, \dots, V_n) \right), \quad (\text{A3})$$

where each F_i is the formula representation of rule R_i and rules R_1, \dots, R_k constitute the

definition of p/n in Π . By $Cdef(\Pi)$, we denote the theory containing the Clark definitions of all predicate symbols. We also define $Clark(\Pi) \stackrel{\text{def}}{=} Cdef(\Pi) \cup \Pi_C$, where Π_C is the set containing the formula representation of all constraints in Π .

About first-order formulas F and G , it is said that F is *strongly equivalent* to G if, for any formula H , any occurrence of F in H , and any list \mathbf{p} of distinct predicate symbols, $SM_{\mathbf{p}}[H]$ is equivalent to $SM_{\mathbf{p}}[H']$, where H' is obtained from H by replacing the occurrence of F by G . About finite first-order theories Γ and Γ' , we say that Γ is *strongly equivalent* to Γ' when the conjunction of all sentences in Γ is strongly equivalent to the conjunction of all sentences in Γ' . First-order theory Γ is strongly equivalent to Γ' iff Γ is equivalent to Γ' in quantified equilibrium logic (Ferraris et al. 2011, Theorem 8). Therefore, Γ is strongly equivalent to Γ' if Γ is equivalent to Γ' in intuitionistic logic.

Lemma 7. *Let Π be a program without constraints. Then, $\tau^*\Pi$ is strongly equivalent to $Cdef(\Pi)$.*

Proof. By definition, $\tau^*\Pi$ contains a formula of the form

$$\forall V_1 \dots V_n \mathbf{U}_i (F_i \rightarrow p(V_1, \dots, V_n)) \quad (\text{A4})$$

for each rule R_i in Π . Note that (A4) is strongly equivalent to

$$\forall V_1 \dots V_n (\exists \mathbf{U}_i F_i \rightarrow p(V_1, \dots, V_n)).$$

Furthermore, since Π is finite, it follows that $\tau^*\Pi$ is finite too and, therefore, we get that $\tau^*\Pi$ is strongly equivalent to Γ , where Γ is the theory containing a formula of the form

$$\bigwedge_{i=1}^k \forall V_1 \dots V_n (\exists \mathbf{U}_i F_i \rightarrow p(V_1, \dots, V_n)) \quad (\text{A5})$$

for each predicate symbol p/n . Finally, since (A3) and (A5) are strongly equivalent, we get that $\tau^*\Pi$ and $Cdef(\Pi)$ are also strongly equivalent. \square

Lemma 8. *For any program Π , $\tau^*\Pi$ is strongly equivalent to $Clark(\Pi)$.*

Proof. Let Π_1, Π_2 be a partition of Π such that Π_2 contains all constraints and Π_1 all the remaining rules. Then, $\tau^*\Pi = \tau^*(\Pi_1 \cup \Pi_2) = \tau^*\Pi_1 \cup \tau^*\Pi_2 = \tau^*\Pi_1 \cup \Pi_C$. Now, the result follows directly from Lemma 7. \square

Proof of Theorem 1. From the Main Lemma for IO-Programs, it follows that \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) iff $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^*\Pi$ and $\mathcal{I}^{in} = \mathbf{i}$. Furthermore,

$$\begin{aligned} & \mathcal{I}^{\mathbf{v}} \text{ is a } \mathbf{p}\text{-stable model with private symbols } \mathbf{h} \text{ of } \tau^*\Pi \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} SM_{\mathbf{p}}[\tau^*\Pi]_{\mathbf{H}}^{\mathbf{h}} \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} SM_{\mathbf{p}}[Clark(\Pi)]_{\mathbf{H}}^{\mathbf{h}} \quad (\text{Lemma 8}) \\ & \text{implies that } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} COMP_{\mathbf{p}}[Clark(\Pi)]_{\mathbf{H}}^{\mathbf{h}} \quad (\text{Proposition 7}) \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} COMP_{\mathbf{p}}[\tau^*\Pi]_{\mathbf{H}}^{\mathbf{h}} \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } COMP[\Omega]. \quad \square \end{aligned}$$

A.4 Theorem 2

To prove Theorem 2, we need the following terminology. An occurrence of a predicate symbol in a formula is called *negated* if it belongs to a subformula of the form $F \rightarrow \perp$ and *nonnegated* otherwise. An occurrence of a predicate symbol in a formula is called *positive* if the number of implications containing that occurrence in the antecedent is even. The *dependency graph* of a theory in Clark normal form is a directed graph that

- has all intensional predicate symbols as vertices and
- has an edge from p to q if q has a positive nonnegated occurrence in G for some sentence of form (A2).

A theory is *tight* if its predicate dependency graph is acyclic. The following is a reformulation of Theorem 11 in (Ferraris et al. 2011) adapted to our notation.

Proposition 8. *For any finite, tight, one-sorted theory Γ in Clark normal form, $\text{SM}_{\mathbf{p}}[\Gamma]$ is equivalent to $\text{COMP}_{\mathbf{p}}[\Gamma]$.*

The following lifts this result to the case of two sorts.

Proposition 9. *For any finite, tight, two-sorted theory Γ in Clark normal form, $\text{SM}_{\mathbf{p}}[\Gamma]$ is equivalent to $\text{COMP}_{\mathbf{p}}[\Gamma]$.*

Proof. Let I be any two-sorted interpretation. From Lemma 1, we get that $I \models \text{COMP}_{\mathbf{p}}[\Gamma]$ iff $I^{\text{us}} \models (\text{COMP}_{\mathbf{p}}[\Gamma])^{\text{us}}$. Furthermore, it is easy to see that $(\text{COMP}_{\mathbf{p}}[\Gamma])^{\text{us}} = \text{COMP}_{\mathbf{p}}[\Gamma^{\text{us}}]$, and thus, we get

$$I \models \text{COMP}_{\mathbf{p}}[\Gamma] \text{ iff } I^{\text{us}} \models \text{COMP}_{\mathbf{p}}[\Gamma^{\text{us}}].$$

Similarly, from Lemma 3, we get

$$I \models \text{SM}_{\mathbf{p}}[\Gamma] \text{ iff } I^{\text{us}} \models \text{SM}_{\mathbf{p}}[\Gamma^{\text{us}}].$$

Finally, from Proposition 8, we get

$$I^{\text{us}} \models \text{COMP}_{\mathbf{p}}[\Gamma^{\text{us}}] \text{ iff } I^{\text{us}} \models \text{SM}_{\mathbf{p}}[\Gamma^{\text{us}}].$$

Consequently, the result holds. \square

Proof of Theorem 2. From the Main Lemma for IO-Programs, it follows that \mathcal{I} is an io-model of Ω for an input (\mathbf{v}, \mathbf{i}) iff $\mathcal{I}^{\mathbf{v}}$ is a \mathbf{p} -stable model with private symbols \mathbf{h} of $\tau^*\Pi$ and $\mathcal{I}^{\text{in}} = \mathbf{i}$. Furthermore,

$$\begin{aligned} & \mathcal{I}^{\mathbf{v}} \text{ is a } \mathbf{p}\text{-stable model with private symbols } \mathbf{h} \text{ of } \tau^*\Pi \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} \text{SM}_{\mathbf{p}}[\tau^*\Pi]_{\mathbf{H}}^{\mathbf{h}} \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} \text{SM}_{\mathbf{p}}[\text{Clark}(\Pi)] \quad (\text{Lemma 7}) \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} \text{COMP}_{\mathbf{p}}[\text{Clark}(\Pi)]_{\mathbf{H}}^{\mathbf{h}} \quad (\text{Proposition 9}) \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \exists \mathbf{H} \text{COMP}_{\mathbf{p}}[\tau^*\Pi]_{\mathbf{H}}^{\mathbf{h}} \\ & \text{iff } \mathcal{I}^{\mathbf{v}} \text{ is a model of } \text{COMP}[\Omega]. \end{aligned}$$

Recall that Ω is tight, and this implies that $\text{Clark}(\tau^*\Pi)$ is also tight. Note that $\text{Clark}(\tau^*\Pi)$ contains a formula of form (A3) for every predicate symbol p/n and that the antecedent of this formula is a disjunction of the formula representations of the bodies of all rules

defining p/n . Therefore, the dependency graph of $\text{Clark}(\tau^*\Pi)$ is identical to the dependency graph of Ω with the exception of the addition of nonpositive edges corresponding to choice rules. \square

A.5 Theorem 3

A *predicate expression* is a lambda expression of the form

$$\lambda X_1 \dots X_n F(X_1, \dots, X_n), \quad (\text{A6})$$

where $F(X_1, \dots, X_n)$ is a formula and X_1, \dots, X_n are object variables. This formula may have free variables other than X_1, \dots, X_n , called the *parameters* of (A6). If E is (A6) and t_1, \dots, t_n are terms, then $E(t_1, \dots, t_n)$ stands for the formula $F(t_1, \dots, t_n)$. If $G(P)$ is a formula containing a predicate constant or variable P and E is a predicate expression of the same arity as P , then $G(E)$ stands for the result of replacing each atom $P(t_1, \dots, t_n)$ in $G(P)$ by $E(t_1, \dots, t_n)$. For any predicate expression E , the formulas

$$\forall P G(P) \rightarrow A(E) \quad \text{and} \quad G(E) \rightarrow \exists P A(P)$$

are theorems of second-order logic.

Lemma 9. *Let $\mathbf{P} = P_1, \dots, P_l$ be a list of predicate variables and let $\mathbf{P}_i = P_1, \dots, P_i$ be a prefix of \mathbf{P} . Let $F_1(\mathbf{P}_1), \dots, F_l(\mathbf{P}_l)$ be formulas such that \mathbf{P}_i contains all free predicate variables occurring in F_i . Let F and G , respectively, be the following two formulas:*

$$\exists \mathbf{P} (F_1(\mathbf{P}_1) \wedge \dots \wedge F_l(\mathbf{P}_l) \wedge F'(\mathbf{P})), \quad (\text{A7})$$

$$\forall \mathbf{P} (F_1(\mathbf{P}_1) \wedge \dots \wedge F_l(\mathbf{P}_l) \rightarrow F'(\mathbf{P})). \quad (\text{A8})$$

Then, $F \equiv G$.

Proof. If $l = 0$, then \mathbf{P} is the empty tuple, and thus, both (A7) and (A8) stand just for the formula $F'(\mathbf{P})$, so the result holds. Otherwise, we proceed by induction. Note that (A7) and (A8) are, respectively, equivalent to

$$\exists P_1 (F_1(P_1) \wedge \exists \mathbf{P}'_l (F_2(P_1, \mathbf{P}'_2) \wedge \dots \wedge F_l(P_1, \mathbf{P}'_l) \wedge F'(P_1, \mathbf{P}'_l))), \quad (\text{A9})$$

$$\forall P_1 (F_1(P_1) \rightarrow \forall \mathbf{P}'_l (F_2(P_1, \mathbf{P}'_2) \wedge \dots \wedge F_l(P_1, \mathbf{P}'_l) \rightarrow F'(P_1, \mathbf{P}'_l))). \quad (\text{A10})$$

where $\mathbf{P}'_i = P_2, \dots, P_i$. That is, $\mathbf{P}_i = P_1, \mathbf{P}'_i$. Then, by induction hypothesis, we get that the following two formulas are equivalent:

$$\exists \mathbf{P}'_l (F_2(P_1, \mathbf{P}'_2) \wedge \dots \wedge F_l(P_1, \mathbf{P}'_l) \wedge F'(P_1, \mathbf{P}'_l)),$$

$$\forall \mathbf{P}'_l (F_2(P_1, \mathbf{P}'_2) \wedge \dots \wedge F_l(P_1, \mathbf{P}'_l) \rightarrow F'(P_1, \mathbf{P}'_l)).$$

Therefore, (A10) is equivalent to

$$\forall P_1 (F_1(P_1) \rightarrow \exists \mathbf{P}'_l (F_2(P_1, \mathbf{P}'_2) \wedge \dots \wedge F_l(P_1, \mathbf{P}'_l) \wedge F'(P_1, \mathbf{P}'_l))). \quad (\text{A11})$$

Hence, it only remains to be shown that (A9) and (A11) are equivalent. Let E be the predicate expression $\lambda X_1 \dots X_n G(X_1, \dots, X_n)$ such that $H(E) = F_1(P_1)$, with $H(Q)$ being the following formula:

$$\forall V_1 \dots V_n (P_1(V_1, \dots, V_n) \leftrightarrow Q(V_1, \dots, V_n)).$$

Then,

$$\begin{aligned}
(\text{A9}) &\Leftrightarrow \exists P_1 (F_1(P_1) \wedge \exists \mathbf{P}_2 (F_2(E, \mathbf{P}'_2) \wedge \cdots \wedge F_l(E, \mathbf{P}'_l) \wedge F'(E, \mathbf{P}'_l))) \\
&\Leftrightarrow \exists P_1 F_1(P_1) \wedge \exists \mathbf{P}_2 (F_2(E, \mathbf{P}'_2) \wedge \cdots \wedge F_l(E, \mathbf{P}'_l) \wedge F'(E, \mathbf{P}'_l)) \\
&\Leftrightarrow \top \wedge \exists \mathbf{P}_2 (F_2(E, \mathbf{P}'_2) \wedge \cdots \wedge F_l(E, \mathbf{P}'_l) \wedge F'(E, \mathbf{P}'_l)) \\
&\Leftrightarrow \exists \mathbf{P}_2 (F_2(E, \mathbf{P}'_2) \wedge \cdots \wedge F_l(E, \mathbf{P}'_l) \wedge F'(E, \mathbf{P}'_l)) \\
&\Leftrightarrow \forall P_1 (F_1(P_1) \rightarrow \exists \mathbf{P}_2 (F_2(E, \mathbf{P}'_2) \wedge \cdots \wedge F_l(E, \mathbf{P}'_l) \wedge F'(E, \mathbf{P}'_l))) \\
&\Leftrightarrow (\text{A11}),
\end{aligned}$$

and the result holds. For the second-to-last equivalence, note that $F_1(P_1)$ is satisfiable and that P_1 does not occur on the right-hand side of the implication. \square

Proof of Theorem 3. Recall that io-program Ω uses private recursion if

- its predicate dependency graph has a cycle such that every vertex in it is a private symbol or
- it includes a choice rule with a private symbol in the head.

This implies that, for a program that does not use private recursion, there is a private predicate symbol that does not depend on any other private predicate symbol. Let us assume without loss of generality that this is the predicate symbol p_1/n_1 . Then, there is a predicate symbol that does not depend on any other private predicate symbol other than p_1/n_1 , which we assume to be the predicate symbol p_2/n_2 , and so on. Therefore, we have an order on the private symbols $p_1/n_1, \dots, p_l/n_l$ such that each predicate symbol p_i/n_i only depends on other predicate symbols that precede them in this order. Then, the completed definition $F_i(\mathbf{P}_i)$ of any private predicate symbol p_i/n_i can be written as

$$\forall V_1 \dots V_{n_i} (P_i(V_1, \dots, V_{n_i}) \leftrightarrow G_i(\mathbf{P}_{i-1})),$$

where $\mathbf{P}_{i-1} = P_1, \dots, P_{i-1}$ contains all free predicate variables in G_i and where we assume that $G_1(\mathbf{P}_0)$ is a first-order formula. Then, (15) and (16) can be, respectively, rewritten as (A7) and (A8). The result follows then directly from Lemma 9. \square